# ON THE QUANTITATIVE SUBSPACE THEOREM

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## ROTH'S THEOREM

For $\xi \in \mathbb{Q}$ we define $H(\xi) = \max(|p|, |q|)$, where $\xi = p/q$, $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$.

**Theorem (Roth, 1955).** *Let $\alpha$ be a real algebraic number and $\delta > 0$. Then the inequality*

(1) $$|\alpha - \xi| \leqslant H(\xi)^{-2-\delta} \text{ in } \xi \in \mathbb{Q}$$

*has only finitely many solutions.*

Quantitative versions (upper bounds for the number of solutions) were given by Davenport& Roth (1955), Mignotte (1972), Bombieri&van der Poorten (1987), Schmidt (1988), E. (1996).

## A QUANTITATIVE ROTH'S THEOREM

Let $\alpha$ be a real algebraic number of degree $d$ and $0 < \delta < 1$.

Define $H(\alpha)$ to be the maximum of the absolute values of the coefficients of the minimal polynomial of $\alpha$.

A solution $\xi$ of (1) is called *large* if

$$H(\xi) \geqslant \max(4^{1/\delta}, H(\alpha))$$

and *small* otherwise.

**Theorem.** *The number of large solutions of* (1) *is at most*

$$A := 2^{25}\delta^{-3} \log 4d \log(\delta^{-1} \log 4d)$$

*and the number of small solutions at most*

$$B := 10\delta^{-1} \log \Big(\delta^{-1} \log(4H(\alpha)\Big).$$

**Proof.** Interval result + gap principle.

## AN INTERVAL RESULT

*There are reals* $Q_1 < \cdots < Q_{[A/2]}$ *such that for every solution* $\xi \in \mathbb{Q}$ *of*

$$(1) \qquad |\alpha - \xi| \leqslant H(\xi)^{-2-\delta}$$

*we have either* $H(\xi) < \max(4^{1/\delta}, H(\alpha))$ *or*

$$H(\xi) \in \left[Q_1, Q_1^{1+\delta/2}\right) \cup \cdots \cup \left[Q_{[A/2]}, Q_{[A/2]}^{1+\delta/2}\right).$$

**Proof.** Roth machinery: construction of auxiliary polynomial, application of Roth's Lemma.

**Remark.** The $Q_i$ cannot be determined effectively.

## A GAP PRINCIPLE

*Let* $Q \geqslant 2$*. Then* (1) *has at most two solutions* $\xi \in \mathbb{Q}$ *with* $Q \leqslant H(\xi) < Q^{1+\delta/2}$*.*

**Proof.** Suppose the contrary. Then there are two solutions of (1) at the same side of $\alpha$, say $\xi_1, \xi_2$. Hence

$$\begin{aligned} Q^{-2(1+\delta/2)} &< \Big(H(\xi_1)H(\xi_2)\Big)^{-1} \\ &\leqslant |\xi_1 - \xi_2| \leqslant \max_i |\alpha - \xi_i| \\ &\leqslant \max_i H(\xi_i)^{-2-\delta} \leqslant Q^{-2-\delta}, \end{aligned}$$

which is impossible.

## SCHMIDT'S SUBSPACE THEOREM

Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers in $\mathbb{C}$ and let

$$L_i(\mathbf{X}) = \alpha_{i1}X_1 + \cdots + \alpha_{in}X_n \quad (i = 1, \ldots, n)$$

be linearly independent linear forms with coefficients $\alpha_{ij} \in \overline{\mathbb{Q}}$.

For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, put
$\|\mathbf{x}\| := \max(|x_1|, \ldots, |x_n|)$.

