EFFECTIVE RESULTS FOR HYPER- AND SUPERELLIPTIC EQUATIONS OVER NUMBER FIELDS

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"To the memory of Professor Antal Bege"

Abstract. Let f be a polynomial with coefficients in the ring O_S of S-integers of a given number field K, b a non-zero S-integer, and m an integer ≥ 2. Suppose that f has no multiple zeros. We consider the equation (*) $by^m = f(x)$ in $x, y \in O_S$. In the present paper we give explicit upper bounds in terms of K, S, b, f, m for the heights of the solutions of (*). Further, we give an explicit bound C in terms of K, S, b, f such that if $m > C$ then (*) has only solutions with $y = 0$ or a root of unity. Our results are more detailed versions of work of Trelina, Brindza, and Shorey and Tijdeman. The results in the present paper are needed in a forthcoming paper of ours on Diophantine equations over integral domains which are finitely generated over $\mathbb{Z}$.

1. Introduction

Let $f \in \mathbb{Z}[X]$ be a polynomial of degree n without multiple roots and m an integer ≥ 2. Siegel proved that the equation

$$y^m = f(x) \tag{1.1}$$

has only finitely many solutions in $x, y \in \mathbb{Z}$ if $m = 2, n \geq 3$ [24] and if $m \geq 3, n \geq 2$ [25]. Siegel's proof is ineffective. In 1969, Baker [1] gave an

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effective proof of Siegel's result. More precisely, he showed that if (x, y) is a solution of (1.1), then

$$\max(|x|, |y|) \leq \begin{cases} \exp\exp\left\{(5m)^{10}(n^{10n}H)^{n^2}\right\} & \text{if } m \geq 3,\ n \geq 2, \\ \exp\exp\exp\left\{(10^{10n}H)^2\right\} & \text{if } m = 2,\ n \geq 3, \end{cases}$$

where H is the maximum of the absolute values of the coefficients of f. In 1976, Schinzel and Tijdeman [21] proved that there is an effectively computable number C, depending only on f, such that (1.1) has no solutions $x, y \in \mathbb{Z}$ with $y \neq 0, \pm 1$ if $m > C$. The proofs of Baker and of Schinzel and Tijdeman are both based on Baker's results on linear forms in logarithms of algebraic numbers.

First Trelina [27] and later in a more general form Brindza [5] generalized the results of Baker to equations of the type (1.1) where the coefficients of f belong to the ring of S-integers O_S of a number field K for some finite set of places S, and where the unknowns x, y are taken from O_S. In their proof they used Baker's result on linear forms in logarithms, as well as a p-adic analogue of this. In fact, Baker, Schinzel and Tijdeman, Trelina and Brindza considered (1.1) also for polynomials f which may have multiple roots. Brindza gave an effective bound for the solutions in the most general situation where (1.1) has only finitely many solutions. This was later improved by Bilu [2] and Bugeaud [6]. Shorey and Tijdeman [22, Theorem 10.2] extended the theorem of Schinzel and Tijdeman to equation (1.1) over the S-integers of a number field. For further related results and applications we refer to [23], [2], [6], [13] and the references given there.

In a forthcoming paper, we will prove effective analogues of the theorems of Baker and Schinzel and Tijdeman for equations of the type (1.1) where the unknowns x, y are taken from an arbitrary finitely generated domain over $\mathbb{Z}$. For this, we need effective finiteness results for Eq. (1.1) over the ring of S-integers of a number field which are more precise than the results of Trelina, Brindza, Bilu, Bugeaud and Shorey and Tijdeman mentioned above. In the present paper, we derive such precise results. Here, we follow improved, updated versions of standard methods. For technical convenience, we restrict ourselves to the case that the polynomial f has no multiple roots. We mention that recently, Gallegos-Ruiz [11] obtained an explicit bound for the heights of the solutions of the hyperelliptic equation $y^2 = f(x)$ in S-integers x, y over $\mathbb{Q}$, but his result is not adapted to our purposes.

In Theorems 2.1 and 2.2 stated below we give for any fixed exponent m effective upper bounds for the heights of the solutions $x, y \in O_S$ of (1.1) which are fully explicit in terms of m, the degree and height of f, the degree and discriminant of K and the prime ideals in S. In Theorem 2.3 below we generalize the Schinzel-Tijdeman Theorem to the effect that if (1.1) has a solution $x, y \in O_S$ with y not equal to 0 or to a root of unity, then m is bounded above by an explicitly given bound depending only on n, the height of f, the degree and discriminant of K and the prime ideals in S.

2. Results

We start with some notation. Let K be a number field. We denote by d, D_K the degree and discriminant of K, by O_K the ring of integers of K and by M_K the set of places of K. The set M_K consists of real infinite places, these are the embeddings $\sigma : K \hookrightarrow \mathbb{R}$; complex infinite places, these are the pairs of conjugate complex embeddings $\{\sigma, \overline{\sigma} : K \hookrightarrow \mathbb{C}\}$, and finite places, these are the prime ideals of O_K. We define normalized absolute values $|\cdot|_v$ ($v \in M_K$) as follows:

$$\left\{ \begin{array}{ll} |\cdot|_v = |\sigma(\cdot)| & \text{if } v = \sigma \text{ is real infinite;} \\ |\cdot|_v = |\sigma(\cdot)|^2 & \text{if } v = \{\sigma, \overline{\sigma}\} \text{ is complex infinite;} \\ |\cdot|_v = (N_K\mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(\cdot)} & \text{if } v = \mathfrak{p} \text{ is finite;} \end{array} \right. \tag{2.1}$$

here $N_K\mathfrak{p} = \#O_K/\mathfrak{p}$ is the norm of $\mathfrak{p}$ and $\operatorname{ord}_{\mathfrak{p}}(x)$ denotes the exponent of $\mathfrak{p}$ in the prime ideal decomposition of x, with $\operatorname{ord}_{\mathfrak{p}}(0) = \infty$.

The logarithmic height of $\alpha \in K$ is defined by

$$h(\alpha) := \frac{1}{[K : \mathbb{Q}]} \log \prod_{v \in M_K} \max(1, |\alpha|_v).$$

Let S be a finite set of places of K containing all (real and complex) infinite places. We denote by O_S the ring of S integers in K, i.e.

$$O_S = \{x \in K \ : \ |x|_v \leq 1 \text{ for } v \in M_K \setminus S\}.$$

Let $s := \# S$ and put

$$P_S = Q_S := 1 \quad \text{if } S \text{ consists only of infinite places,}$$
$$P_S = \max_{i=1,\dots,t} N_K\mathfrak{p}_i, \quad Q_S := \prod_{i=1}^{t} N_K\mathfrak{p}_i$$
$$\text{if } \mathfrak{p}_1, \dots, \mathfrak{p}_t \text{ are the prime ideals in } S.$$

We are now ready to state our results. In what follows,

$$f(X) = a_0X^n + a_1X^{n-1} + \cdots + a_n \in O_S[X] \tag{2.2}$$

is a polynomial of degree $n \geq 2$ without multiple roots and b is a non-zero element of O_S. Put

$$\widehat{h} := \frac{1}{d} \sum_{v \in M_K} \log \max(1, |b|_v, |a_0|_v, \dots, |a_n|_v).$$

Our first result concerns the superelliptic equation

$$f(x) = by^m \qquad \text{in } x, y \in O_S. \tag{2.3}$$

with a fixed exponent $m \geq 3$.

