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Distances between the conjugates of an algebraic number

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In memory of Professor Béla Brindza

Abstract. Let K be a given number field of degree $r \geqslant 3$, denote by $\xi \mapsto \xi^{(i)}$ $(i = 1, \dots, r)$ the isomorphic embeddings of K into $\mathbb{C}$, and let Σ be a subset of $\{1, \dots, r\}$ of cardinality at least 2. Denote by $M(\alpha)$ the Mahler measure of an algebraic number α. By an elementary argument one shows that $(*)$ $\prod_{\{i,j\}\subset\Sigma} |\alpha^{(i)} - \alpha^{(j)}| \geqslant C \cdot M(\alpha)^{-\kappa}$ holds for all α with $K = \mathbb{Q}(\alpha)$, with $C = 2^{-r(r-1)/2}$ and $\kappa = r - 1$. In the present paper we deduce inequalities $(*)$ with $\kappa < r - 1$ and with a constant C depending on K which are valid for all α with $\mathbb{Q}(\alpha) = K$. We obtain such inequalities with an ineffective constant C, using arguments and results from [7], [8], and with an effective constant C using a result from [9], [11].

Define $\kappa(\Sigma)$ to be the infimum of all real numbers κ for which there exists a constant $C > 0$ such that $(*)$ holds for every α with $\mathbb{Q}(\alpha) = K$. Then clearly $\kappa(\Sigma) \leqslant r - 1$. We describe the sets Σ for which $\kappa(\Sigma) = r - 1$ and we give upper bounds for $\kappa(\Sigma)$ in case that it is smaller than $r - 1$. For cubic fields we give the precise value of $\kappa(\Sigma)$ for each set Σ. This solves a problem posed by Mignotte and Payafar [12, p. 187].

1. Introduction

Given an algebraic number α of degree r, we denote by $\alpha^{(1)}, \dots, \alpha^{(r)}$ the conjugates of α. Letting a_0 be the positive integer such that the

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polynomial $a_0\prod_{i=1}^r(X-\alpha^{(i)})$ has integer coefficients with greatest common divisor 1, we define the Mahler measure and discriminant of α by

$$M(\alpha) := a_0\prod_{i=1}^{r}\max\left(1,|\alpha^{(i)}|\,\right), \tag{1.1}$$

$$D(\alpha) := a_0^{2r-2}\prod_{1\leqslant i<j\leqslant r}\left(\alpha^{(i)}-\alpha^{(j)}\right)^2, \tag{1.2}$$

respectively.

Let Σ be a subset of $\{1,\dots,r\}$ of cardinality $|\Sigma|\geqslant 2$. Then, taking the product over all 2-element subsets of Σ,

$$\begin{aligned}\prod_{\{i,j\}\subset\Sigma}|\alpha^{(i)}-\alpha^{(j)}| &\geqslant \prod_{1\leqslant i<j\leqslant r}\frac{|\alpha^{(i)}-\alpha^{(j)}|}{2\max(1,|\alpha^{(i)}|)\max(1,|\alpha^{(j)}|)}\\ &= 2^{-r(r-1)/2}|D(\alpha)|^{1/2}M(\alpha)^{1-r} \qquad (1.3)\\ &\geqslant 2^{-r(r-1)/2}M(\alpha)^{1-r}\end{aligned}$$

where the last inequality follows from the fact that $D(\alpha)$ is a non-zero integer.

Our purpose is to obtain improvements of (1.3) with an exponent on $M(\alpha)$ larger than $1-r$. More specifically, one could think of improvements

$$\prod_{\{i,j\}\subset\Sigma}|\alpha^{(i)}-\alpha^{(j)}|\geqslant C(r)M(\alpha)^{-\kappa} \tag{1.4}$$

with $\kappa<r-1$ and a constant $C(r)>0$ depending only on r which are valid for all algebraic numbers of degree r, or, for a given number field K of degree r,

$$\prod_{\{i,j\}\subset\Sigma}|\alpha^{(i)}-\alpha^{(j)}|\geqslant C(K)M(\alpha)^{-\kappa} \tag{1.5}$$

with $\kappa<r-1$ and a constant $C(K)>0$ depending on K, which are valid for all α with $\mathbb{Q}(\alpha)=K$. Apart from a few special cases settled in the literature, it seems to be difficult to obtain improvements of the shape (1.4). In this paper we consider only (1.5).

We recall some results from the literature dealing with the case $|\Sigma|=2$, i.e., inequalities of the shape

$$|\alpha^{(i)}-\alpha^{(j)}|\geqslant C\cdot M(\alpha)^{-\kappa}\,, \tag{1.6}$$

where $\Sigma = \{i, j\}$, $\kappa < r - 1$ and either $C = C(r)$ where $r = \deg \alpha$ or $C = C(K)$ where $K = \mathbb{Q}(\alpha)$. MIGNOTTE and PAYAFAR [12, Theorems 1, 2] proved (1.6) with $\kappa = (r-1)/2$ and $C = 2^{1-r(r-1)/4}$ if $\alpha^{(i)}, \alpha^{(j)} \notin \mathbb{R}$ and $\alpha^{(j)} \neq \overline{\alpha^{(i)}}$; with $\kappa = (r-1)/3$ and $C = 2^{(4-r(r-1))/6}$ if $\alpha^{(i)} \in \mathbb{R}$, $\alpha^{(j)} \notin \mathbb{R}$; and with $\kappa = 2$ and $C = 2^{1-r}$ if $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a normal extension. Further, the author [7, Theorem 4] obtained (1.6) with $\kappa = \frac{41}{42}(r-1)$ and with a constant $C = C(K)$ depending on $K = \mathbb{Q}(\alpha)$, where no restrictions on $\mathbb{Q}(\alpha)$, $\alpha^{(i)}$, $\alpha^{(j)}$ are imposed. Here C is not effectively computable from the method of proof. Let $\kappa(r)$ be the infimum of all κ for which there is a constant C such that (1.6) holds for all algebraic numbers α of degree r and all i, j. Computations of COLLINS [6] suggest that $\kappa(r) = r/2$. BUGEAUD and MIGNOTTE [5] gave an example showing that if r is even and $r \geqslant 6$ then $\kappa(r) \geqslant r/2$. More generally, Bugeaud and Mignotte gave an example showing that for all integers k, n with $k \geqslant 2$, $n \geqslant 3$ there are algebraic numbers α of degree $r = kn$ and of arbitrarily large Mahler measure, and sets Σ of cardinality k, such that

$$\prod_{\{i,j\} \subset \Sigma} |\alpha^{(i)} - \alpha^{(j)}| < c(n,k) M(\alpha)^{-(1-k^{-1})r}.$$

Estimates for the distances between the conjugates of an algebraic number play an important role in complexity analyses of algorithms for polynomials. Further, they are of crucial importance in the study of the difference $w_n(\xi) - w_n^*(\xi)$, where $w_n(\xi)$, $w_n^*(\xi)$ are quantities introduced by Mahler and Koksma, respectively, measuring how well a given transcendental complex number ξ can be approximated by algebraic numbers of degree n, see the three recent papers by BUGEAUD [1], [2], [3].