**Theorem (W.M. Schmidt, 1972).**
*Let $\delta > 0$. Then the set of solutions of*

$$(2) \qquad |L_1(\mathbf{x}) \cdots L_n(\mathbf{x})| \leqslant \|\mathbf{x}\|^{-\delta} \text{ in } \mathbf{x} \in \mathbb{Z}^n$$

*is contained in a union of finitely many proper linear subspaces of $\mathbb{Q}^n$.*

## SYSTEMS OF INEQUALITIES

By a combinatorial argument, inequality (2) can be reduced to finitely many systems of inequalities of the shape

$$(3)\quad |L_1(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_1}, \ldots, |L_n(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n,$$

where $C > 0$, $c_1 + \cdots + c_n > 0$.

Our quantitative results will be formulated for systems (3).

# TECHNICAL ASSUMPTIONS

We consider

(3) $$|L_1(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_1}, \ldots, |L_n(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_n}$$
$$\text{in } \mathbf{x} \in \mathbb{Z}^n,$$

where

$$L_i = \alpha_{i1}X_1 + \cdots + \alpha_{in}X_n \quad (i = 1, \ldots, n),$$

$C$ and $c_1, \ldots, c_n$ satisfy the following:

(4) $$\begin{cases} L_1, \ldots, L_n \text{ lin. independent;} \\ H(\alpha_{ij}) \leqslant H, \ \deg \alpha_{ij} \leqslant D \quad \forall i, j; \\ C \leqslant |\det(L_1, \ldots, L_n)|^{1/n}; \\ c_1 + \cdots + c_n \geqslant \delta \text{ with } 0 < \delta \leqslant 1; \\ \min(c_1, \ldots, c_n) = -1. \end{cases}$$

## A QUANTITATIVE SUBSPACE THEOREM

Assume (4) and consider the system

$$(3) \quad |L_1(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_1}, \ldots, |L_n(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_n}$$

in $\mathbf{x} \in \mathbb{Z}^n$.

**Theorem (Schlickewei, E., 2002)**
*The set of solutions of* (3) *with*

$$\|\mathbf{x}\| \geqslant \max(n^{2n/\delta}, H)$$

*is contained in a union of at most*

$$4^{(n+9)^2}\delta^{-n-4} \log 4D \log\log 4D$$

*proper linear subspaces of* $\mathbb{Q}^n$.

## HISTORY

• The first quantitative version of the Subspace Theorem was obtained by Schmidt (1989). This was later improved and generalized by Schlickewei, E. The previous result is a consequence of a more general result over number fields by Schlickewei, E.

• The method of proof is a quantification of Schmidt's proof from 1972, based on geometry of numbers, a construction of an auxiliary polynomial, and an application of Roth's Lemma.

• In 1994, Faltings and Wüstholz gave another proof of the Subspace Theorem, based on Faltings' Product Theorem, stability theory for vector spaces, and Diophantine approximation on non-linear projective varieties. With their method of proof it is also possible to compute an upper bound for the number of subspaces, which is however much larger.

## A FURTHER IMPROVEMENT

Assume (4) and consider again

(3) $|L_1(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_1}, \ldots, |L_n(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_n}$ in $\mathbf{x} \in \mathbb{Z}^n$.

**Theorem 1 (Ferretti, E.)**

**(i)** *The set of solutions of* (3) *with*

$$\|\mathbf{x}\| \geqslant \max(n^{2n/\delta}, H)$$

*is contained in a union of at most*

$$A := 2^{2n}(10n)^{20}\delta^{-3}\log\delta^{-1}\log 4D\log(\delta^{-1}\log 4D)$$

*proper linear subspaces of* $\mathbb{Q}^n$.

**(ii)** *The set of solutions of* (3) *with*

$$\|\mathbf{x}\| < \max(n^{2n/\delta}, H)$$

*is contained in a union of at most*

$$B := 10^{3n}\delta^{-1}\log(\delta^{-1}\log 4H)$$

*proper linear subspaces of* $\mathbb{Q}^n$.

## ABOUT THE PROOF

The proof uses ideas from both the methods of Schmidt (1972) and Faltings and Wüstholz (1994).

