WIRSING SYSTEMS AND RESULTANT INEQUALITIES

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1. Introduction.

We give a survey on recent results about Wirsing systems and resultant inequalities.

We define the Mahler measure of $f(X) = a_0(X-\xi_1)\cdots(X-\xi_t) \in \mathbf{C}[X]$ by $M(f) = |a_0|\prod_{i=1}^t \max(1,|\xi_i|)$. A polynomial $f \in \mathbf{Z}[X]$ is called primitive if its coefficients have gcd 1 and if its leading coefficient is > 0. The minimal polynomial of an algebraic number ξ is the primitive irreducible polynomial $f \in \mathbf{Z}[X]$ with $f(\xi) = 0$. We define the Mahler measure $M(\xi)$ of an algebraic number α to be the Mahler measure of its minimal polynomial. Algebraic numbers are always supposed to belong to $\mathbf{C}$. We choose for every algebraic number ξ of degree t over $\mathbf{Q}$ an ordering of its conjugates $\xi^{(1)},\ldots,\xi^{(t)}$.

We first introduce Wirsing systems (named after Wirsing who studied such systems in [19]). Let I be a non-empty subset of $\{1,\ldots,t\}$. Further, let γ_i $(i \in I)$ be algebraic numbers and φ_i $(i \in I)$ non-negative reals. Then a *Wirsing system* is a system of inequalities

$$|\gamma_i - \xi^{(i)}| \le M(\xi)^{-\varphi_i} \quad (i \in I) \qquad \text{in algebraic numbers } \xi \text{ of degree } t. \tag{1.1}$$

Second, we introduce resultant inequalities. The resultant of two polynomials $f, g \in \mathbf{C}[X]$ of degrees r, t, respectively, say $f = a_0X^r + a_1X^{r-1} + \cdots + a_r$ with $a_0 \ne 0$ and $g = b_0X^t + \cdots + b_t$ with $b_0 \ne 0$, is defined by the determinant of order $r+t$,

$$R(f,g) = \begin{vmatrix} a_0 & a_1 & \cdots & \cdots & a_r & & & \\ & \ddots & & & & \ddots & & \\ & & a_0 & a_1 & \cdots & \cdots & a_r \\ b_0 & b_1 & \cdots & b_t & & & \\ & \ddots & & & \ddots & & \\ & & \ddots & & & \ddots & \\ & & & b_0 & b_1 & \cdots & b_t \end{vmatrix} \tag{1.2}$$

where the first t rows consist of coefficients of f and the last r rows of coefficients of g. It is well-known that if $f(X) = a_0 \prod_{i=1}^{r}(X - \alpha_i)$, $g(X) = b_0 \prod_{j=1}^{t}(X - \xi_j)$, then

$$R(f,g) = a_0^t b_0^r \prod_{i=1}^{r} \prod_{j=1}^{t} (\alpha_i - \xi_j). \tag{1.3}$$

This implies at once

$$|R(f,g)| \le 2^{rt} M(f)^t M(g)^r. \tag{1.4}$$

By a *resultant inequality* we mean a Diophantine inequality of the shape

$$0 < |R(f,g)| < M(g)^{r-\kappa} \quad \text{in polynomials } g \in \mathbf{Z}[X] \text{ of degree } t \tag{1.5}$$

where $\kappa > 0$ and where f is a fixed polynomial in $\mathbf{Z}[X]$ of degree r. Clearly, (1.5) is unsolvable if $\kappa > r$.

Wirsing systems and resultant inequalities are closely related. Roughly speaking, if ξ is an algebraic number of degree t satisfying (1.1) then its minimal polynomial g satisfies (1.5), where $f \in \mathbf{Z}[X]$ is a non-zero polynomial of degree r with zeros γ_i $(i \in I)$, and where $\kappa = \sum_{i \in I} \varphi_i$. Conversely, to any inequality (1.5) we can associate a finite number of systems (1.1) in which the numbers γ_i $(i \in I)$ are zeros of f and $\sum_{i \in I} \varphi_i$ is slightly smaller than κ, such that any *primitive, irreducible* polynomial g of degree t with (1.5) is the minimal polynomial of a solution ξ of one of the corresponding systems (1.1). We have made this correspondence concrete in Proposition 2.1 in Section 2. More precisely, with the method of proof of Proposition 2.1 one may deduce from an upper bound for the number of solutions ξ of (1.1) an upper bound for the number of primitive, irreducible polynomials g with (1.5) and vice versa.