In the present paper we are seeking for improvements of the shape (1.5). Thus, let K be a given number field of degree $r \geqslant 3$. Denote by $\xi \mapsto \xi^{(i)}$ $(i = 1, \ldots, r)$ the isomorphic embeddings of K into $\mathbb{C}$. The embedding $\xi \mapsto \xi^{(i)}$ is called real if it maps K into $\mathbb{R}$ and complex if it does not map K into $\mathbb{R}$. Further, two embeddings $\xi \mapsto \xi^{(i)}$, $\xi \mapsto \xi^{(j)}$ are called complex conjugate if $\xi^{(j)} = \overline{\xi^{(i)}}$ for $\xi \in K$.

Definition. Let Σ be a subset of $\{1, \ldots, r\}$ of cardinality $\geqslant 2$. We define $\kappa(\Sigma)$ to be the infimum of all reals κ with the property that there exists a constant $C(K) > 0$ such that

$$\prod_{\{i,j\}\subset\Sigma} |\alpha^{(i)} - \alpha^{(j)}| \geqslant C(K) \cdot M(\alpha)^{-\kappa} \quad \text{for every } \alpha \text{ with } \mathbb{Q}(\alpha) = K. \tag{1.5}$$

From (1.3) it is clear that $\kappa(\Sigma) \leqslant r-1$. If K is a cubic field, it is possible to give the exact values for the quantities $\kappa(\Sigma)$. Our first result is as follows.

Theorem 1.1. *Let K be a number field of degree 3, and Σ a subset of $\{1,2,3\}$.*

(i) *Suppose that either $\Sigma = \{1,2,3\}$, or K is totally real and $|\Sigma| = 2$, or $\Sigma = \{i,j\}$ where $\xi \mapsto \xi^{(i)}$ and $\xi \mapsto \xi^{(j)}$ are complex conjugate. Then $\kappa(\Sigma) = 2$.*

(ii) *Suppose that $\Sigma = \{i,j\}$, where one of the embeddings $\xi \mapsto \xi^{(i)}$, $\xi \mapsto \xi^{(j)}$ is real and the other complex. Then $\kappa(\Sigma) = \frac{2}{3}$.*

We mention that this result solves a problem of Mignotte and Payafar [12, bottom of p. 187]. Further, Bugeaud [3, Theorem 1] applied this to obtain an almost optimal result on the set of values assumed by the difference $w_3 - w_3^*$, where w_3 and w_3^* are Mahler's and Koksma's quantities, respectively, mentioned above for cubic numbers.

In the case that the number field K has degree $r \geqslant 4$, we have been able to determine which sets Σ have $\kappa(\Sigma) = r-1$ and to give non-trivial (but far from best possible) upper bounds for $\kappa(\Sigma)$ for the other sets Σ.

Theorem 1.2. *Let K be a number field of degree $r \geqslant 4$, and Σ a subset of $\{1,\dots,r\}$.*

(i) *Suppose that either $\Sigma = \{1,\dots,r\}$ or $\Sigma = \{1,\dots,r\}\backslash\{i_0\}$ where $\xi \mapsto \xi^{(i_0)}$ is real. Then $\kappa(\Sigma) = r-1$.*

(ii) *Suppose that either $2 \leqslant |\Sigma| \leqslant r-2$ or $\Sigma = \{1,\dots,r\}\backslash\{i_0\}$ where $\xi \mapsto \xi^{(i_0)}$ is complex. Then*

$$\kappa(\Sigma) \leqslant r - 1 - \frac{(r-|\Sigma|)^2}{135r}.$$

For instance if $|\Sigma| = 2$ part (ii) gives $\kappa(\Sigma) \leqslant r-1-(r-2)^2/135r = r-1-O(r)$ which is comparable to the author's result $\kappa(\Sigma) \leqslant \frac{41}{42}(r-1)$ mentioned above. In the other extremal situation $|\Sigma| = r-1$ part (ii) gives $\kappa(\Sigma) \leqslant r-1-1/135r$.

Our proof of part (ii) of Theorem 1.2 is ineffective. More precisely, we prove an inequality of the shape (1.5) where $\kappa = r-1-(r-|\Sigma|)^2/135r$ and $C(K)$ is not effectively computable by our method of proof. Below we give an effective version, but obviously with a value of κ much closer to $r-1$. We denote by D_K the discriminant of a number field K.

Theorem 1.3. *Let K be a number field of degree $r \geqslant 4$ and let Σ be a subset of $\{1,\dots,r\}$ such that either $2 \leqslant |\Sigma| \leqslant r-2$ or $\Sigma = \{1,\dots,r\}\backslash\{i_0\}$ where $\xi \mapsto \xi^{(i_0)}$ is complex. Then for every α with $\mathbb{Q}(\alpha) = K$ we have*

$$\prod_{\{i,j\}\subset\Sigma} |\alpha^{(i)} - \alpha^{(j)}| \geqslant C(K)\cdot M(\alpha)^{-\kappa}$$

with

$$\kappa = r-1-(c_1 r)^{-c_2 r^4}|D_K|^{-2r^3}, \quad C(K) = \exp\big(-(c_3 r)^{c_4 r^4}|D_K|^{2r^3}\big) \tag{1.7}$$

where c_1, c_2, c_3, c_4 are effectively computable absolute constants.