Basically, we follow Schmidt's method, but replace Schmidt's construction of an auxiliary polynomial by that of Faltings and Wüstholz.

## A REFINEMENT OF THE SUBSPACE THEOREM

Consider again

(3) $|L_1(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_1}, \ldots, |L_n(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_n}$

in $\mathbf{x} \in \mathbb{Z}^n$

where $c_1 + \cdots + c_n > 0$.

**Theorem (Vojta, 1989; Schmidt, 1993; Faltings&Wüstholz, 1994)** *There is an effectively determinable, proper linear subspace $T_0$ of $\mathbb{Q}^n$, that can be chosen from a finite set depending only on $L_1, \ldots, L_n$, such that* (3) *has only finitely many solutions outside $T_0$.*

This can be deduced from the basic Schmidt's Subspace Theorem.

## ABOUT $T_0$

In case $L_1, \ldots, L_n$ have real algebraic coefficients, the space $T_0$ can be described as follows.

For any linear subspace $T$ of $\mathbb{Q}^n$, define $\nu(T)$ to be the maximum of the quantities

$$c_{i_1} + \cdots + c_{i_m}, \quad m = \dim T,$$

taken over all subsets $\{i_1, \ldots, i_m\}$ of $\{1, \ldots, n\}$ such that

$$\text{rank } \{L_{i_1}|_T, \ldots, L_{i_m}|_T\} = m.$$

Then $T_0$ is the unique subspace of $\mathbb{Q}^n$ such that

- $\nu(T_0) = \min\{\nu(T) : T \text{ lin. subsp. of } \mathbb{Q}^n\}$;
- among all spaces with minimal $\nu$-value, $T_0$ has the largest dimension.

## AN INTERVAL RESULT FOR THE REFINEMENT

Assume (4) and consider again

(3) $\quad |L_1(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_1}, \ldots, |L_n(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_n}$

in $\mathbf{x} \in \mathbb{Z}^n$.

**Theorem 2 (Ferretti, E.)**
*Let*

$$A := 2^{2n}(10n)^{20}\delta^{-3} \log \delta^{-1} \log 4D \log(\delta^{-1} \log 4D).$$

*Then there are positive reals* $Q_1 < \cdots < Q_{[A]-1}$ *such that for every solution* $\mathbf{x} \in \mathbb{Z}^n$ *of* (3) *with* $\mathbf{x} \notin T_0$ *we have*

$$\|\mathbf{x}\| < \max(n^{2n/\delta}, H)$$

*or*

$$\|\mathbf{x}\| \in \left[Q_1, Q_1^{1+\delta/n}\right) \cup \cdots \cup \left[Q_{[A]-1}, Q_{[A]-1}^{1+\delta/n}\right).$$

## A GAP PRINCIPLE

Theorem 1 is deduced from Theorem 2 and the following gap principle.

Again we assume (4) and consider

$$(3)\quad |L_1(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_1}, \ldots, |L_n(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

**Gap principle.**
*Let $Q \geqslant 2$. Then the set of solutions of* (3) *with*

$$Q \leqslant \|\mathbf{x}\| < Q^{1+\delta/n}$$

*is contained in*

- **one** *proper linear subspace of $\mathbb{Q}^n$ if $Q \geqslant n^{2n/\delta}$;*
- *a union of at most $300^n$ proper linear subspaces of $\mathbb{Q}^n$ if $Q < n^{2n/\delta}$.*

## AN OPEN PROBLEM

What about the number of solutions of

$$(3)\quad |L_1(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_1}, \ldots, |L_n(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_n}$$
$$\text{in } \mathbf{x} \in \mathbb{Z}^n$$

outside $T_0$?

An observation of Schmidt from 1990 suggests, that an explicit upper bound for the number of solutions of (3) outside $T_0$ would lead to very strong *effective* Diophantine inequalities.