A particular instance of (1.1) is the single inequality

$$|\gamma - \xi| \le M(\xi)^{-\varphi} \quad \text{in algebraic numbers } \xi \text{ of degree } t$$

where γ is a fixed algebraic number and φ a positive real. Wirsing [19] showed that this inequality has only finitely many solutions if $\varphi > 2t$ and Schmidt [14] improved this to $\varphi > t + 1$ which is best possible. In the special case that $t = 1$ we can rewrite (1.5) as a Thue inequality

$$0 < |F(u,v)| \le \big(\max(|u|,|v|)\big)^{r-\kappa} \quad \text{in } u, v \in \mathbf{Z},$$

where $g = uX + v$ with $u, v \in \mathbf{Z}$ and where F is a binary form of degree r with coefficients in $\mathbf{Z}$. By a theorem of Roth [11], this latter inequality has only finitely many solutions if $\kappa > 2$ and F is irreducible. Hence (1.5) has only finitely many solutions if $t = 1$, $\kappa > 2$ and f is irreducible.

The following result follows from work of Wirsing, Schmidt and Ru and Wong:

Theorem 1.1. *(i) Let $f \in \mathbf{Z}[X]$ be a polynomial of degree r without multiple zeros, let t be a positive integer and let $\kappa > 2t$. Then (1.5) has only many solutions in (not necessarily primitive or irreducible) polynomials $g \in \mathbf{Z}[X]$ of degree t.*
(ii) Let I be a non-empty subset of $\{1, \ldots, t\}$ and γ_i $(i \in I)$ algebraic numbers. Further, let φ_i $(i \in I)$ be positive reals with $\sum_{i \in I} \varphi_i > 2t$. Then (1.1) has only finitely many solutions in algebraic numbers ξ of degree t.

Wirsing [19] showed that for any tuple of algebraic numbers γ_i $(i \in I)$ and any tuple of non-negative reals φ_i $(i \in I)$ with

$$\sum_{i \in I} \varphi_i > 2t \cdot \sum_{k=1}^{\#I} \frac{1}{2k - 1}$$

system (1.1) has only finitely many solutions in algebraic numbers ξ of degree t. Further, he proved that if f is any polynomial in $\mathbf{Z}[X]$ of degree r without multiple zeros and if $\kappa > 2t \cdot \sum_{k=1}^{t} \frac{1}{2k-1}$ then (1.5) has only finitely many solutions in polynomials $g \in \mathbf{Z}[X]$ of degree t. By applying his Subspace Theorem, Schmidt [16] extended Wirsing's result on (1.5) to $\kappa > 2t$, i.e., proved part (i) of Theorem 1.1, but only for polynomials $f \in \mathbf{Z}[X]$ having no irreducible factors in $\mathbf{Z}[X]$ of degree $\leq t$. Finally, Ru and Wong [13] proved a general result (Theorem 4.1 on p. 212 of their paper) which contains as a special case part (i) of Theorem 1.1 without Schmidt's constraint on f. We mention that from the result of Ru and Wong one may deduce a generalization of part (i) of Theorem 2.1 involving p-adic absolute values. Also the result of Ru and Wong is a consequence of Schmidt's Subspace Theorem. Either by combining part (i) of Theorem 1.1 with Proposition 2.1 from Section 2, or by applying directly the result of Ru and Wong, one obtains part (ii) of Theorem 1.1.

In the other direction, Schmidt [16] proved that if $\kappa < t + 1$ and if f is any polynomial in $\mathbf{Z}[X]$ of degree r without multiple zeros and with a complex conjugate pair of zeros of degree $> t$, then (1.5) has infinitely many solutions in polynomials $g \in \mathbf{Z}[X]$ of degree $\leq t$. Further, in [16] Schmidt gave for every $t \geq 1$ examples of polynomials $f \in \mathbf{Z}[X]$ such that for every $\kappa < 2t$ inequality (1.5) has infinitely many solutions in polynomials $g \in \mathbf{Z}[X]$ of degree t. These results of Schmidt on resultant inequalities do not have consequences for Wirsing systems (indeed, by Proposition 2.1, a particular Wirsing system has infinitely many solutions in algebraic numbers of degree t only if the corresponding resultant inequality has infinitely many solutions in primitive, irreducible polynomials g of degree t; but in his examples Schmidt did not show that the polynomials g under consideration are primitive or irreducible). In Section 3, we show that for every $t \geq 1$ there is a tuple of algebraic numbers $(\gamma_i : i \in I)$ with the property that for every $\kappa < 2t$ there are non-negative reals φ_i $(i \in I)$ with $\sum_{i \in I} \varphi_i = \kappa$ such that (1.1) has infinitely many solutions in algebraic numbers of degree t. So for certain polynomials f and for certain tuples of algebraic numbers γ_i $(i \in I)$ Theorem 1.1 is best possible.