Our proofs consist of modifications of arguments from [8]. We prove Theorem 1.1 and part (i) of Theorem 1.2 in Section 2. Further, we prove part (ii) of Theorem 1.2 and Theorem 1.3 in Section 3.

In our proofs we use properties of equivalence classes of algebraic numbers. Two algebraic numbers α, α^* are called equivalent if

$$\alpha^* = \frac{a\alpha+b}{c\alpha+d} \quad \text{for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}).$$

In Section 2 we show that if Σ satisfies the conditions of part (i) of Theorem 1.2, then for every $\delta > 0$ and every α^* with $\mathbb{Q}(\alpha^*) = K$ there are infinitely many α which are equivalent to α^* and satisfy

$$\prod_{\{i,j\}\subset\Sigma} |\alpha^{(i)} - \alpha^{(j)}| \leqslant M(\alpha)^{1-r+\delta}.$$

This implies at once that $\kappa(\Sigma) = r-1$. We use an argument from [8], based on Roth's Theorem. The proof of Theorem 1.1 is along the same lines.

Two equivalent algebraic numbers have the same discriminant. The author [7] proved that every algebraic number α with $\mathbb{Q}(\alpha) = K$ is equivalent to an algebraic number α^* such that

$$M(\alpha^*) \leqslant A(K)|D(\alpha)|^{21/(r-1)}, \tag{1.8}$$

where $A(K)$ is some ineffective constant depending on K. Thus in (1.3) we may replace the term $|D(\alpha)|^{1/2}$ by a positive power of $M(\alpha^*)$, but $M(\alpha^*)$ may be much smaller than $M(\alpha)$.

Provided Σ satisfies the conditions from part (ii) of Theorem 1.2, we deduce a refinement of (1.3) (Lemma 3.3 in Section 3) which allows us to replace the positive power of $M(\alpha^*)$ coming from the discriminant by a positive power of $M(\alpha)$. This yields at once our upper bound for $\kappa(\Sigma)$.

To prove Theorem 1.3, we use a recent result by Győry [11] (which is a slight improvement of an older result by Győry and the author [9]) stating in a precise form that every algebraic number α is equivalent to a number α^* with

$$M(\alpha^*) \leqslant A(K)|D(\alpha)|^{a(K)} \tag{1.9}$$

where both $A(K)$, $a(K)$ are effectively computable in terms of K. Then the proof of Theorem 1.3 is completed similarly as that of part (ii) of Theorem 1.2.

We mention that both (1.8) and (1.9) were deduced from an inequality of the following type. Let K be a number field of degree r and a, b, c nonzero integers of K with $a + b = c$. Then

$$\prod_{i=1}^{r} \max\left(|a^{(i)}|, |b^{(i)}|, |c^{(i)}|\right) \leqslant U \cdot |N_{K/\mathbb{Q}}(abc)|^{V}, \tag{1.10}$$

where $\xi \mapsto \xi^{(i)}$ $(i = 1, \dots, r)$ denote as usual the isomorphic embeddings of K into $\mathbb{C}$, and U, V are constants. Inequality (1.8) follows from a version of (1.10) in which $V = 1+\varepsilon$ for any $\varepsilon > 0$ and $U = U(K, \varepsilon)$ is some ineffective constant (see [7, Lemma 11]). This version is in turn a consequence of Roth's Theorem over number fields. Inequality (1.9) was deduced from a version of (1.10) in which both U, V are effectively computable in terms of K, but V is rather large (see [10, Theorem], [4, Corollary]). The latter is proved by means of linear forms in logarithms estimates.

As mentioned before, it is as yet open to obtain an inequality of the shape (1.4) with $\kappa < r - 1$ and some constant $C(r)$ depending on r. We discuss how this is related to certain other open problems. Assume Σ satisfies the condition of part (ii) of Theorem 1.2. Then following the reasoning of the proof of part (ii) of Theorem 1.2 one can deduce (1.4) with $\kappa = \kappa(r) < r - 1$ and $C(r) > 0$ from a conjectural improvement of

(1.9) stating that there are numbers $A(r)$, $a(r)$ depending only on r such that every algebraic number α of degree r is equivalent to a number α^* for which

$$M(\alpha^*) \leqslant A(r)|D(\alpha)|^{a(r)}. \tag{1.11}$$

Speculating further, by going through the arguments from [7] it would be possible to deduce (1.11) from a version of (1.10) in which

$$U = c_1(r)|D_K|^{c_2(r)}, \quad V = c_3(r)$$

where $c_1(r)$, $c_2(r)$, $c_3(r)$ depend only on r. We mention that such a version of (1.10), and hence also (1.11) and (1.4), can be deduced from a sharpening of Roth's Theorem over number fields conjectured by Vojta [13, §3, p. 65].

2. Proofs of Theorem 1.1 and part (i) of Theorem 1.2

Our basic tool is the following.

Lemma 2.1. *Let α be a real, irrational algebraic number and let $\beta_1, \dots, \beta_n$ be different complex numbers different from α. Then for every $\delta > 0$ and every Q which is sufficiently large in terms of δ, there is a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$ such that*

$$\begin{cases} Q^{-1-\delta} \leqslant |\alpha a + b|, \quad |\alpha c + d| \leqslant Q^{-1+\delta}, \\ Q^{1-\delta} \leqslant |\beta_i a + b|, \quad |\beta_i c + d| \leqslant Q^{1+\delta} \quad (i = 1, \dots, n). \end{cases} \tag{2.1}$$

Proof. This lemma is a special case of [8, Lemma 4.4]. For convenience of the reader we give the proof.