## AN EXAMPLE

Let $\alpha$ be a real algebraic number of degree $\geqslant 4$ and $\delta > 0$. Consider the system of inequalities

$$(5) \qquad \begin{cases} |x_1 + \alpha x_2 + \alpha^2 x_3| \leqslant \|\mathbf{x}\|^{-2-\delta}, \\ |x_2| \leqslant \|\mathbf{x}\|, \ \ |x_3| \leqslant \|\mathbf{x}\| \end{cases}$$

$$\text{in } \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3.$$

This system has only finitely many solutions ($T_0 = (0)$).

**Fact.** *Let $N$ be an upper bound for the number of solutions of* (5) *with* $\gcd(x_1, x_2, x_3) = 1$*. Then for every $\xi \in \mathbb{Q}$ we have*

$$|\alpha - \xi| \geqslant \Big(2(1 + |\alpha|)N\Big)^{-2-\delta} \cdot H(\xi)^{-3-\delta}.$$

## A SEMI-EFFECTIVE RESULT

Assume (4) and consider again

$$(3)\quad |L_1(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_1}, \ldots, |L_n(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

Let $K$ be the algebraic number field generated by the coefficients of $L_1, \ldots, L_n$.

**Theorem 3 (Ferretti, E.)**
*There are an effectively computable constant* $c^{\text{eff}}(n, \delta, D) > 0$ *and an ineffective constant* $B^{\text{ineff}}(n, \delta, K) > 0$ *such that for every solution* $\mathbf{x} \in \mathbb{Z}^n$ *of* (3) *with* $\mathbf{x} \notin T_0$ *we have*

$$\|\mathbf{x}\| \leqslant \max\left(B^{\text{ineff}}(n, \delta, K), H^{c^{\text{eff}}(n,\delta,D)}\right).$$

This may be compared with the Subspace Theorem with moving targets of Ru and Vojta (1997).

## ANOTHER OPEN PROBLEM

Prove the same result with $B^{\mathsf{ineff}}(n, \delta, D)$ instead of $B^{\mathsf{ineff}}(n, \delta, K)$.

## ABOUT THE PROOF OF THM. 1

Consider again

$$(3) \quad |L_1(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_1}, \ldots, |L_n(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

Using geometry of numbers, for every solution $\mathbf{x}$ we can construct, for some $R \leqslant 2^n$, a parallelepiped

$$\Pi(\mathbf{x}) := \Big\{ \mathbf{z} \in \mathbb{R}^R : \ |M_i(\mathbf{z})| \leqslant \|\mathbf{x}\|^{-d_i(\mathbf{x})} \quad (i = 1, \ldots, R) \Big\}$$

such that $\Pi(\mathbf{x}) \cap \mathbb{Z}^R$ generates a linear subspace $T(\mathbf{x})$ of $\mathbb{Q}^R$ of dimension $R-1$.

Here $M_1, \ldots, M_R$ are linearly independent linear forms in $R$ variables with real algebraic coefficients which are independent of $\mathbf{x}$, but the exponents $d_i(\mathbf{x})$ may depend on $\mathbf{x}$.

## THE APPROACH OF SCHLICKEWEI&E. (AND BEFORE)

• The solutions of (3) can be divided into at most $\gamma^{n^2}\delta^{-n}$ classes (where $\gamma$ is an absolute constant and $\delta = c_1 + \cdots + c_n$) such that for any two solutions $\mathbf{x}$, $\mathbf{x}'$ in the same class, we have $d_i(\mathbf{x}) \approx d_i(\mathbf{x}')$ for $i = 1, \ldots, R$.

• Pick solutions $\mathbf{x}_1, \ldots, \mathbf{x}_m$ from the same class, and construct via Schmidt's approach an auxiliary polynomial

$$P(\mathbf{Z}_1, \ldots, \mathbf{Z}_m) \in \mathbb{Z}[\mathbf{Z}_1, \ldots, \mathbf{Z}_m]$$

in $m$ blocks of $R$ variables which vanishes with high multiplicity at all points in

$$T(\mathbf{x}_1) \times \cdots \times T(\mathbf{x}_m).$$

• Derive a contradiction using Roth's Lemma.