However, in many cases, Theorem 1.1 can be improved. An irreducible polynomial $f \in \mathbf{Z}[X]$ is said to be t times transitive if for any two ordered tuples $(\gamma_{i_1}, \ldots, \gamma_{i_t})$, $(\gamma_{j_1}, \ldots, \gamma_{j_t})$ consisting of distinct zeros of f, there is a $\mathbf{Q}$-isomorphism σ with $\sigma(\gamma_{i_k}) = \gamma_{j_k}$ for $k = 1, \ldots, t$. Then we have:

Theorem 1.2. *(i) Let $f \in \mathbf{Z}[X]$ be an irreducible polynomial of degree r which is t times transitive and let $\kappa > t + 1$. Then (1.5) has only finitely many solutions in polynomials $g \in \mathbf{Z}[X]$ of degree t.*
(ii) Suppose that γ_i $(i \in I)$ are zeros of an irreducible polynomial $f \in \mathbf{Z}[X]$ which is t times transitive and let $\sum_{i \in I} \varphi_i > t + 1$. Then (1.1) has only finitely many solutions in algebraic numbers ξ of degree t.

Part (i) follows from a general result of Schmidt on norm form inequalities (cf. [15], Theorem 3 or [17], Theorem 10A, p. 237) and part (ii) follows from part (i) and Proposition 2.1. We believe that if $\sum_{i \in I} \varphi_i > t + 1$ or $\kappa > t + 1$ then the general pattern is, that (1.1), (1.5) have only finitely many solutions, and that only for numbers γ_i or polynomials f of a special shape the number of solutions is infinite. As yet we are not able to make this more precise.

Recently, Győry and Ru [7] obtained, as a special case of a more general result of theirs, a result for inequality (1.5) in which also the polynomials f are allowed to vary within a limited range. Their proof is based on the "Subspace Theorem with moving targets," due to Ru and Vojta [12]. From the result of Győry and Ru it is possible to deduce the following generalization of Theorem 1.1:

Theorem 1.3. *Let K be a fixed number field and let r, t be positive integers.*

(i) Let $\kappa > 2t$. Then there does not exist an infinite sequence of pairs of polynomials $\{f_n, g_n\}_{n=1}^{\infty}$ in $\mathbf{Z}[X]$ such that for $n = 1, 2, \ldots,$

$$\begin{aligned}&0 < |R(f_n, g_n)| < M(g_n)^{r-\kappa},\\ &f_n \text{ has splitting field } K,\ f_n \text{ has no multiple zeros,}\\ &\deg f_n = r,\ \deg g_n = t\end{aligned}$$

and such that $\lim_{n\to\infty}\big(\log M(f_n)/\log M(g_n)\big) = 0$.
(ii) Let I be a non-empty subset of $\{1, \ldots, t\}$. Further, let φ_i $(i \in I)$ be non-negative reals with $\sum_{i\in I}\varphi_i > 2t$. Then there does not exist an infinite sequence of tuples of algebraic numbers $(\xi_n; \gamma_{in}\,(i \in I))_{n=1}^{\infty}$ such that for $n = 1, 2, \ldots$ and for $i \in I$,

$$|\gamma_{in} - \xi_n^{(i)}| < M(\xi_n)^{-\varphi_i}, \quad \gamma_{in} \in K, \deg \xi_n = t,$$

and such that $\lim_{n\to\infty}\big(\log \max_{i\in I} M(\gamma_{in})/\log M(\xi_n)\big) = 0$.

Part (i) follows directly from Theorem 6 of [7] (which gives in fact a p-adic generalization), while part (ii) is obtained by combining part (i) with Proposition 2.1. Győry's survey paper [6] contains more information about resultant inequalities (1.5), resultant equations $R(f, g) = c$ in polynomials $g \in \mathbf{Z}[X]$ of degree t where f is a fixed polynomial of degree r and c is a constant, and their applications.