First we prove the following assertion. For every ε with $0 < \varepsilon < 1/2$ and every sufficiently large Q, the following holds: if (x, y) is any non-zero point of $\mathbb{Z}^2$ satisfying

$$|\alpha x + y| \leqslant Q^{-1+\varepsilon}, \quad |\beta_i x + y| \leqslant Q^{1+\varepsilon} \quad (i = 1, \dots, n), \tag{2.2}$$

then (x, y) satisfies also

$$|\alpha x + y| \geqslant Q^{-1-2\varepsilon}, \quad |\beta_i x + y| \geqslant Q^{1-2\varepsilon} \ (i = 1, \dots, n). \tag{2.3}$$

Below, constants implied by the Vinogradov symbols $\ll$, $\gg$ depend on α, $\beta_1, \ldots, \beta_n$ and ε. Let (x, y) be a non-zero point in $\mathbb{Z}^2$ satisfying (2.2) but not (2.3). Then $x \neq 0$. First assume that $|\alpha x + y| < Q^{-1-2\varepsilon}$. Then from (2.2) we infer $|x| \ll Q^{1+\varepsilon}$ and so

$$|\alpha x + y| \ll |x|^{-(1+2\varepsilon)/(1+\varepsilon)}.$$

By Roth's Theorem, $|x|$ is bounded. But then, Q is bounded for otherwise there are fixed integers x, y with $x \neq 0$ satisfying (2.2) for arbitrarily large Q, hence $\alpha x + y = 0$, which contradicts our assumption that $\alpha \notin \mathbb{Q}$.

Now suppose that $|\beta_i x + y| < Q^{1-2\varepsilon}$ for some i. Then by using the first inequality in (2.2) twice, we obtain first $|x| \ll Q^{1-2\varepsilon}$ and then

$$|\alpha x + y| \ll |x|^{-(1-\varepsilon)/(1-2\varepsilon)}.$$

Again by Roth's Theorem, $|x|$ and hence Q is bounded. This proves our assertion.

Now consider the symmetric convex body $S(Q) \subset \mathbb{R}^2$, given by

$$|\alpha x + y| \leqslant Q^{-1}, \quad |\beta_i x + y| \leqslant Q \quad (i = 1, \ldots, n).$$

$S(Q)$ contains the set of points $(x, y) \in \mathbb{R}^2$ with $|\alpha x + y| \leqslant Q^{-1}$, $|y| \ll Q$, therefore its area is $\gg 1$. So by Minkowski's Theorem, for the successive minima λ_1, λ_2 of $S(Q)$ we have

$$\lambda_1 \lambda_2 \ll 1. \tag{2.4}$$

Recall that $\mathbb{Z}^2$ has a basis (a, b), (c, d) (i.e., $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$) such that $(a, b) \in \lambda_1 S(Q)$, $(c, d) \in \lambda_2 S(Q)$. Here λ_1, λ_2, (a, b), (c, d) depend on Q.

Let $0 < \varepsilon < 1/6$. Assuming Q is sufficiently large we have $\lambda_1 \geqslant Q^{-2\varepsilon}$, since otherwise the point (a, b) would satisfy (2.2) but not (2.3), contradicting the assertion proved above. But then by (2.4) we have $\lambda_2 \ll Q^{2\varepsilon}$, and hence $\lambda_2 \leqslant Q^{3\varepsilon}$, assuming that Q is large enough to absorb the constant implied by $\ll$. This means that both (a, b), (c, d) satisfy (2.2) with 3ε instead of ε, and then by our assertion they satisfy also (2.3) with 3ε instead of ε, provided Q is sufficiently large. Now choose $\varepsilon < \min(1, \delta)/6$. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfies (2.1) and our lemma follows. □

Proof of part (i) of Theorems 1.1 and 1.2. Notice that part (i) of Theorem 1.1 is precisely part (i) of Theorem 1.2 with $r = 3$. We prove parts (i) of Theorems 1.1 and 1.2 simultaneously.

Let K be a number field of degree $r \geqslant 3$. Without loss of generality we assume that either $\Sigma = \{1, \dots, r\}$ or $\Sigma = \{1, \dots, r\} \backslash \{1\}$, where $\xi \mapsto \xi^{(1)}$ is real. As mentioned in Section 1, we pick α^* with $\mathbb{Q}(\alpha^*) = K$ and consider numbers which are equivalent to α^*. Constants implied by $\ll, \gg$ depend on α^*, K and another parameter δ introduced later. Let a_0 be the integer such that $a_0 \prod_{i=1}^{r}(X - \alpha^{*(i)})$ has integer coefficients with greatest common divisor 1. We use that for the Mahler measures of the numbers equivalent to α^* we have

$$M\left(\frac{a\alpha^* + b}{c\alpha^* + d}\right) = a_0 \prod_{i=1}^{r} \max\left(|a\alpha^{*(i)} + b|, |c\alpha^{*(i)} + d|\right)$$
$$\text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}). \tag{2.5}$$

First suppose $\Sigma = \{1, \dots, r\}$. We consider numbers $\alpha_d = (\alpha^* + d)^{-1}$ with $d \in \mathbb{Z}$. By (2.5) we have $|d|^r \ll M(\alpha_d) \ll |d|^r$, so $M(\alpha_d)$ tends to ∞ with $|d|$. Moreover, for every $d \in \mathbb{Z}$ we have

$$\prod_{1 \leqslant i < j \leqslant r} |\alpha_d^{(i)} - \alpha_d^{(j)}| = \prod_{1 \leqslant i < j \leqslant r} \frac{|\alpha^{*(i)} - \alpha^{*(j)}|}{|\alpha^{*(i)} + d| \cdot |\alpha^{*(j)} + d|} \ll |d|^{-r(r-1)}$$
$$\ll M(\alpha_d)^{1-r}.$$

Hence $\kappa(\Sigma) = r - 1$.

Now assume that $\Sigma = \{1, \dots, r\} \backslash \{1\}$ where $\xi \mapsto \xi^{(1)}$ is real. We prove that for every $\delta > 0$ there are infinitely many numbers α which are equivalent to α^* and satisfy

$$\prod_{\{i,j\} \subset \Sigma} |\alpha^{(i)} - \alpha^{(j)}| \leqslant M(\alpha)^{1-r+\delta}. \tag{2.6}$$

This proves $\kappa(\Sigma) = r - 1$.