The term $\gamma^{n^2}\delta^{-n}$ dominates the resulting upper bound for the number of subspaces.

## THE IMPROVEMENT

Using the method of Faltings and Wüstholz, we can construct an auxiliary polynomial which works also for solutions $\mathbf{x}_1, \ldots, \mathbf{x}_m$ from different classes.

Thus, it is not necessary to divide the solutions into classes, and we can save a factor $\gamma^{n^2}\delta^{-n}$ in the upper bound for the number of subspaces.

## CONSTRUCTION OF THE POLYNOMIAL

We have to construct a multihomogeneous polynomial

$$P(\mathbf{Z}_1, \ldots, \mathbf{Z}_m) \in \mathbb{Z}[\mathbf{Z}_1, \ldots, \mathbf{Z}_m]$$

of small height in $m$ blocks $\mathbf{Z}_1, \ldots, \mathbf{Z}_m$ of $R$ variables, which is homogeneous of degree $d_h$ in the block $\mathbf{Z}_h$ for $h = 1, \ldots, m$.

This polynomial can be expressed as

$$\sum_{\mathbf{i}} c(\mathbf{i}) \prod_{h=1}^{m} \prod_{j=1}^{R} M_j(\mathbf{Z}_h)^{i_{hj}}$$

where the summation is over tuples $\mathbf{i} = (i_{hj})$ such that $\sum_{j=1}^{R} i_{hj} = d_h$ for $h = 1, \ldots, m$.

**Schmidt's construction:**
Construct $P$ with small height such that

$$c(\mathbf{i}) = 0 \text{ if } \max_{1 \leqslant j \leqslant n} \left| \Big( \sum_{h=1}^{m} \frac{i_{hj}}{d_h} \Big) - \frac{m}{R} \right| \geqslant \varepsilon.$$

**Construction of Faltings and Wüstholz:**
Let $\alpha_{hj} \in \mathbb{R}$ with $|\alpha_{hj}| \leqslant 1$ for $h = 1, \ldots, m$, $j = 1, \ldots, R$. Construct $P$ with small height such that

$$c(\mathbf{i}) = 0 \text{ if } \left| \sum_{h=1}^{m} \sum_{j=1}^{R} \alpha_{hj} \Big( \frac{i_{hj}}{d_h} - \frac{1}{R} \Big) \right| \geqslant \varepsilon.$$

By the law of large numbers, in both cases, the number of $\mathbf{i}$ with $c(\mathbf{i}) = 0$ is much smaller than the number of coefficients of $P$.

Thus, the existence of $P$ with small height follows from Siegel's Lemma.

The above results are consequences of quantitative versions of the *Parametric Subspace Theorem*, which is a Diophantine approximation result for parametrized classes of *twisted heights*.

## HEIGHTS

Fix an algebraic number field $K$. Denote by $M_K$ its set of places. Normalize the absolute values $|\cdot|_v$ ($v \in M_K$) in such a way that if $v$ lies above $p \in \{\infty\} \cup \{\text{primes}\}$, then $|x|_v = |x|_p^{[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]}$ for $x \in \mathbb{Q}$.

Define the height of $\mathbf{z} = (z_1, \ldots, z_R) \in K^R$ or $\mathbb{P}^{R-1}(K)$ by

$$H(\mathbf{z}) := \prod_{v \in M_K} \max_{1 \leqslant i \leqslant R} |z_i|_v.$$

This can be extended to a height on $\overline{\mathbb{Q}}^R$ or $\mathbb{P}^{R-1}(\overline{\mathbb{Q}})$.

For a linear subspace $X$ of $\overline{\mathbb{Q}}^R$ define

$$H(X) := H(\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_n)$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is any basis of $X$.