Theorem 2.1. *Assume that $m \geq 3$, $n \geq 2$. If $x, y \in O_S$ is a solution to the equation (2.3) then we have*

$$h(x), h(y) \leq (6ns)^{14m^3n^3s} |D_K|^{2m^2n^2} Q_S^{3m^2n^2} e^{8m^2n^3d\widehat{h}}. \tag{2.4}$$

We now consider the hyperelliptic equation

$$f(x) = by^2 \qquad \text{in } x, y \in O_S. \tag{2.5}$$

Theorem 2.2. *Assume that $n \geq 3$. If $x, y \in O_S$ is a solution to the equation (2.5) then we have*

$$h(x), h(y) \leq (4ns)^{212n^4s} |D_K|^{8n^3} Q_S^{20n^3} e^{50n^4d\widehat{h}}. \tag{2.6}$$

Our last result is an an explicit version of the Schinzel-Tijdeman theorem over the S-integers.

Theorem 2.3. *Assume that (2.3) has a solution $x, y \in O_S$ where y is neither 0 nor a root of unity. Then*

$$m \leq (10n^2s)^{40ns} |D_K|^{6n} P_S^{n^2} e^{11nd\widehat{h}}. \tag{2.7}$$

3. Notation and auxiliary results

We denote by d, D_K, h_K, R_K the degree, discriminant, class number and regulator, and by O_K the ring of integers of K. Further, we denote by $\mathcal{P}(K)$ the collection of non-zero prime ideals of O_K. For a non-zero fractional ideal $\mathfrak{a}$ of O_K we have the unique factorization

$$\mathfrak{a} = \prod_{\mathfrak{p}\in\mathcal{P}(K)} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{a}},$$

where there are only finitely many prime ideals $\mathfrak{p} \in \mathcal{P}(K)$ with $\mathrm{ord}_{\mathfrak{p}}\,\mathfrak{a} \neq 0$. Given $\alpha_1, \dots, \alpha_n \in K$, we denote by $[\alpha_1, \dots, \alpha_n]_K$ the fractional ideal of O_K generated by $\alpha_1, \dots, \alpha_n$. For a polynomial $f \in K[X]$ we denote by $[f]_K$ the fractional ideal generated by the coefficients of f. We denote by $N_K\mathfrak{a}$ the absolute norm of a fractional ideal of O_K. In case that $\mathfrak{a} \subseteq O_K$ we have $N_K\mathfrak{a} = \#O_K/\mathfrak{a}$.

We define $\log^* x := \max(1, \log x)$ for $x \geq 0$.

3.1. Discriminant estimates.

Let L be a finite extension of K. Recall that the relative discriminant ideal $\mathfrak{d}_{L/K}$ of L/K is the ideal of O_K generated by the numbers

$$D_{L/K}(\omega_1, \dots, \omega_n) \ \text{ with } \omega_1, \dots \omega_n \in O_L,$$

where $n := [L : K]$.

Lemma 3.1. *Suppose that $L = K(\alpha)$ and let $f \in K[X]$ be a square-free polynomial of degree m with $f(\alpha) = 0$. Then*

$$\mathfrak{d}_{L/K} \supseteq \frac{[D(f)]_K}{[f]_K^{2m-2}}. \tag{3.1}$$

Proof. We have inserted a proof for lack of a good reference. We write $[\cdot]$ for $[\cdot]_K$. Let $g \in K[X]$ be the monic minimal polynomial of α. Then $f = g_1 g_2$ with $g_2 \in K[X]$. Let $n := \deg g_1$ and $k := \deg h_1$. Then

$$D(f) = D(g_1)D(g_2)R(g_1, g_2)^2,$$

where $R(g_1, g_2)$ is the resultant of g_1 and g_2. Using determinantal expressions for $D(g_1)$, $D(g_2)$, $R(g_1, g_2)$ we get

$$D(g_1) \in [g_1]^{2n-2}, \quad D(g_2) \in [g_2]^{2k-2}, \quad R(g_1, g_2) \in [g_1]^k [g_2]^n,$$

and by Gauss' Lemma, $[f] = [g_1] \cdot [g_2]$. Hence

$$\frac{[D(f)]}{[f]^{2m-2}} = \frac{[D(g_1)]}{[g_1]^{2n-2}} \frac{[D(g_2)]}{[g_2]^{2k-2}} \frac{[R(g_1, g_2)]}{[g_1]^k [g_2]^n} \subseteq \frac{[D(g_1)]}{[g_1]^{2n-2}}.$$

Therefore, it suffices to prove

$$\mathfrak{d}_{L/K} \supset \frac{[D(g_1)]}{[g_1]^{2n-2}}.$$

Note that $[g_1]^{-1}$ consists of all $\lambda \in K$ with $\lambda g_1 \in O_K[X]$. Hence the ideal $[D(g_1)] \cdot [g_1]^{-2n+2}$ is generated by the numbers $\lambda^{2n-2} D(g_1) = D(\lambda g_1)$ such that $\lambda g_1 \in O_K[X]$. Writing $h := \lambda g_1$, we see that it suffices to prove that if $h \in O_K[X]$ is irreducible in $K[X]$ and $h(\alpha) = 0$ with $L = K(\alpha)$, then

$$D(h) \in \mathfrak{d}_{L/K}.$$

To prove this, we use an argument of Birch and Merriman [3]. Let $h(X) = b_0 X^m + b_1 x^{m-1} + \cdots + b_m \in O_K[X]$ with $h(\alpha) = 0$. Put

$$\omega_i := b_0 \alpha^i + b_1 \alpha^{i-1} + \cdots + b_i \quad (i = 0, 1, \ldots, n).$$

We show by induction on i that $\omega_i \in O_L$. For $i = 0$ this is clear. Assume that we have proved that $\omega_i \in O_L$ for some $i \geq 0$. By $h(\alpha) = 0$ we clearly have

$$\omega_i \alpha^{n-i} + b_{i+1} \alpha^{n-i-1} + \cdots + b_n = 0.$$

By multiplying this expression with ω_i^{n-i-1}, we see that $\omega_i \alpha$ is a zero of a monic polynomial from $O_L[X]$, hence belongs to O_L. Therefore, $\omega_{i+1} = \omega_i \alpha + b_{i+1} \in O_L$.