Let $\varepsilon > 0$ be a number depending on δ, but much smaller than δ, which will be specified later. Let $Q > 1$. According to Lemma 2.1, assuming that Q is sufficiently large in terms of ε, there is a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$ such that

$$\begin{cases} Q^{-1-\varepsilon} \leqslant |\alpha^{*(1)}a+b|, & |\alpha^{*(1)}c+d| \leqslant Q^{-1+\varepsilon}, \\ Q^{1-\varepsilon} \leqslant |\alpha^{*(i)}a+b|, & |\alpha^{*(i)}c+d| \leqslant Q^{1+\varepsilon} \quad (i=2,\dots,r). \end{cases} \tag{2.7}$$

Let $\alpha_Q = \frac{a\alpha^*+b}{c\alpha^*+d}$; then α_Q is equivalent to α^*. By (2.5), (2.7) we have

$$Q^{r-2-r\varepsilon} \ll M(\alpha_Q) \ll Q^{r-2+r\varepsilon}, \tag{2.8}$$

where a_0 has been inserted into the constants implied by $\ll$. Further, by (2.7), (2.8),

$$\begin{aligned} \prod_{\{i,j\}\subset\Sigma} |\alpha_Q^{(i)} - \alpha_Q^{(j)}| &= \prod_{2\leqslant i<j\leqslant r} |\alpha_Q^{(i)} - \alpha_Q^{(j)}| \\ &= \prod_{2\leqslant i<j\leqslant r} \frac{|\alpha^{*(i)} - \alpha^{*(j)}|}{|\alpha^{*(i)}c+d|\cdot|\alpha^{*(j)}c+d|} \ll Q^{-(r-1)(r-2)(1-\varepsilon)} \\ &\ll M(\alpha_Q)^{-(r-1)(r-2)(1-\varepsilon)/(r-2-r\varepsilon)}. \end{aligned}$$

Now taking ε sufficiently small in terms of δ and then letting $Q \to \infty$ we infer that α_Q satisfies (2.6) and, in view of (2.8), that $M(\alpha_Q) \to \infty$. Hence (2.6) has infinitely many solutions equivalent to α^*. This completes our proof of part (i) of Theorems 1.1 and 1.2. □

Proof of part (ii) of Theorem 1.1. Let K be a cubic field. Without loss of generality we assume that $\Sigma = \{1,2\}$, where $\xi \mapsto \xi^{(1)}$ is real, $\xi \mapsto \xi^{(2)}$ is complex and $\xi^{(3)} = \overline{\xi^{(2)}}$ for $\xi \in K$.

We recall an argument of Mignotte and Payafar [12]. Let α with $\mathbb{Q}(\alpha) = K$. Then

$$|\alpha^{(1)} - \alpha^{(3)}| = |\alpha^{(1)} - \alpha^{(2)}|,$$

$$|\alpha^{(2)} - \alpha^{(3)}| \leqslant |\alpha^{(1)} - \alpha^{(2)}| + |\alpha^{(1)} - \alpha^{(3)}| = 2\cdot|\alpha^{(1)} - \alpha^{(2)}|,$$

hence

$$|\alpha^{(1)} - \alpha^{(2)}| \geqslant \left(\frac{1}{2}\prod_{1\leqslant i<j\leqslant 3} |\alpha^{(i)} - \alpha^{(j)}|\right)^{1/3} = \left(\frac{1}{2}a_0^{-2}|D(\alpha)|^{1/2}\right)^{1/3}$$

$$\geqslant 2^{-1/3} M(\alpha)^{-2/3}$$

where a_0 has the meaning from (1.1), (1.2). This proves $\kappa(\Sigma) \leqslant 2/3$.

To prove the reverse inequality we proceed as in the case $\Sigma = \{1, \ldots, r\}$ above. Choose α^* with $\mathbb{Q}(\alpha^*) = K$ and for $d \in \mathbb{Z}$ define $\alpha_d = (\alpha^* + d)^{-1}$. Then by (2.5) we have $|d|^3 \ll M(\alpha_d) \ll |d|^3$ for $d \in \mathbb{Z}$. Therefore, $M(\alpha_d)$ tends to ∞ as $|d| \to \infty$. Moreover,

$$|\alpha_d^{(1)} - \alpha_d^{(2)}| = \frac{|\alpha^{*(1)} - \alpha^{*(2)}|}{|\alpha^{*(1)} + d| \cdot |\alpha^{*(2)} + d|} \ll |d|^{-2} \ll M(\alpha_d)^{-2/3}.$$

Hence $\kappa(\Sigma) \geqslant 2/3$. This completes the proof of Theorem 1.1. □

3. Proofs of part (ii) of Theorem 1.2 and Theorem 1.3

We first state two results of crucial importance for us which are easy consequences of the literature. Recall that two equivalent algebraic numbers have the same discriminant.

Lemma 3.1. *Let K be a number field of degree $r \geqslant 4$. Then every α with $\mathbb{Q}(\alpha) = K$ is equivalent to a number α^* for which*

$$M(\alpha^*) \leqslant A_1(K) \cdot |D(\alpha)|^{21/(r-1)}, \tag{3.1}$$

where $A_1(K)$ is a constant depending only on K (which is not effectively computable from our method of proof).

Lemma 3.2. *Let K be a number field of degree $r \geqslant 4$. Then every α with $\mathbb{Q}(\alpha) = K$ is equivalent to a number α^* for which*

$$M(\alpha^*) \leqslant A_2(K) \cdot |D(\alpha)|^{a(K)} \tag{3.2}$$

with

$$A_2(K) = \exp\left((c_5 r)^{c_6 r^4} |D_K|^{4r^3}\right), \quad a(K) = (c_7 r)^{c_8 r^4} |D_K|^{2r^3}, \tag{3.3}$$

where c_5, c_6, c_7, c_8 are effectively computable absolute constants.

Proof. Lemma 3.2 is precisely Corollary 5 of [11] (this is in fact a slight improvement of a special case of Theorem 3' of [9]). As for Lemma 3.1, to each α with $\mathbb{Q}(\alpha) = K$ we associate the binary form $F_\alpha(X, Y) := a_0 \prod_{i=1}^r (X - \alpha^{(i)})$, where a_0 is the positive integer such that F_α has integer coefficients with greatest common divisor 1. Now Lemma 3.1 follows by applying [6, Theorem 1] to F_α and observing that two algebraic numbers α, α^* are equivalent if and only if F_α, F_{α^*} are equivalent. □

Our last tool is an improvement of (1.3).