## TWISTED HEIGHTS

Let $\mathbf{c} = (c_{iv} : v \in M_K, i = 1, \ldots, R)$ be a tuple of reals such that

$$c_{1v} = \cdots = c_{Rv} = 0 \quad \text{for all but finitely many } v \in M_K$$

and let $Q \geqslant 1$.

Define the twisted height of $\mathbf{z} = (z_1, \ldots, z_R) \in K^R$ by

$$H_{Q,\mathbf{c}}(\mathbf{z}) := \prod_{v \in M_K} \max_{1 \leqslant i \leqslant R} |z_i|_v Q^{c_{iv}}.$$

More generally, if $\mathbf{z} \in \overline{\mathbb{Q}}^R$, choose any finite extension $L$ of $K$ with $\mathbf{z} \in L^R$ and put

$$H_{Q,\mathbf{c}}(\mathbf{z}) := \prod_{w \in M_L} \max_{1 \leqslant i \leqslant R} |z_i|_w Q^{c_{iw}}$$

where $c_{iw} = \frac{[L_w:K_v]}{[L:K]} \cdot c_{iv}$ if $w|v$.

This defines a twisted height on $\overline{\mathbb{Q}}^R$ or $\mathbb{P}^{R-1}(\overline{\mathbb{Q}})$.

## TWISTED HEIGHT FOR SUBSPACES

Let $X$ be a linear subspace of $\overline{\mathbb{Q}}^R$.
Choose a basis $\mathbf{a}_i = (a_{i1}, \ldots, a_{iR})$ $(i = 1, \ldots, n)$ of $X$.
The coordinates of $\mathbf{a}_1 \wedge \cdots \wedge \mathbf{a}_n$ are, in some order,

$$\Delta_I = \det(a_{i,i_j})_{i,j=1,\ldots,n}$$

for all tuples $I = (i_1, \ldots, i_n)$ with
$1 \leqslant i_1 < \cdots < i_n \leqslant R$.

Choose a finite extension $L$ of $K$ with $\mathbf{a}_1, \ldots, \mathbf{a}_n \in L^n$ and define

$$H_{Q,\mathbf{c}}(X) = \prod_{w \in M_L} \max_I |\Delta_I|_w Q^{\sum_{i \in I} c_{iw}}.$$

## GEOMETRY OF NUMBERS

Let again $\mathbf{c} = (c_{iv} : \ v \in M_K, i = 1, \ldots, R)$ be a tuple of reals such that $c_{1v} = \cdots = c_{Rv} = 0$ for all but finitely many $v \in M_K$ and let $Q \geqslant 1$.

**Theorem (Roy&Thunder, S. Zhang)**
*There is a constant $C(R) > 0$ such that for every $n$-dimensional linear subspace $X$ of $\overline{\mathbb{Q}}^R$ there is a non-zero $\mathbf{z} \in X(\overline{\mathbb{Q}})$ with*

$$H_{Q,\mathbf{c}}(\mathbf{z}) \leqslant C(R) H_{Q,\mathbf{c}}(X)^{1/n}.$$

## THE PARAMETRIC SUBSPACE THEOREM

Let $\mathbf{c} = (c_{iv} : \ v \in M_K, i = 1, \ldots, R)$ be a tuple of reals such that

$c_{1v} = \cdots = c_{Rv} = 0$
for all but finitely many $v \in M_K$.

Let $\theta > 0$ and let $X$ be a linear subspace of $\overline{\mathbb{Q}}^R$ of dimension $n$ defined over $K$.