Now on the one hand, $D_{L/K}(1, \omega_1, \ldots, \omega_{n-1}) \in \mathfrak{d}_{L/K}$, on the other hand,

$$\begin{aligned} D_{L/K}(1, \omega_1, \ldots, \omega_{n-1}) &= b_0^{2n-2} D_{L/K}(1, \alpha, \ldots, \alpha^{n-1}) \\ &= b_0^{2n-2} \prod_{1 \leq i < j \leq 0} (\alpha^{(i)} - \alpha^{(j)})^2 = D(h). \end{aligned}$$

Hence $D(h) \in \mathfrak{d}_{L/K}$. □

Put $u(n) := \mathrm{lcm}(1, 2, \ldots, n)$. For the possible prime factors of the discriminant $\mathfrak{d}_{L/K}$ we have:

Lemma 3.2. *Let $[L : K] = n$. Then for every prime ideal $\mathfrak{p} \in \mathcal{P}(K)$ with $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{d}_{L/K}) > 0$ we have*

$$\mathrm{ord}_{\mathfrak{p}}(\mathfrak{d}_{L/K}) \leq n \cdot (1 + \mathrm{ord}_{\mathfrak{p}}(u(n))).$$

Proof. Let $\mathfrak{D}_{L/K}$ denote the different of L/K. According to J. Neukirch [19, p. 210, Theorem 2.6], we have for every prime ideal $\mathfrak{P}$ of L lying above $\mathfrak{p}$

$$\begin{aligned}\mathrm{ord}_{\mathfrak{P}}(\mathfrak{D}_{L/K}) &\leq e(\mathfrak{P}|\mathfrak{p}) - 1 + \mathrm{ord}_{\mathfrak{P}}(e(\mathfrak{P}|\mathfrak{p})) \\ &\leq e(\mathfrak{P}|\mathfrak{p}) - 1 + e(\mathfrak{P}|\mathfrak{p})\,\mathrm{ord}_{\mathfrak{p}}(e(\mathfrak{P}|\mathfrak{p})),\end{aligned}$$

where $e(\mathfrak{P}|\mathfrak{p})$, $f(\mathfrak{P}|\mathfrak{p})$ denote the ramification index and residue class degree of $\mathfrak{P}$ over $\mathfrak{p}$. Using $\mathfrak{d}_{L/K} = N_{L/K}\mathfrak{D}_{L/K}$, $N_{L/K}\mathfrak{P} = \mathfrak{p}^{f(\mathfrak{P}|\mathfrak{p})}$, $\sum_{\mathfrak{P}|\mathfrak{p}} e(\mathfrak{P}|\mathfrak{p})f(\mathfrak{P}|\mathfrak{p}) = [L:K] \leq n$, we infer

$$\begin{aligned}\mathrm{ord}_{\mathfrak{p}}(\mathfrak{d}_{L/K}) = \mathrm{ord}_{\mathfrak{p}}(N_{L/K}\mathfrak{D}_{L/K}) &= \sum_{\mathfrak{P}|\mathfrak{p}} f(\mathfrak{P}|\mathfrak{p})\,\mathrm{ord}_{\mathfrak{P}}(\mathfrak{D}_{L/K}) \\ &\leq \sum_{\mathfrak{P}|\mathfrak{p}} f(\mathfrak{P}|\mathfrak{p})e(\mathfrak{P}|\mathfrak{p})(1 + \mathrm{ord}_{\mathfrak{p}}(e(\mathfrak{P}|\mathfrak{p})) \\ &\leq n(1 + \mathrm{ord}_{\mathfrak{p}}(u(n))).\end{aligned}$$

□

Lemma 3.3. (i) *Let $M \supset L \supset K$ be a tower of finite extensions. Then we have*

$$\mathfrak{d}_{M/K} = N_{L/K}(\mathfrak{d}_{M/L})\mathfrak{d}_{L/K}^{[M:L]}.$$

(ii) *Let L_1, L_2 be finite extensions of K. Then for their compositum $L_1 \cdot L_2$ we have*

$$\mathfrak{d}_{L_1L_2/K} \supseteq \mathfrak{d}_{L_1/K}^{[L_1L_2:L_1]}\mathfrak{d}_{L_2/K}^{[L_1L_2:L_2]}.$$

Proof. For (i) see Neukirch [19, p. 213, Korollar 2.10]. For (ii) apply Stark [26, Lemma 6] and take norms. □

Lemma 3.4. *Let $m \in \mathbb{Z}_{\geq 0}$, $\gamma \in K^*$ and $L := K(\sqrt[m]{\gamma})$. Further, let $\mathfrak{p} \in \mathcal{P}(K)$ be a prime ideal with*

$$\mathrm{ord}_{\mathfrak{p}}(m) = 0, \qquad \mathrm{ord}_{\mathfrak{p}}(\gamma) \equiv 0 \pmod{m}.$$

Then L/K is unramified at $\mathfrak{p}$, i.e.

$$\mathrm{ord}_{\mathfrak{p}}(\mathfrak{d}_{L/K}) = 0.$$

Proof. Choose $\tau \in K^*$ such that $\mathrm{ord}_{\mathfrak{p}}(\tau) = 1$. Then $\gamma = \tau^{mt}\varepsilon$ with $t \in \mathbb{Z}$ and $\mathrm{ord}_{\mathfrak{p}}(\varepsilon) = 0$. We clearly have $L = K(\sqrt[m]{\varepsilon})$, hence

$$\mathfrak{d}_{L/K} \supseteq \frac{[D(X^m - \varepsilon)]}{[1,\varepsilon]^{2m-2}} = \frac{[m^m\varepsilon^{m-1}]}{[1,\varepsilon]^{2m-2}}.$$

This implies $\mathrm{ord}_{\mathfrak{p}}(\mathfrak{d}_{L/K}) = 0$. □

3.2. S**-integers.** Let K be an algebraic number field and denote by M_K its set of places. We keep using throughout the absolute values defined by (2.1). Recall that these absolute values satisfy the product formula

$$\prod_{v\in M_K} |\alpha|_v = 1 \quad \text{for } \alpha \in K^*.$$

If L is a finite extension of K, and v, w places of K, L, respectively, we say that w lies above v, notation $w|v$, if the restriction of $|\cdot|_w$ to K is a power of $|\cdot|_v$, and in that case we have

$$|\alpha|_w = |\alpha|_v^{[L_w:K_v]} \quad \text{for } \alpha \in K,$$

where K_v, L_w denote the completions of K at v, L at w, respectively. In case that $v = \mathfrak{p}$, $w = \mathfrak{P}$ are prime ideals of O_K, O_L, respectively, we have $w|v$ if and only if $\mathfrak{p} \subset \mathfrak{P}$.