We now turn to quantitative results. Let again I be a non-empty subset of $\{1, \ldots, t\}$, let γ_i $(i \in I)$ be algebraic numbers with

$$\max_{i\in I} M(\gamma_i) \le M, \quad [\mathbf{Q}(\gamma_i : i \in I) : \mathbf{Q}] = r \tag{1.6}$$

and let φ_i $(i \in I)$ be non-negative reals. Further, let $0 < \delta < 1$. Put $\theta := \big(\sum_{i\in I}\varphi_i\big) - 2t$. By making explicit Wirsing's arguments, Evertse [2] (Theorem 2) proved that if

$$\sum_{i\in I}\varphi_i \ge (2t + \delta)\sum_{k=1}^{\#I}\frac{1}{2k-1} \tag{1.7}$$

then the Wirsing system (1.1) has at most

$$2\times 10^7 \cdot t^7\delta^{-4}\log 4r \cdot \log\log 4r \tag{1.8}$$

solutions with

$$M(\xi) \ge \max(4^{t(t+1)/\theta}, M) \tag{1.9}$$

and at most

$$2^{t^2+3t+\theta+4}\Big(1+\frac{\log(2+\theta^{-1})}{\log(1+\theta/t)}\Big)+t\cdot\frac{\log\log 4M}{\log(1+\theta/t)}$$

solutions with $M(\xi)<\max(4^{t(t+1)/\theta},M)$.
As a consequence, Evertse ([2], Thm. 3) derived the following result for resultant inequalities: if $f\in\mathbf{Z}[X]$ is a polynomial of degree r without multiple zeros and if

$$\kappa\geq(2t+\delta)\sum_{k=1}^{t}\frac{1}{2k-1} \tag{1.10}$$

then there are at most

$$10^{15}(\delta^{-1})^{t+3}(100r)^t\log 4r\log\log 4r$$

primitive, irreducible polynomials $g\in\mathbf{Z}[X]$ of degree t satisfying (1.5) and

$$M(g)\geq\big(2^{8r^2t}M(f)^{4(r-1)t}\big)^{\delta^{-1}(1+\frac{1}{3}+\cdots+\frac{1}{2t-1})^{-1}}.$$

We mention that independently, Locher [10] obtained for the single inequality $|\gamma-\xi|\leq M(\xi)^{-\varphi}$ in algebraic numbers ξ of degree t a quantitative result similar to the one mentioned above. More generally, Locher considered inequalities involving p-adic absolute values.

It was to be expected that in Evertse's quantitative results on (1.1), (1.5), respectively, condition (1.7) could be relaxed to $\sum_{i\in I}\varphi_i>2t$ and condition (1.10) to $\kappa>2t$, but Wirsing's method did not provide for that.

By a method very different from Wirsing's Evertse proved the following result for resultant inequalities. Evertse's proof is basically to go through the proof of the Quantitative Subspace Theorem as in [1] and to show that in the particular case under consideration all occurring subspaces are one-dimensional.

Theorem 1.4. ([3]) *Suppose $f\in\mathbf{Z}[X]$ is a polynomial of degree r without multiple zeros. Let $\kappa=2t+\delta$ with $0<\delta<1$. Then there are at most*

$$2^{9t+60}t^{2t+20}\delta^{-t-5}r^t\log 4r\log\log 4r \tag{1.11}$$

primitive, irreducible polynomials $g\in\mathbf{Z}[X]$ of degree t satisfying (1.5) and

$$M(g)\geq\Big(2^{2r^2}M(f)^{4r-4}\Big)^{t/\delta}.$$

Let I be a non-empty subset of $\{1,\dots,t\}$ and let φ_i $(i\in I)$ be non-negative reals as in (1.1). Assume that $\sum_{i\in I}\varphi_i > 2t$. By combining Theorem 1.4 with Proposition 2.1 in a more explicit form one obtains an upper bound for the number of"large solutions" (i.e. solutions with (1.9)) of the Wirsing system (1.1) which is similar to (1.11). Evertse's method of proof of Theorem 1.4, when applied directly to Wirsing systems instead of resultant inequalities, might produce a bound with a better dependence on r and t than (1.11), but with the same dependence on δ, i.e., δ^{-t}.