Lemma 3.3. *Let α be an algebraic number of degree $r \geqslant 4$. Let α^* be equivalent to α and suppose that $M(\alpha^*) \leqslant M(\alpha)$. Further, let Σ be a subset of $\{1, \ldots, r\}$ such that either $2 \leqslant |\Sigma| \leqslant r - 2$ or $\Sigma = \{1, \ldots, r\} \backslash \{i_0\}$ where $\alpha^{(i_0)} \notin \mathbb{R}$. Then*

$$\prod_{\{i,j\} \subset \Sigma} |\alpha^{(i)} - \alpha^{(j)}|$$

$$\geqslant 2^{-2r^2} \cdot \frac{|D(\alpha)|^{1/2}}{M(\alpha)^{r-1}} \cdot \max\left(1, \frac{|D(\alpha)|^{1/2}}{M(\alpha^*)^{r-1}} \cdot \left(\frac{M(\alpha)}{M(\alpha^*)}\right)^{4(r-|\Sigma|)^2/9r}\right). \quad (3.4)$$

Proof. Write

$$\alpha^* = \frac{a\alpha + b}{c\alpha + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}).$$

Define

$$\varphi_i := \max(|a\alpha^{(i)} + b|, |c\alpha^{(i)} + d|), \quad f_i := \frac{\max(1, |\alpha^{(i)}|)}{\varphi_i} \quad (i = 1, \ldots, r),$$

and

$$g_{ij} := \frac{|\alpha^{(i)} - \alpha^{(j)}|}{\max(1, |\alpha^{(i)}|) \max(1, |\alpha^{(j)}|)} \quad (i, j = 1, \ldots, r).$$

We first deduce some relations and inequalities for these quantities. Let a_0 be the positive integer such that $a_0 \prod_{i=1}^r (X - \alpha^{(i)})$ has integer coefficients with greatest common divisor 1. Then

$$M(\alpha) = a_0 \prod_{i=1}^r \max(1, |\alpha^{(i)}|), \quad M(\alpha^*) = a_0 \prod_{i=1}^r \varphi_i,$$

hence

$$f_1 \cdots f_r = \frac{M(\alpha)}{M(\alpha^*)}. \tag{3.5}$$

It is obvious that

$$g_{ij} \leqslant 2 \quad \text{for } i,j = 1,\dots,r. \tag{3.6}$$

Further, since $ad - bc = \pm 1$ we have

$$|\alpha^{(i)} - \alpha^{(j)}| = |(a\alpha^{(i)} + b)(c\alpha^{(j)} + d) - (a\alpha^{(j)} + b)(c\alpha^{(i)} + d)| \leqslant 2\varphi_i\varphi_j,$$

hence

$$g_{ij} f_i f_j \leqslant 2 \quad \text{for } i,j = 1,\dots,r. \tag{3.7}$$

From (1.1), (1.2) it is obvious that

$$\prod_{1 \leqslant i < j \leqslant r} g_{ij} = \frac{|D(\alpha)|^{1/2}}{M(\alpha)^{r-1}} \tag{3.8}$$

and together with (3.5) this implies

$$\prod_{1 \leqslant i < j \leqslant r} (g_{ij} f_i f_j) = \frac{|D(\alpha)|^{1/2}}{M(\alpha^*)^{r-1}}. \tag{3.9}$$

Lastly, let $i, j \in \{1,\dots,r\}$ be such that $f_i \leqslant f_j$. By (3.5) there is $k \in \{1,\dots,r\}$ with $f_k \geqslant (M(\alpha)/M(\alpha^*))^{1/r}$. From the vector identity

$$(\alpha^{(i)} - \alpha^{(j)}) \begin{pmatrix} 1 \\ \alpha^{(k)} \end{pmatrix} = (\alpha^{(i)} - \alpha^{(k)}) \begin{pmatrix} 1 \\ \alpha^{(j)} \end{pmatrix} + (\alpha^{(k)} - \alpha^{(j)}) \begin{pmatrix} 1 \\ \alpha^{(i)} \end{pmatrix}$$

we infer

$$\begin{aligned} &|\alpha^{(i)} - \alpha^{(j)}| \cdot \max(1, |\alpha^{(k)}|) \\ &\qquad \leqslant |\alpha^{(i)} - \alpha^{(k)}| \cdot \max(1, |\alpha^{(j)}|) + |\alpha^{(k)} - \alpha^{(j)}| \cdot \max(1, |\alpha^{(i)}|) \end{aligned}$$

and so $g_{ij} \leqslant g_{ik} + g_{kj}$. Now invoking (3.7) and our assumption $f_i \leqslant f_j$ we obtain

$$g_{ij} f_i f_j f_k \leqslant g_{ik} f_i f_k f_j + g_{kj} f_k f_j f_i \leqslant 2f_j + 2f_i \leqslant 4f_j,$$

and by dividing by f_j and using our assumption on k we arrive at

$$g_{ij} f_i \cdot \left(\frac{M(\alpha)}{M(\alpha^*)}\right)^{1/r} \leqslant 4 \ \text{ for } i,j \in \{1,\dots,r\} \ \text{ with } f_i \leqslant f_j. \tag{3.10}$$

Having finished our preparations, we now commence with our proof. By (3.8) we have

$$\prod_{\{i,j\}\subset\Sigma} |\alpha^{(i)} - \alpha^{(j)}| \geqslant \prod_{\{i,j\}\subset\Sigma} g_{ij} = \frac{|D(\alpha)|^{1/2}}{M(\alpha)^{r-1}} \cdot \prod_{\{i,j\}\not\subset\Sigma} g_{ij}^{-1}.$$

By (3.6) we have $\prod_{\{i,j\}\not\subset\Sigma} g_{ij}^{-1} \geqslant 2^{-r(r-1)/2}$. So in order to prove (3.4), it suffices to prove that

$$\prod_{\{i,j\}\not\subset\Sigma} g_{ij}^{-1} \geqslant 2^{-2r^2} \frac{|D(\alpha)|^{1/2}}{M(\alpha^*)^{r-1}} \cdot \left(\frac{M(\alpha)}{M(\alpha^*)}\right)^{4(r-|\Sigma|)^2/9r}. \tag{3.11}$$

We distinguish two cases.