Let S be a finite set of places of K containing all infinite places. The non-zero fractional ideals of the ring of S-integers O_S (i.e., finitely generated O_S-submodules of K) form a group under multiplication, and there is an isomorphism from the multiplicative group of non-zero fractional ideals of O_S to the group of fractional ideals of O_K composed of prime ideals outside S given by $\mathfrak{a} \mapsto \mathfrak{a}^*$, where $\mathfrak{a} = \mathfrak{a}^* O_S$. We define the S-norm of a fractional ideal of O_S by

$$N_S(\mathfrak{a}) := N_K\mathfrak{a}^* = \text{absolute norm of } \mathfrak{a}^*.$$

Given $\alpha_1, \dots, \alpha_r \in K$ we denote by $[\alpha_1, \dots, \alpha_r]_S$ the fractional ideal of O_S generated by $\alpha_1, \dots, \alpha_r$. We have

$$N_S([\alpha_1, \dots, \alpha_r]_S) = \prod_{v\in M_K\setminus S} \max(|\alpha_1|_v, \dots, |\alpha_r|_v)^{-1}. \tag{3.2}$$

Further, for $\alpha \in K$ we define $N_S(\alpha) := N_S([\alpha]_S)$. By the product formula,

$$N_S(\alpha) = \prod_{v\in S} |\alpha|_v \quad \text{for } \alpha \in K. \tag{3.3}$$

Let L be a finite extension of K, and T the set of places of L lying above the places in S. Then the ring of T-integers O_T is the integral closure in L of O_S. Every fractional ideal $\mathfrak{A}$ of O_T can be expressed uniquely as $\mathfrak{A} = \mathfrak{A}^* O_T$ where $\mathfrak{A}^*$ is a fractional ideal of O_L composed of prime ideals outside T. We put

$$N_T\mathfrak{A} := N_L\mathfrak{A}^*, \quad N_{T/S}\mathfrak{A} := (N_{L/K}\mathfrak{A}^*)O_S.$$

Then

$$(3.4)\qquad \begin{cases} N_T\mathfrak{A} = N_S(N_{T/S}\mathfrak{A}), \\ N_T(\mathfrak{a}O_T) = N_S\mathfrak{a}^{[L:K]} \text{ for a fractional ideal } \mathfrak{a} \text{ of } O_S. \end{cases}$$

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the prime ideals in S and put $Q_S := \prod_{i=1}^{t} N_K\mathfrak{p}_i$. Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_{t'}$ be the prime ideals in T and put $Q_T := \prod_{i=1}^{t'} N_K\mathfrak{P}_i$. Then for every prime ideal $\mathfrak{p}$ of O_K we have

$$\prod_{\mathfrak{P}|\mathfrak{p}} N_L\mathfrak{P} = \prod_{\mathfrak{P}|\mathfrak{p}} (N_K\mathfrak{p})^{f_{\mathfrak{P}|\mathfrak{p}}} \leq \prod_{\mathfrak{P}|\mathfrak{p}} (N_K\mathfrak{p})^{e_{\mathfrak{P}|\mathfrak{p}}\cdot f_{\mathfrak{P}|\mathfrak{p}}} \leq (N_K\mathfrak{p})^{[L:K]},$$

where the product is over all prime ideals $\mathfrak{P}$ of O_L dividing $\mathfrak{p}$ and where $e(\mathfrak{P}|\mathfrak{p})$, $f(\mathfrak{P}|\mathfrak{p})$ denote the ramification index and residue class degree of $\mathfrak{P}$ over $\mathfrak{p}$. Hence

$$(3.5)\qquad Q_T \leq Q_S^{[L:K]}.$$

3.3. **Class number and regulator.** Let again K be a number field.

Lemma 3.5. *For the regulator R_K and class number h_K of K we have the following estimates:*

$$(3.6)\qquad R_K \geq 0.2,$$

$$(3.7)\qquad h_K R_K \leq |D_K|^{\frac{1}{2}} (\log^* |D_K|)^{d-1}.$$

Proof. Statement (3.6) is a result of Friedman [10]. Inequality (3.7) follows from Louboutin [17], see also (59) in Győry and Yu [14]. □

Let S be a finite set of places of K consisting of the infinite places and of the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. Then the S-regulator R_S is given by

$$(3.8)\qquad R_S = h_S R_K \prod_{i=1}^{t} \log N_K\mathfrak{p}_i,$$

where h_S is the order of the group generated by the ideal classes of $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ and where h_S and the product are 1 if S consists only of the infinite places. Together with Lemma 3.5 this implies

$$(3.9)\qquad \tfrac{1}{5} \ln 2 \leq R_S \leq |D_K|^{\frac{1}{2}} (\log^* |D_K|)^{d-1} \cdot (\log P_S)^t,$$

where the last factor has to be interpreted as 1 if $t = 0$.

3.4. **Heights.** We define the absolute logarithmic height of $\alpha \in \overline{\mathbb{Q}}$ by

$$h(\alpha) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} \max(0, \log |\alpha|_v),$$

where K is any number field with $K \ni \alpha$. More generally, we define the logarithmic height of a polynomial $f(X) = a_0 x^n + \cdots + a_n \in \overline{\mathbb{Q}}[X]$ by

$$h(f) := \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} \log \max(1, |a_0|_v, \dots, |a_n|_v)$$

where K is any number field with $f \in K[X]$. These heights do not depend on the choice of K.

We will frequently use the inequalities

$$h(\alpha_1 \cdots \alpha_n) \le \sum_{i=1}^n h(\alpha_i), \quad h(\alpha_1 + \cdots + \alpha_n) \le \sum_{i=1}^n h(\alpha_i) + \log n$$

for $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ and the equality

$$h(\alpha^m) = |m| h(\alpha) \;\text{ for } \alpha \in \overline{\mathbb{Q}}^*,\ m \in \mathbb{Z}.$$

(see Waldschmidt [29, Chapter 3]). Further we frequently use the trivial fact that if α belongs to a number field K and S is a finite set of places of K containing the infinite places, then

$$h(\alpha) \ge \frac{1}{[K:\mathbb{Q}]} \log N_S(\alpha).$$

We have collected some further facts.