On the other hand, Hirata-Kohno discovered another method to estimate from above the number of large solutions of (1.1), which combines techniques from the proof of the Quantitative Subspace Theorem with ideas of Ru and Wong [13] and with the notion of Nochka weight (see [4], Section 2–4). This is work in preparation; see [8]. It is an open problem to obtain an upper bound for the number of large solutions of (1.1) of a similar quality as (1.8) if the tuple of reals $(\varphi_i : i\in I)$ does not satisfy (1.7).

It does not seem to be possible to prove quantitative versions of Theorems 1.2 and 1.3. Further, it should be noted that Theorem 1.4 gives only a bound for the number of primitive, irreducible polynomials g, whereas part (i) of Theorem 1.1 gives a qualitative finiteness result for polynomials g without this constraint. We explain below that it is likely to be very difficult to get an explicit upper bound for the number of reducible or non-primitive polynomials g with (1.5).

First suppose we have an upper bound N for the number of not necessarily primitive polynomials $g\in\mathbf{Z}[X]$ of degree t satisfying (1.5) when $\kappa > 2t$. Pick a primitive polynomial g^* of degree t with (1.5). Then for each nonzero integer d with

$$|d| \le \Big(M(g^*)^{r-\kappa}|R(f,g^*)|^{-1}\Big)^{1/\kappa} \tag{1.12}$$

the polynomial $g=dg^*$ satisfies (1.5). Consequently, the right-hand side of (1.11) is at most N, that is,

$$|R(f,g^*)| \ge N^{-1}M(g^*)^{r-\kappa}.$$

Hence any explicit upper bound for the number of non-primitive polynomials with (1.5) would yield an effective lower bound for $|R(f,g^*)|$ which is far beyond what is possible with the presently available techniques.

Now suppose for instance that $t = 2$, $\kappa > 2t = 4$, and let N be an upper bound for the number of primitive, reducible polynomials $g \in \mathbf{Z}[X]$ of degree 2 with (1.5). We use the notation $|a, b| := \max(|a|, |b|)$. Fix a polynomial $g_1 = q_1 X - p_1$ with $p_1, q_1 \in \mathbf{Z}$ and $\gcd(p_1, q_1) = 1$. Note that $M(g_1) = |p_1, q_1|$. Then for every polynomial $g_2 = q_2 X - p_2$ with

$$p_2, q_2 \in \mathbf{Z}, \quad \gcd(p_2, q_2) = 1, \tag{1.13}$$

$$|p_2, q_2| \leq \Big(2^{-r} M(f)^{-1} |R(f, g_1)|^{-1} \cdot M(g_1)^{r-\kappa}\Big)^{1/\kappa} \tag{1.14}$$

we have by (1.4),

$$\begin{aligned} |R(f, g_1 g_2)| &= |R(f, g_1)| \cdot |R(f, g_2)| \\ &\leq 2^{-r} M(f)^{-1} M(g_1)^{r-\kappa} M(g_2)^{-\kappa} \cdot 2^r M(f) M(g_2)^r \\ &= M(g_1 g_2)^{r-\kappa}. \end{aligned}$$

Now the number of pairs (p_2, q_2) with (1.13), (1.14) is bounded from below by the square of the right-hand side of (1.14) multiplied by some absolute constant. This last number is bounded from above by N, since each pair (p_2, q_2) with (1.13), (1.14) yields a solution $g = g_1 g_2$ of (1.5) which is primitive and reducible. This implies that for some effective constant $c_1 = c_1(r)$ depending only on r we have

$$|R(f, g_1)| \geq c_1 N^{-\kappa/2} M(f)^{-1} M(g_1)^{r-\kappa}. \tag{1.15}$$

Now let $f = a_0 (X - \alpha_1) \cdots (X - \alpha_r)$ and $\alpha \in \{\alpha_1, \ldots, \alpha_r\}$. Then by (1.4) we have

$$\begin{aligned} |R(f, g_1)| &= |a_0 (p_1 - \alpha_1 q_1) \cdots (p_1 - \alpha_r q_1)| \\ &\leq 2^r M(f) |\alpha - (p_1/q_1)| \cdot |p_1, q_1|^r. \end{aligned}$$

Together with (1.15) and $M(g_1) = |p_1, q_1|$, this implies that there is an effective constant $c_2 = c_2(r)$ such that

$$|\alpha - \frac{p_1}{q_1}| \geq c_2 N^{-\kappa/2} M(f)^{-2} \cdot |p_1, q_1|^{-\kappa}.$$

Therefore, any explicit upper bound N for the number of primitive, reducible polynomials g of degree 2 with (1.5) would yield a very strong effective improvement of Liouville's inequality.