First assume that $2 \leqslant |\Sigma| \leqslant r-2$. Put $l := r - |\Sigma|$. Choose $j_0 \in \Sigma$. Without loss of generality we may assume that $\{j_0\} \cup \{1,\dots,r\}\backslash\Sigma = \{1,\dots,l+1\}$ and that

$$f_1 \leqslant f_2 \leqslant \cdots \leqslant f_{l+1}. \tag{3.12}$$

Notice that if $1 \leqslant i < j \leqslant l+1$ then $\{i,j\} \not\subset \Sigma$. Denote by A the collection of pairs of indices (i,j) with $2 \leqslant i < j \leqslant \min(2i-1, l+1)$ and by B the collection of pairs (i,j) such that $1 \leqslant i < j \leqslant r$, $(i,j) \notin A$ and $\{i,j\} \not\subset \Sigma$. By an easy computation we have $|A| = l^2/4$ if l is even, $|A| = (l^2-1)/4$ if l is odd and so for both l even or odd (using $l \geqslant 3$ if l is odd),

$$|A| \geqslant 2l^2/9 = 2(r-|\Sigma|)^2/9. \tag{3.13}$$

First take $(i,j) \in A$. Then $1 \leqslant 2i-j < i < j \leqslant l+1$, and so by (3.10), (3.12),

$$\begin{aligned} g_{ij}^{-1} &\geqslant g_{ij}^{-1} \cdot \frac{1}{4} g_{2i-j,i} f_{2i-j} \left(\frac{M(\alpha)}{M(\alpha^*)}\right)^{1/r} \cdot \frac{1}{4} g_{ij} f_i \left(\frac{M(\alpha)}{M(\alpha^*)}\right)^{1/r} \\ &= \frac{1}{16} g_{2i-j,i} f_{2i-j} f_i \cdot \left(\frac{M(\alpha)}{M(\alpha^*)}\right)^{2/r}. \end{aligned}$$

For $(i,j) \in B$ we use (3.6). Thus we obtain

$$\prod_{\{i,j\}\not\subset\Sigma} g_{ij}^{-1} \geqslant 2^{-|B|-4|A|} \cdot \left(\frac{M(\alpha)}{M(\alpha^*)}\right)^{2|A|/r} \cdot \prod_{(i,j)\in A} (g_{2i-j,i} f_{2i-j} f_i). \tag{3.14}$$

Thanks to the fact that the sets $\{2i-j,i\}$ $((i,j)\in A)$ are distinct (which is crucial and the main motivation for our set-up), we infer from (3.9), (3.7),

$$\prod_{(i,j)\in A}(g_{2i-j,i}f_{2i-j}f_i)\geqslant 2^{|A|-r(r-1)/2}\cdot\frac{|D(\alpha)|^{1/2}}{M(\alpha^*)^{r-1}}.$$

By inserting this and (3.13) into (3.14), and using our assumption $M(\alpha)\geqslant M(\alpha^*)$ we arrive at

$$\begin{aligned}\prod_{\{i,j\}\not\subset\Sigma}g_{ij}^{-1}&\geqslant 2^{-|B|-3|A|-r(r-1)/2}\cdot\frac{|D(\alpha)|^{1/2}}{M(\alpha^*)^{r-1}}\cdot\left(\frac{M(\alpha)}{M(\alpha^*)}\right)^{2|A|/r}\\&\geqslant 2^{-2r^2}\cdot\frac{|D(\alpha)|^{1/2}}{M(\alpha^*)^{r-1}}\cdot\left(\frac{M(\alpha)}{M(\alpha^*)}\right)^{4(r-|\Sigma|)^2/9r}\end{aligned}$$

which is (3.11).

We now treat the case $\Sigma=\{1,\dots,r\}\backslash\{i_0\}$ where $\alpha^{(i_0)}\notin\mathbb{R}$. Without loss of generality we assume that $i_0=1$ and that $\alpha^{(2)}=\overline{\alpha^{(1)}}$. Then $f_1=f_2$ and so by (3.10),

$$\begin{aligned}g_{12}^{-1}&\geqslant g_{12}^{-1}\cdot\frac{1}{4}g_{12}f_1\left(\frac{M(\alpha)}{M(\alpha^*)}\right)^{1/r}\cdot\frac{1}{4}g_{12}f_2\left(\frac{M(\alpha)}{M(\alpha^*)}\right)^{1/r}\\&=\frac{1}{16}\cdot\left(\frac{M(\alpha)}{M(\alpha^*)}\right)^{2/r}\cdot g_{12}f_1f_2.\end{aligned}$$

Now by (3.9), (3.7) we have

$$g_{12}f_1f_2\geqslant 2^{1-r(r-1)/2}\frac{|D(\alpha)|^{1/2}}{M(\alpha^*)^{r-1}}.$$

Hence

$$\begin{aligned}\prod_{\{i,j\}\not\subset\Sigma}g_{ij}^{-1}&=\prod_{j=2}^{r}g_{1j}^{-1}\geqslant 2^{2-r}g_{12}^{-1}\geqslant 2^{-2-r}g_{12}f_1f_2\cdot\left(\frac{M(\alpha)}{M(\alpha^*)}\right)^{2/r}\\&\geqslant 2^{-1-r-r(r-1)/2}\cdot\frac{|D(\alpha)|^{1/2}}{M(\alpha^*)^{r-1}}\cdot\left(\frac{M(\alpha)}{M(\alpha^*)}\right)^{2/r}\end{aligned}$$

which implies (3.11). This completes the proof of Lemma 3.3. □

In what follows, Let K, Σ, r be as in part (ii) of Theorem 1.2. Take α with $\mathbb{Q}(\alpha) = K$. From the equivalence class of α we choose an element α^* of minimal Mahler measure. Thus, $M(\alpha^*) \leqslant M(\alpha)$ hence all conditions of Lemma 3.3 are satisfied. Further, α^* satisfies the inequalities (3.1) and (3.2) in Lemma 3.1, Lemma 3.2, respectively. Put

$$u := 4(r - |\Sigma|)^2/9r.$$

Let $0 \leqslant \theta \leqslant 1$. Then (3.4) implies

$$\begin{aligned}\prod_{\{i,j\}\notin\Sigma} &|\alpha^{(i)} - \alpha^{(j)}| \\ &\geqslant 2^{-2r^2} \frac{|D(\alpha)|^{1/2}}{M(\alpha)^{r-1}} \cdot \left(\frac{|D(\alpha)|^{1/2}}{M(\alpha^*)^{r-1}} M(\alpha)^u M(\alpha^*)^{-u} \right)^{\theta} \qquad (3.15) \\ &= 2^{-2r^2} \cdot |D(\alpha)|^{(1+\theta)/2} \cdot M(\alpha^*)^{-\theta(r-1+u)} \cdot M(\alpha)^{1-r+\theta u}.\end{aligned}$$

We prove first part (ii) of Theorem 1.2 and then Theorem 1.3 by combining (3.15) with (3.1), (3.2), respectively, and choosing an appropriate value for θ.