Lemma 3.6. *Let $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ and $f = (X - \alpha_1) \cdots (X - \alpha_n)$. Then*

$$|h(f) - \sum_{i=1}^n h(\alpha_i)| \le n \log 2.$$

Proof. See Bombieri and Gubler [4, p.28, Thm.1.6.13]. □

Lemma 3.7. *Let K be a number field and $f = a_0 X^n + a_1 X^{n-1} + \cdots + a_n \in K[X]$ a polynomial of degree n with discriminant $D(f) \ne 0$. Then*

(i) $|D(f)|_v \le n^{(2n-1)s(v)} \max(|a_0|_v, \dots, |a_n|_v)^{2n-2}$ *for $v \in M_K$,*

(ii) $h(D(f)) \le (2n-1) \log n + (2n-2) h(f)$,

where $s(v) = 1$ if v is real, $s(v) = 2$ if v is complex, $s(v) = 0$ if v is finite.

Proof. Inequality (ii) is an immediate consequence of (i). For finite v, inequality (i) follows from the ultrametric inequality, noting that $D(f)$ is a homogeneous polynomial of degree $2n-2$ in the coefficients of f with integer coefficients. For infinite v, inequality (i) follows from a a result of Lewis and Mahler [16, p. 335]). □

Lemma 3.8. *Let K be an algebraic number field and S a finite set of places of K, which consists of the infinite places and of the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. Then for every $\alpha \in O_S \setminus \{0\}$ and $m \in \mathbb{N}$ there exists an S-unit $\eta \in O_S^*$ with*

$$h(\alpha\eta^m) \leq \frac{1}{d}\log N_S(\alpha) + m \cdot \left(cR_K + \frac{h_K}{d}\log Q_S\right),$$

where $c := 39d^{d+2}$ and $Q_S := \prod_{i=1}^{t} N_K\mathfrak{p}_i$.

Proof. This is a slightly weaker version of Lemma 3 of Győry and Yu [14]. The result was essentially proved (with a larger constant) in [9] and [12]. □

Lemma 3.9. *Let α be a non-zero algebraic number of degree d which is not a root of unity. Then*

$$h(\alpha) \geq m(d) := \begin{cases} \log 2 & \text{if} \quad d = 1, \\ 2/d(\log 3d)^3 & \text{if} \quad d \geq 2. \end{cases}$$

Proof. See Voutier [28]. □

3.5. **Baker's method.** Let K be an algebraic number field, and denote by M_K the set of places of K. Let $\alpha_1, \ldots, \alpha_n$ be $n \geq 2$ non-zero elements of K, and $b_1, \ldots, b_n$ are rational integers, not all zero. Put

$$\begin{aligned} \Lambda &:= \alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1, \\ \Theta &:= \prod_{i=1}^{n} \max\big(h(\alpha_i), m(d)\big), \\ B &:= \max(3, |b_1|, , \ldots, |b_n|), \end{aligned}$$

where $m(d)$ is the lower bound from Lemma 3.9 (i.e., the maximum is $h(\alpha_i)$ unless α_i is a root of unity). For a place $v \in M_K$, we write

$$N(v) = \begin{cases} 2 & \text{if } v \text{ is infinite} \\ N_K\mathfrak{p} & \text{if } v = \mathfrak{p} \text{ is finite.} \end{cases}$$

Proposition 3.10. *Suppose that $\Lambda \neq 0$. Then for $v \in M_K$ we have*

$$\log|\Lambda|_v > -\,c_1(n,d)\frac{N(v)}{\log N(v)}\Theta\log B, \tag{3.10}$$

where $c_1(n,d) = 12(16ed)^{3n+2}(\log^* d)^2$.

Proof. First assume that v is infinite. Without loss of generality, we assume that $K \subset \mathbb{C}$ and $|\cdot|_v = |\cdot|^{s(v)}$ where $s(v) = 1$ if $K \subset \mathbb{R}$ and $s(v) = 2$ otherwise. Denote by log the principal natural logarithm on $\mathbb{C}$ (with $|\mathrm{Im}\,\log z| \le \pi$ for $z \in \mathbb{C}^*$. Let b_0 be the rational integer such that $|\mathrm{Im}\,\Xi| \le \pi$, where

$$\Xi := b_1\log\alpha_1 + \cdots + b_n\log\alpha_n + 2b_0\log(-1), \quad \log(-1) = \pi i.$$

Thus,

$$B' := \max(|2b_0|, |b_1|, \ldots, |b_n|) \le 1 + nB.$$

A result of Matveev [18, Corollary 2.3] implies that

$$\log|\Xi| \ge -\,s(v)^{-1}\big(\tfrac{1}{2}e(n+1)\big)^{s(v)}(n+1)^{3/2}30^{n+4}d^2(\log ed)\Omega\log(eB'),$$

where

$$\Omega := \pi\prod_{i=1}^{n}\max(h(\alpha_i), \pi).$$

Assuming, as we may, that $|\Lambda| \le \frac{1}{2}$, we get $|\Xi| = |\log(1+\Lambda)| \le 2|\Lambda| \le 1$. Further, $\Omega \le \pi^{n+1}m(d)^{-n}\Theta$. By combining this with Matveev's lower bound we obtain a lower bound for $|\Lambda|_v$ which is better than (3.10).

Now assume that v is finite, say $v = \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of O_K. By a result of K. Yu [30] (consequence of Main Theorem on p. 190) we have

$$\mathrm{ord}_{\mathfrak{p}}(\Lambda) \le (16ed)^{2n+2}n^{3/2}\log(2nd)\log(2d)e_{\mathfrak{p}}^n\cdot\frac{N_K\mathfrak{p}}{(\log N_K\mathfrak{p})^2}\cdot\Theta\log B,$$

where $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$. Using that $\log|\Lambda|_{\mathfrak{p}} = -\,\mathrm{ord}_{\mathfrak{p}}(\Lambda)\log N_K\mathfrak{p}$ and $e_{\mathfrak{p}} \le d$, we obtain a lower bound for $\log|\Lambda|_{\mathfrak{p}}$ which is better than (3.10). □

3.6. **Thue equations and Pell equations.** Let K be an algebraic number field of degree d, discriminant D_K, regulator R_K and class number h_K, and denote by O_K its ring of integers. Let S be a finite set of places of K containing all infinite places. Denote by s the cardinality of S and by O_S the ring of S integers in K. Further denote by R_S the S-regulator, let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the prime ideals in S, and put

$$P_S := \max\{N_K\mathfrak{p}_1, \dots, N_K\mathfrak{p}_t\}, \quad Q_S := N_K(\mathfrak{p}_1 \cdots \mathfrak{p}_t),$$

with the convention that $P_S = Q_S = 1$ if S contains no finite places.