We recall that earlier, Schmidt [18] gave another example of a Diophantine inequality, an explicit upper bound for the number of whose solutions implies a very strong effective improvement of some Liouville-type inequality.

2. Wirsing systems vs. resultant inequalities.

We prove the following:

Proposition 2.1. *Let $f \in \mathbf{Z}[X]$ be a polynomial of degree r without multiple zeros. Let $\kappa_0 > 0$. Then the following two assertions are equivalent:*
(i) for any $\kappa > \kappa_0$ inequality (1.5) *has only finitely many solutions in* primitive, irreducible *polynomials $g \in \mathbf{Z}[X]$ of degree t;*
(ii) for any non-empty subset I of $\{1, \dots, t\}$, for any tuple $(\gamma_i : i \in I)$ consisting of (possibly equal) zeros of f and any tuple $(\varphi_i : i \in I)$ consisting of non-negative reals with $\sum_{i \in I} \varphi_i > \kappa_0$, system (1.1) *has only finitely many solutions in algebraic numbers ξ of degree t.*

We keep the notation introduced in Section 1. Let $f \in \mathbf{Z}[X]$ be a polynomial of degree r without multiple zeros. We write $f = a_0 \prod_{i=1}^{r} (X - \alpha_i)$. Constants implied by $\ll$, $\gg$ depend on f. We write again $|a, b|$ for $\max(|a|, |b|)$.

First assume (i). Consider system (1.1) where $\sum_{i \in I} \varphi_i =: \kappa > \kappa_0$ and where γ_i $(i \in I)$ are not necessarily distinct zeros of f. Let g be the minimal polynomial of a solution ξ of (1.1). Then $g = b_0 \prod_{j=1}^{t} (X - \xi^{(j)})$ with $b_0 \in \mathbf{Z}$. We may rewrite (1.1) as

$$|\alpha_i - \xi^{(j)}| \le M(g)^{-\varphi^*_{ij}} \quad ((i,j) \in J)$$

where J is a subset of $V := \{1, \dots, r\} \times \{1, \dots, t\}$ and where $\sum_{(i,j) \in J} \varphi^*_{ij} = \kappa$. For the pairs $(i,j) \in V \backslash J$ we use $|\alpha_i - \xi^{(j)}| \ll |1, \xi^{(j)}|$. Together with (1.3) this gives

$$\begin{aligned} |R(f,g)| &\ll |b_0|^r \prod_{(i,j) \in V \backslash J} |1, \xi^{(j)}| \cdot \prod_{(i,j) \in J} |\alpha_i - \xi^{(j)}| \\ &\ll M(g)^r \prod_{(i,j) \in J} |\alpha_i - \xi^{(j)}| \ll M(g)^{r - \sum_{(i,j) \in J} \varphi^*_{ij}} \\ &\ll M(g)^{r - \kappa}. \end{aligned}$$

By (i), the latter inequality has only finitely many solutions in primitive, irreducible polynomials $g \in \mathbf{Z}[X]$ of degree t. Hence system (1.1) has only finitely many solutions in algebraic numbers ξ of degree t. This proves (ii).

Now assume that (ii) holds. Let $\kappa > \kappa_0$. The argument is basically Wirsing's [19]. Let $g \in \mathbf{Z}[X]$ be a primitive, irreducible polynomial of degree t satisfying (1.5). Denote the zeros of g (which are all conjugates of some number ξ) by $\xi^{(1)}, \dots, \xi^{(t)}$. Let b_0 be the leading coefficient of g. For $j = 1, \dots, t$, let α_{i_j} be the zero of f which is closest to $\xi^{(j)}$. Then for any other zero α_i of f we have

$$|\alpha_i - \xi^{(j)}| \geq \frac{1}{2}(|\alpha_i - \xi^{(j)}| + |\alpha_{i_j} - \xi^{(j)}|) \geq \frac{1}{2}|\alpha_i - \alpha_{i_j}| \gg 1.$$