We consider for varying $Q$ the set

$$S(Q, \theta) = \left\{ \mathbf{z} \in X(\overline{\mathbb{Q}}) : H_{Q,\mathbf{c}}(\mathbf{z}) \leqslant H_{Q,\mathbf{c}}(X)^{1/n} Q^{-\theta} \right\}.$$

**Theorem (Schlickewei, E., 2002)**
*There are $Q_0$, and a finite collection $\{T_1, \ldots, T_t\}$ of proper linear subspaces of $X$, defined over $K$, such that for every $Q \geqslant Q_0$, there is $T_i \in \{T_1, \ldots, T_t\}$ with*

$$S(Q, \theta) \subset T_i.$$

## THE QUANTITATIVE PST

Let $R, n, X$ be as before, and assume that $0 < \theta < 1$ and

- $c_{1v} = \cdots = c_{Rv} = 0$ for all but finitely many $v \in M_K$;
- $c_{iv} \geqslant 0$ for all $i, v$;
- $\sum_{v \in M_K} \sum_{i=1}^{R} c_{iv} = 1$.

**Theorem 4 (Ferretti, E.)**
*The previous Theorem holds with*

$Q_0 = \max\left(n^{2/\theta}, H(X)\right)$,

$t \leqslant 2^{2n}(10n)^{16}\theta^{-3} \log \theta^{-1} \log 4R \log(\theta^{-1} \log 4R)$.

## FROM SYSTEMS OF INEQUALITIES TO THE QUANTITATIVE PST

Assume (4) and consider again

$$(3)\quad |L_1(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_1}, \ldots, |L_n(\mathbf{x})| \leqslant C\|\mathbf{x}\|^{-c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

Let $K$ be a finite normal, extension of $\mathbb{Q}$ containing the coefficients of $L_1, \ldots, L_n$.

Let $M_1, \ldots, M_R$ be the distinct linear forms among $\sigma(L_i)$ ($i = 1, \ldots, n$, $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$). Define the map

$$\varphi : \mathbf{x} \mapsto \Big(M_1(\mathbf{x}), \ldots, M_R(\mathbf{x})\Big)$$

and let $X = \varphi(\overline{\mathbb{Q}}^n)$.

Then there are $\mathbf{c}$, $\theta > 0$, such that if $\mathbf{x} \in \mathbb{Z}^n$ satisfies (3), then

$$\varphi(\mathbf{x}) \in \Big\{\mathbf{z} \in X(\overline{\mathbb{Q}}) : H_{Q,\mathbf{c}}(\mathbf{z}) \leqslant H_{Q,\mathbf{c}}(X)^{1/n} Q^{-\theta}\Big\}$$

with $Q = \|\mathbf{x}\|$.

Thus, if $\mathbf{x}$ is a 'large' solution of (3), then $\varphi(\mathbf{x}) \in T_1 \cup \cdots \cup T_t$, where $T_1, \ldots, T_t$ are the subspaces from the quantitative PST.

Hence $\mathbf{x} \in \varphi^{-1}(T_1) \cup \cdots \cup \varphi^{-1}(T_t)$.

# EXTENSIONS TO NUMBER FIELDS

Let $K$ be an algebraic number field.
Normalize the absolute values $|\cdot|_v$ in such a way that if $v$ lies above $p \in \{\infty\} \cup \{\text{primes}\}$, then $|x|_v = |x|_p^{[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]}$ for $x \in \mathbb{Q}$.
For $\mathbf{x} = (x_1, \dots, x_n) \in K^n$, put

$$H(\mathbf{x}) = \prod_v \|\mathbf{x}\|_v, \quad \text{where } \|\mathbf{x}\|_v = \max_i |x_i|_v.$$

The previously mentioned results have been extended to systems of inequalities

$$\frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} \leqslant C_v H(\mathbf{x})^{-c_{iv}} \ (v \in S, \, i = 1, \dots, n)$$

$$\text{in } \mathbf{x} \in K^n$$

where
- $S$ is a finite set of places of $K$,
- for each $v \in S$, $L_{1v}, \dots, L_{nv}$ are linearly independent linear forms with coefficients in $K$,
- $\sum_{v \in S} \sum_{i=1}^{n} c_{iv} > n$.