We state effective results on Thue equations and on systems of Pell equations which are easy consequences of a general effective result on decomposable form equations by Győry and Yu [14]. In both results we use the constant

$$c_1(s,d) := s^{2s+4}2^{7s+60}d^{2s+d+2}.$$

Proposition 3.11. *Let $\beta \in K^*$ and let $F(X,Y) = \sum_{i=0}^{n} a_i X^{n-i}Y^i \in K[X,Y]$ be a binary form of degree $n \geq 3$ with non-zero discriminant which splits into linear factors over K. Suppose that*

$$\max_{0 \leq i \leq n} h(a_i) \leq A, \quad h(\beta) \leq B.$$

Then for the solutions of

$$F(x,y) = \beta \quad \text{in } x, y \in O_S \tag{3.11}$$

we have

$$\max(h(x), h(y)) \leq c_1(s,d)n^6 P_S R_S \left(1 + \frac{\log^* R_S}{\log^* P_S}\right) \cdot \left(R_K + \frac{h_K}{d}\log Q_S + ndA + B\right). \tag{3.12}$$

Proof. Győry and Yu [14, p. 16, Corollary 3] proved this with instead of our $c_1(s,d)$ a smaller bound $5d^2n^5 \cdot 50(n-1)c_1c_3$, where c_1, c_3 are given respectively in [14, Theorem 1], and in [14, bottom of page 11]. □

Proposition 3.12. *Let $\gamma_1, \gamma_2, \gamma_3, \beta_{12}, \beta_{13}$ be non-zero elements of K such that*

$$\beta_{12} \neq \beta_{13}, \quad \sqrt{\gamma_1/\gamma_2},\ \sqrt{\gamma_1/\gamma_3} \in K,$$
$$h(\gamma_i) \leq A \text{ for } i = 1,2,3, \quad h(\beta_{12}), h(\beta_{13}) \leq B.$$

Then for the solutions of the system

$$\gamma_1 x_1^2 - \gamma_2 x_2^2 = \beta_{12}, \quad \gamma_1 x_1^2 - \gamma_3 x_3^2 = \beta_{13} \qquad \text{in } x_1, x_2, x_3 \in O_S \tag{3.13}$$

we have

$$\begin{aligned} &\max(h(x_1), h(x_2), h(x_3)) \\ &\leq c_1(s,d) P_S R_S \left(1 + \frac{\log^* R_S}{\log^* P_S}\right) \cdot \Big(R_K + \frac{h_K}{d}\log Q_S + dA + B\Big). \end{aligned} \tag{3.14}$$

Proof. Put $\beta_{23} := \beta_{13} - \beta_{12}$, $\beta := \beta_{12}\beta_{13}\beta_{23}$ and define

$$F := (\gamma_1 X_1^2 - \gamma_2 X_2^2)(\gamma_1 X_1^2 - \gamma_3 X_3^2)(\gamma_2 X_2^2 - \gamma_3 X_3^2).$$

Thus, every solution of (3.13) satisfies also

$$F(x_1, x_2, x_3) = \beta \quad \text{in } x_1, x_2, x_3 \in O_S. \tag{3.15}$$

By assumption, $\beta \neq 0$. Further, F is a decomposable form of degree 6 with splitting field K, i.e., $F = l_1 \cdots l_6$ where $l_1, \ldots, l_6$ are linear forms with coefficients in K. We make a graph on $\{l_1, \ldots, l_6\}$ by connecting two linear forms l_i, l_j if there is a third linear form l_k such that $l_k = \lambda l_i + \mu l_j$ for certain non-zero $\lambda, \mu \in K$. Then this graph is connected. Further, $\operatorname{rank}\{l_1, \ldots, l_6\} = 3$. Hence F satisfies all the conditions of Theorem 3 of Győry and Yu [14]. According to this Theorem, the solutions x_1, x_2, x_3 of (3.15), and so also the solutions of (3.13), satisfy (3.14) but with instead of $c_1(s,d)$ the smaller number $375 c_1 c_3$, where c_1, c_3 are given respectively in [14, Theorem 1], and on [14, bottom of page 11]. □

4. Proof of the results in the case of fixed exponent

Let K be an algebraic number field, put $d := [K : \mathbb{Q}]$, and let D_K denote the discriminant of K. Further, let S be a finite set of places of K containing all infinite places.

Lemma 4.1. *Let $f(X) \in K[X]$ be a polynomial of degree n and discriminant $D(f) \neq 0$. Suppose that f factorizes over an extension of K as $a_0(X - \alpha_1) \ldots (X - \alpha_n)$ and let $L := K(\alpha_1, \ldots, \alpha_k)$. Then for the discriminant of L we have*

$$|D_L| \leq \left(n \cdot e^{h(f)}\right)^{2kn^k d} \cdot |D_K|^{n^k}.$$

For the case $k=1$ we have the sharper estimate

$$|D_L| \le n^{(2n-1)d} \cdot e^{(2n-2)d \cdot h(f)} \cdot |D_K|^{[L:K]}.$$

Proof. By Lemma 3.3 (i), we have

$$|D_L| = N_K \mathfrak{d}_{L/K} \cdot |D_K|^{[L:K]} \le N_K \mathfrak{d}_{L/K} \cdot |D_K|^{n^k}. \tag{4.1}$$

Applying Lemma 3.3 (ii) to $L = K(\alpha_1)\cdots K(\alpha_k)$ yields

$$\mathfrak{d}_{L/K} \supseteq \prod_{i=1}^{k} \left(\mathfrak{d}_{K(\alpha_i)/K}\right)^{[L:K(\alpha_i)]}. \tag{4.2}$$

Further, since α_i is a root of f we have by Lemma 3.1,

$$\mathfrak{d}_{K(\alpha_i)/K} \supseteq \frac{[D(f)]}{[f]^{2n-2}},$$

and so

$$N_K \mathfrak{d}_{K(\alpha_i)/K} \le N_K \left(\frac{[D(f)]}{[f]^{2n-2}}\right). \tag{4.3}$$

By Lemma 3.7 we have

$$\begin{aligned} |N_K(D(f))| &= \prod_{v \in M_K^\infty} |D(f)|_v \le \prod_{v \in M_K^\infty} \left(n^{2n-1}\right)^{s(v)} |f|_v^{2n-2} \\ &\le n^{(2n-1)d} \prod_{v \in M_K^\infty} |f|_v^{2n-2} \end{aligned}$$

where $|f|_v$ is the maximum of the v-adic absolute values of the coefficients of f; moreover,

$$N_K([f]^{-2n+2}) = \prod_{v \in M_K \setminus M_K^\infty} |f|_v^{2n-2}.$$

Thus, we obtain

$$N_K \left(\frac{[D(f)]}{[f]^{2n-2}}\right) \le \left(n^{2n-1} \cdot e^{(2n-2)h(f)}\right)^d. \tag{4.4}$$