By distinguishing between the two cases $|\xi^{(j)}| \geq 2|1, \alpha_i|$ and $|\xi^{(j)}| < 2|1, \alpha_i|$ we obtain $|\alpha_i - \xi^{(j)}| \gg |1, \xi^{(j)}|$. Let I be the set of indices $j \in \{1, \dots, t\}$ for which $|\alpha_{i_j} - \xi^{(j)}| < 1$. For $j \in I$ we have $|\xi^{(j)}| \ll 1$, while for $j \notin I$ we have $|\alpha_{i_j} - \xi^{(j)}| \gg |1, \xi^{(j)}|$. Consequently, by (1.3),

$$\begin{aligned} M(g)^{r-\kappa} &\gg |b_0|^r \prod_{i=1}^{r} \prod_{j=1}^{s} |\alpha_i - \xi^{(j)}| \\ &\gg |b_0|^r \prod_{i=1}^{r} \prod_{j=1}^{s} |1, \xi^{(j)}| \cdot \prod_{j \in I} |\alpha_{i_j} - \xi^{(j)}| = M(g)^r \prod_{j \in I} |\alpha_{i_j} - \xi^{(j)}|. \end{aligned}$$

Hence

$$\prod_{j \in I} |\alpha_{i_j} - \xi^{(j)}| \ll M(\xi)^{-\kappa}.$$

Denote the factor with index j in the product on the left-hand side by A_j. We have $A_j < 1$ for $j \in I$, hence if $M(\xi)$ is sufficiently large there are reals ψ_j ($j \in I$) with $A_j \leq M(\xi)^{-\psi_j}$, $\psi_j > 0$ for $j \in I$ and $\sum_{j \in I} \psi_j = \kappa_1$ with $\kappa_0 < \kappa_1 < \kappa$. The ψ_j vary with ξ, but we may choose φ_j from a finite set such that $0 \leq \varphi_j \leq \psi_j$ for $j \in I$ and $\kappa_0 < \sum_{j \in I} \varphi_j < \kappa_1$. (We may cover the bounded set of points $(\psi_i : i \in I) \in \mathbf{R}^{\# I}$ with $\psi_i \geq 0$, $\kappa_0 \leq \sum_{i \in I} \psi_i \leq \kappa$ by a finite number of very small cubes and choose for $(\varphi_i : i \in I)$ the lower left vertex of the cube containing $(\psi_i : i \in I)$; see [2], pp. 79–82 for a more explicit argument). Thus ξ satisfies one of finitely many systems (1.1) each of which has by (ii) only finitely many solutions. Hence (1.5) has only finitely many solutions in primitive, irreducible polynomials g of degree t. This proves (i). □

3. Wirsing systems with infinitely many solutions.

We prove:

Proposition 3.1. *For every $t \geq 1$, there are algebraic numbers $\gamma_1, \ldots, \gamma_t$ and a constant D such that the system of inequalities*

$$|\gamma_i - \xi^{(i)}| \leq D \cdot M(\xi)^{-2} \quad (i = 1, \ldots, t) \tag{3.1}$$

has infinitely many solutions in algebraic numbers ξ of degree t.

It follows that for every $\delta > 0$ there are non-negative reals $\varphi_1, \ldots, \varphi_t$ with $\varphi_1 + \cdots + \varphi_t = 2t - \delta$ such that (1.1) with $I = \{1, \ldots, t\}$ has infinitely many solutions in algebraic numbers ξ of degree t.

Let α be a real algebraic irrational number. By Dirichlet's Theorem, there is a constant $C > 1$ depending on α such that the inequality

$$|\alpha - \eta| \leq C \cdot M(\eta)^{-2} \quad \text{in } \eta \in \mathbf{Q} \tag{3.2}$$

has infinitely many solutions. Let $p(X), q(X) \in \mathbf{Z}[X]$ be polynomials without common zeros and with $\max(\deg p, \deg q) = t$, such that the equation $p(x)/q(x) = \alpha$ has exactly t distinct solutions, $\gamma_1, \ldots, \gamma_t$, say. So these are the zeros of $p(X) - \alpha q(X)$. For $\eta = a/b$ with $a, b \in \mathbf{Z}$, $b > 0$, $\gcd(a, b) = 1$, define the polynomial $g_\eta := bp(X) - aq(X)$. For instance, by [9], pp. 59–61, there are constants C_1, C_2, depending only on t, such that for a polynomial $f \in \mathbf{C}[X]$ of degree t one has $C_1 \leq M(f)/H(f) \leq C_2$, where $H(f)$ is the maximum of the absolute values of the coefficients of f. Since $M(\eta) = |a, b|$, this implies that

$$M(g_\eta) \gg\ll M(\eta), \tag{3.3}$$

where here and below constants implied by $\ll$, $\gg$ depend only on α, p and q. We prove a few auxiliary results.