Proof of part (ii) of Theorem 1.2. By (3.1) we have

$$|D(\alpha)| \geqslant A_1(K)^{-(r-1)/21} M(\alpha^*)^{(r-1)/21}.$$

We insert this into (3.15) and then choose θ to make the exponent on $M(\alpha^*)$ equal to 0. Thus,

$$\begin{aligned}\prod_{\{i,j\}\notin\Sigma} &|\alpha^{(i)} - \alpha^{(j)}| \\ &\geqslant 2^{-2r^2} A_1(K)^{-\frac{r-1}{42}(1+\theta)} \cdot M(\alpha^*)^{\frac{r-1}{42}(1+\theta)-\theta(r-1+u)} \cdot M(\alpha)^{1-r+\theta u} \\ &= 2^{-2r^2} A_1(K)^{-\frac{r-1}{42}(1+\theta)} \cdot M(\alpha)^{1-r+\theta u},\end{aligned}$$

where $\frac{r-1}{42}(1+\theta) = \theta(r-1+u)$, that is,

$$\theta = \frac{1}{41 + 42u/(r-1)}.$$

Consequently, using $u \leqslant 4(r-1)/9$,

$$\begin{aligned}\kappa(\Sigma) &\leqslant r-1-\theta u \leqslant r-1-\frac{4(r-|\Sigma|)^2/9r}{41+42\times 4/9}\\ &\leqslant r-1-\frac{(r-|\Sigma|)^2}{135r}.\end{aligned}$$

This proves part (ii) of Theorem 1.2. □

Proof of Theorem 1.3. By (3.2) we have

$$|D(\alpha)| \geqslant A_2(K)^{-1/a(K)} M(\alpha^*)^{1/a(K)}.$$

Similarly as above, we insert this into (3.15), and choose θ such that the exponent on $M(\alpha^*)$ becomes 0. Thus,

$$\begin{aligned}&\prod_{\{i,j\}\notin\Sigma} |\alpha^{(i)}-\alpha^{(j)}|\\ &\qquad \geqslant 2^{-2r^2} A_2(K)^{-\frac{1+\theta}{2a(K)}} \cdot M(\alpha^*)^{\frac{1+\theta}{2a(K)}-\theta(r-1+u)} \cdot M(\alpha)^{1-r+\theta u} \qquad (3.16)\\ &\qquad = 2^{-2r^2} A_2(K)^{-\frac{1+\theta}{2a(K)}} \cdot M(\alpha)^{1-r+\theta u},\end{aligned}$$

where $\frac{1+\theta}{2a(K)} = \theta(r-1+u)$, that is,

$$\theta = \frac{1}{(2r-2+2u)a(K)-1}.$$

With this choice of θ we have

$$\begin{aligned}2^{-2r^2} A_2(K)^{-(1+\theta)/2a(K)} &\geqslant 2^{-2r^2} A_2(K)^{-1/a(K)}\\ &\geqslant 2^{-2r^2} \exp\Big(-(c_5 r)^{c_6 r^4} |D_K|^{4r^3} (c_7 r)^{-c_8 r^4} |D_K|^{-2r^3}\Big)\\ &\geqslant \exp\Big(-(c_3 r)^{c_4 r^4} |D_K|^{2r^3}\Big)\end{aligned}$$

and, using $4/9r \leqslant u \leqslant 4(r-1)/9$,

$$\begin{aligned}\theta u &\geqslant \frac{4}{9r} \cdot \Big\{(2r-2+\tfrac{8}{9}(r-1))(c_5 r)^{c_6 r^4} |D_K|^{2r^3}\Big\}^{-1}\\ &\geqslant (c_1 r)^{-c_2 r^4} |D_K|^{-2r^3}.\end{aligned}$$

By inserting this into (3.16), Theorem 1.3 follows. □

References

[1] Y. Bugeaud, Mahler's classification of numbers compared with Koksma's, *Acta Arith.* **110** (2003), 89–105.

[2] Y. Bugeaud, Mahler's classification of numbers compared with Koksma's, II, (*preprint*).

[3] Y. Bugeaud, Mahler's classification of numbers compared with Koksma's, III, *Publ. Math. Debrecen* **65** (2004), 305–316.

[4] Y. Bugeaud and K. Győry, Bounds for the solutions of unit equations, *Acta Arith.* **74** (1996), 67–80.

[5] Y. Bugeaud and M. Mignotte, On the distance between roots of integer polynomials, *Proc. Edinburgh Math. Soc.* **47** (2004), (to appear).

[6] G. E. Collins, Polynomial minimum root separation, *J. Symbol. Comp.* **32** (2001), 467–473.

[7] J.-H. Evertse, Estimates for reduced binary forms, *J. reine angew. Math.* **434** (1993), 159–190.

[8] J.-H. Evertse, Symmetric improvements of Liouville's inequality, *J. reine angew. Math.* **527** (2000), 69–95.

[9] J.-H. Evertse and K. Győry, Effective finiteness results for binary forms with given discriminant, *Compos. Math.* **79** (1991), 169–204.

[10] K. Győry, On the solutions of linear Diophantine equations in algebraic integers of bounded norm, *Ann. Univ. Sci. Budapest. Eötvös, Sect. Math.* **22–23** (1979–80), 225–233.

[11] K. Győry, Polynomials and binary forms with given discriminant, (*preprint*).

[12] M. Mignotte and M. Payafar, Distance entre les racines d'un polynôme, *R.A.I.R.O. Analyse numérique* **13** (1979), 181–192.

[13] P. A. Vojta, Diophantine approximations and value distribution theory, Lecture Notes in Mathematics 1239, *Springer Verlag*, 1987.

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