Together with (4.1), (4.3) this implies the sharper upper bound for $|D_L|$ in the case $k=1$. For arbitrary k, combining (4.2), (4.3), (4.4) and the estimate $[L : K(\alpha_i)] \le (n-1)(n-2)\cdots(n-k+1)$ gives

$$\begin{aligned} N_K \mathfrak{d}_{L/K} &\le \left(n^{2n-1} \cdot e^{(2n-2)h(f)}\right)^{k(n-1)(n-2)\cdots(n-k+1)d} \\ &\le n^{k(2n-1)n^{k-1}d} \cdot e^{k(2n-2)n^{k-1}d \cdot h(f)} \le \left(n \cdot e^{h(f)}\right)^{2kn^k d}. \end{aligned}$$

This in turn, together with (4.1) proves Lemma 4.1. □

Let

$$f = a_0X^n + a_1X^{n-1} + \cdots + a_n \in O_S[X]$$

be a polynomial of degree $n \geq 2$ with discriminant $D(f) \neq 0$. Let b be a non-zero element of O_S, m an integer ≥ 2 and consider the equation

$$f(x) = by^m \qquad \text{in } x, y \in O_S. \tag{4.5}$$

Put

$$\widehat{h} := \frac{1}{d} \sum_{v \in M_K} \log \max(1, |b|_v, |a_0|_v, \ldots, |a_n|_v). \tag{4.6}$$

Let G be the splitting field of f over K. Then

$$f = a_0(X - \alpha_1) \cdots (X - \alpha_n) \text{ with } \alpha_1, \ldots, \alpha_n \in G.$$

For $i = 1, \ldots, n$, let $L_i = K(\alpha_i)$ and denote by T_i the set of places of L_i lying above the places of S. We denote by $[\beta_1, \ldots, \beta_r]_{T_i}$ the fractional of O_{T_i} generated by $\beta_1, \ldots, \beta_r$. Then we have the following Lemma:

Lemma 4.2. *Let $x, y \in O_S$ be a solution of equation (4.5) with $y \neq 0$. Then for $i = 1, \ldots, n$ we have the following:*
(i) There are ideals $\mathfrak{C}_i$, $\mathfrak{A}_i$ of O_{T_i} such that

$$[a_0(x - \alpha_i)]_{T_i} = \mathfrak{C}_i \mathfrak{A}_i^m, \quad \mathfrak{C}_i \supseteq [a_0 b D(f)]_{T_i}^{m-1}. \tag{4.7}$$

(ii) There are γ_i, ξ_i with

$$\begin{cases} x - \alpha_i = \gamma_i \xi_i^m, \quad \gamma_i \in L_i^*, \, \xi \in O_{T_i}, \\ h(\gamma_i) \leq m(n^3 d)^{nd} e^{2nd\widehat{h}} |D_K|^n \cdot \Big(80(dn)^{dn+2} + \frac{1}{d} \log Q_S\Big). \end{cases} \tag{4.8}$$

Proof. It suffices to prove the Lemma for $i = 1$. We suppress the index 1 and write $\alpha, T, L, \gamma, \xi$ for $\alpha_1, T_1, L_1, \gamma_1, \xi_1$. Let $g := (X - \alpha_2) \ldots (X - \alpha_n)$. By $[\cdot]$ we denote fractional ideals in G with respect to the integral closure of O_T in G. Clearly,

$$\frac{[x - \alpha]}{[1, \alpha]} + \frac{[x - \alpha_i]}{[1, \alpha_i]} \supseteq \frac{[\alpha - \alpha_i]}{[1, \alpha][1, \alpha_i]}$$

for $i = 2, \ldots, n$. This implies

$$\frac{[x - \alpha]}{[1, \alpha]} + \prod_{i=2}^{n} \frac{[x - \alpha_i]}{[1, \alpha_i]} \supseteq \prod_{i=2}^{n} \frac{[\alpha - \alpha_i]}{[1, \alpha][1, \alpha_i]}$$

Noting that by Gauss' Lemma we have $[f] = [a_0]\prod_{i=1}^{n}[1,\alpha_i]$, we see that the right-hand side contains

$$\prod_{j=1}^{n}\prod_{i\neq j}\frac{[\alpha_j-\alpha_i]}{[1,\alpha_j][1,\alpha_i]} = \frac{[D(f)]}{[f]^{2n-2}}.$$

Using also $[g] = \prod_{i=2}^{n}[1,\alpha_i]$ we obtain

$$\frac{[x-\alpha]}{[1,\alpha]} + \frac{[g(x)]}{[g]} \supseteq \frac{[D(f)]}{[f]^{2n-2}}. \tag{4.9}$$

Writing equation (4.5) as equation of ideals, we get

$$[b][f]^{-1}[y]^m = \frac{[x-\alpha]}{[1,\alpha]}\cdot\frac{[g(x)]}{[g]}. \tag{4.10}$$

Note that the ideals occurring in (4.9), (4.10) are all defined over L, so we may view them as ideals of O_T. Henceforth, we use $[\cdot]$ to denote ideals of O_T.

Now let $\mathfrak{P}$ be a prime ideal of O_T not dividing $a_0bD(f)$. Note that $D(f) \in [f]^{2n-2}$, hence $\mathfrak{P}$ does not divide $[f]$ either. By (4.9), the prime ideal $\mathfrak{P}$ divides at most one of the ideals $\frac{[x-\alpha_1]}{[1,\alpha_1]}$ and $\frac{[g(x)]}{[g]}$, and we get

$$\mathrm{ord}_{\mathfrak{P}}\frac{[x-\alpha]}{[1,\alpha]} \equiv 0 \pmod{m}.$$

But $[a_0][1,\alpha]$ is not divisible by $\mathfrak{P}$ since it contains a_0. Hence

$$\mathrm{ord}_{\mathfrak{P}}(a_0(x-\alpha)) \equiv 0 \pmod{m}.$$

Applying division with remainder to the exponents of the prime ideals dividing $a_0bD(f)$ in the factorization of $a_0(x-\alpha)$, we obtain that there are ideals $\mathfrak{C}$, $\mathfrak{A}$ of O_T, with $\mathfrak{C}$ dividing $(ba_0D(f))^{m-1}$ such that $[a_0(x-\alpha)] = \mathfrak{C}\mathfrak{A}^m$. This proves (i).