Lemma 3.2. *Let $\eta \in \mathbf{Q}$ with $\eta \neq 0$. Then for each solution ξ of $p(x)/q(x) = \eta$ there is a solution γ_i of $p(x)/q(x) = \alpha$ such that*

$$|\gamma_i - \xi| \ll |\alpha - \eta|.$$

Proof. First suppose that $|\eta| \geq 2 \cdot |1, \alpha|$. Then by (3.3) we have for any solution γ of $p(x)/q(x) = \alpha$,

$$|\gamma - \xi| \ll |1, \xi| \ll |\eta| \ll |\alpha - \eta|$$

which implies Lemma 3.2.

Now assume that $|\eta| \leq 2 \cdot |1, \alpha|$. Then $|\xi| \ll 1$ by (3.3). Choose γ_i from $\gamma_1, \ldots, \gamma_t$ which is closest to ξ. Then for γ_j with $j \neq i$ we have $|\gamma_j - \xi| \geq \frac{1}{2}(|\gamma_j - \xi| + |\gamma_i - \xi|) \gg |\gamma_j - \gamma_i| \gg 1$. Hence

$$|\gamma_i - \xi| \ll \prod_{j=1}^{t} |\gamma_j - \xi| \ll |p(\xi) - \alpha q(\xi)| = |q(\xi)| \cdot |\alpha - \eta| \ll |\alpha - \eta|$$

which is Lemma 3.2. □

Lemma 3.3 *There are only finitely many solutions* $\eta \in \mathbf{Q}$ *of* (3.2) *such that the polynomial* g_η *is reducible in* $\mathbf{Q}[X]$.

Proof. Let $\eta \in \mathbf{Q}$ be a solution of (3.2) for which g_η is reducible in $\mathbf{Q}[X]$. Because of the multiplicativity of the Mahler measure, we may choose an irreducible factor $g_{1,\eta} \in \mathbf{Z}[X]$ of g_η such that $M(g_{1,\eta}) \leq M(g_\eta)^{1/2}$. So if ξ is a zero of $g_{1,\eta}$ we have by (3.3),

$$M(\xi) \ll M(\eta)^{1/2}. \tag{3.4}$$

Let $\xi^{(1)}, \ldots, \xi^{(s)}$ be the conjugates of ξ, where $s = \deg g_{1,\eta} < t$. Then by Lemma 3.2 and (3.2), (3.4) there are solutions $\gamma_1, \ldots, \gamma_s$ of $p(x)/q(x) = \alpha$ such that

$$|\gamma_j - \xi^{(j)}| \ll |\alpha - \eta| \ll M(\eta)^{-2} \ll M(\xi)^{-4} \quad \text{for } j = 1, \ldots, s.$$

Now this is a system of the shape (1.1) with $t = s$ and with the sum of the φ_i's equal to $4s > 2s$. Hence, by part (ii) of Theorem 1.1, this system has only finitely many solutions in algebraic numbers ξ of degree s. But since for the ξ under consideration we have $\eta = p(\xi)/q(\xi)$ this gives only finitely many possibilities for η. This proves Lemma 3.3. □

Proof of Proposition 3.1. Lemma 3.3 implies that (3.2) has infinitely many solutions η for which g_η is irreducible. For such η, let ξ be a zero of g_η and let $\xi^{(1)}, \ldots, \xi^{(t)}$ be the conjugates of ξ. Now by (3.3) we have $M(\xi) \ll M(\eta)$. Together with Lemma 3.2 and (3.2) this implies that there are solutions $\gamma_1, \ldots, \gamma_t$ of $p(x)/q(x) = \alpha$ with

$$|\gamma_j - \xi^{(j)}| \ll |\alpha - \eta| \ll M(\eta)^{-2} \ll M(\xi)^{-2}$$

for $j = 1, \ldots, t$. This is a system of type (3.1). Now it is clear, that for some choice of the γ_i this system has infinitely many solutions. □

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