We prove (ii). The ideal $\mathfrak{A}$ of O_T may be written as $\mathfrak{A} = \mathfrak{A}^*O_T$ with an ideal $\mathfrak{A}^*$ of O_L composed of prime ideals outside T, and further, we may choose non-zero $\xi_1 \in \mathfrak{A}^*$ with $|N_{L/\mathbb{Q}}(\xi_1)| \le |D_L|^{1/2}N_L\mathfrak{A}^*$ (see Lang [15, pp. 119/120]. This implies $N_T(\xi_1) \le |D_L|^{1/2}N_T\mathfrak{A}$, i.e., $[\xi_1] = \mathfrak{B}\mathfrak{A}$ where $\mathfrak{B}$ is an ideal of O_T with $N_T\mathfrak{B} \le |D_L|^{1/2}$. Similarly, there exists $\gamma_1 \in L$ with $[\gamma_1] = \mathfrak{D}\mathfrak{C}$, where $\mathfrak{D}$ is an ideal of O_T with $N_T\mathfrak{D} \le |D_L|^{1/2}$. As a consequence, we have

$$a_0(x-\alpha) = \frac{\gamma_1}{\gamma_2}\xi_1^m,$$

where $\gamma_1, \gamma_2 \in O_T$, and

$$[\gamma_2] = \mathfrak{D}\mathfrak{B}^m.$$

Using (i) and the choice of $\mathfrak{B}$, $\mathfrak{D}$, we get

$$N_T(\gamma_1) \le |D_L|^{1/2} N_T(a_0 b D(f))^{m-1}, \quad N_T(\gamma_2) \le |D_L|^{(m+1)/2}. \tag{4.11}$$

According to Lemma 3.8 we can find T-units $\eta_1, \eta_2 \in O_T^*$ such that

$$h(\gamma_i \eta_i^m) \le d_L^{-1} \log N_T(\gamma_i) + m \cdot \left(cR_L + \frac{h_L}{d_L} \log Q_T \right) \quad \text{for } i = 1, 2$$

where $d_L = [L : \mathbb{Q}]$, $c := 39 d_L^{d_L+2}$ and $Q_T := \prod_{\substack{\mathfrak{P} \in T \\ \mathfrak{P} \text{ finite}}} N_L \mathfrak{P}$. Putting

$$\gamma := a_0^{-1} \gamma_1 \gamma_2^{-1} (\eta_1 \eta_2^{-1})^m, \quad \xi = \eta_2 \eta_1^{-1} \xi_1,$$

and invoking (4.11) we obtain $x - \alpha = \gamma \xi^m$, with $\xi \in O_T$, $\gamma \in L^*$ and

$$\begin{aligned} h(\gamma) \ \le\ & h(a_0) + d_L^{-1} \Big(\frac{m+1}{2} \log |D_L| + m \log N_T(abD(f)) \Big) + \\ & +2m \cdot \Big(cR_L + \frac{h_L}{d_L} \log Q_T \Big). \end{aligned} \tag{4.12}$$

It remains to estimate from above the right-hand side of (4.12). First, we have by (3.4) and Lemma 3.7,

$$\begin{aligned} d_L^{-1} \log N_T(a_0 b D(f)) \ &= \ d^{-1} \log N_S(a_0 b D(f)) \le h(a_0 b D(f)) \\ &\le \ (2n-1) \log n + 2n\widehat{h}. \end{aligned} \tag{4.13}$$

Together with Lemma 4.1 this implies

$$\begin{aligned} & h(a_0) + d_L^{-1} \Big(\frac{m+1}{2} \log |D_L| + m \log N_T(abD(f)) \Big) \\ & \qquad \le m(4n \log n + 4n\widehat{h} + \log |D_K|). \end{aligned} \tag{4.14}$$

Next, by Lemma 3.5, Lemma 4.1 and $d_L \le nd$ we have

$$\begin{aligned} \max(h_L, R_L) \ &\le\ 5|D_L|^{1/2} (\log^* |D_L|)^{nd-1} \le (nd)^{nd} |D_L| \\ &\le\ (n^3 d)^{nd} e^{(2n-2)d\widehat{h}} |D_K|^n. \end{aligned} \tag{4.15}$$

By inserting the bounds (4.14), (4.15), together with (3.5) and the estimate $c \le 39(nd)^{nd+2}$ into (4.12), one easily obtains the upper bound for $h(\gamma)$ given by (ii). □

Let f, b, m be as above, and let $x, y \in O_S$ be a solution of (4.5) with $y \ne 0$. Let $\gamma_1, \ldots, \gamma_n$, $\xi_1, \ldots, \xi_n$ be as in Lemma 4.2.

Lemma 4.3. *(i) Let $m \geq 3$ and $M = K(\alpha_1, \alpha_2, \sqrt[m]{\gamma_1/\gamma_2}, \rho)$, where ρ is a primitive m-th root of unity. Then*

$$|D_M| \leq 10^{m^3n^2d} n^{4m^2n^3d} |D_K|^{m^2n^2} Q_S^{m^2n^2} e^{4m^2n^3d\widehat{h}}. \tag{4.16}$$

(ii) Let $m = 2$ and $M = K(\alpha_1, \alpha_2, \alpha_3, \sqrt{\gamma_1/\gamma_2}, \sqrt{\gamma_1/\gamma_3})$. Then

$$|D_M| \leq n^{40n^4d} Q_S^{8n^3} |D_K|^{4n^3} e^{25n^4d\widehat{h}}. \tag{4.17}$$

Proof. We start with (i). Define the fields $L = K(\alpha_1, \alpha_2)$, $M_1 = L(\sqrt[m]{\gamma_1/\gamma_2})$, $M_2 = L(\rho)$. Then $M = M_1M_2$. By Lemma 3.3 (i) we have

$$|D_M| = N_L\mathfrak{d}_{M/L}|D_L|^{[M:L]}. \tag{4.18}$$

By Lemma 3.1, we have $\mathfrak{d}_{M_2/L} \supseteq [m]^m$, where $[m] = mO_L$. Together with Lemma 3.3 (ii), this implies

$$\mathfrak{d}_{M/L} \supseteq \mathfrak{d}_{M_1/L}^{[M:M_1]}\mathfrak{d}_{M_2/L}^{[M:M_2]} \supseteq m^{m^2}\mathfrak{d}_{M_1/L}^m.$$

Inserting this into (4.18), noting that $[L : \mathbb{Q}] \leq n^2d$, $[M : L] \leq m^2$, we obtain

$$|D_M| \leq m^{m^2n^2d}(N_L\mathfrak{d}_{M_1/L})^m|D_L|^{m^2}. \tag{4.19}$$

We estimate $N_L\mathfrak{d}_{M_1/L}$. Let $\mathfrak{P}$ be a prime ideal of O_L not dividing a prime ideal from S and not dividing $ma_0bD(f)$. Then by Lemma 4.2,

$$\mathrm{ord}_{\mathfrak{P}}(\gamma_1\gamma_2^{-1}) \equiv \mathrm{ord}_{\mathfrak{P}}\left(\frac{a_0(x-\alpha_1)}{a_0(x-\alpha_2)}\right) \equiv 0 \pmod{m},$$

and so by Lemma 3.4, M_1/L is unramified at $\mathfrak{P}$. Consequently, $\mathfrak{d}_{M_1/L}$ is composed of prime ideals from U, where U is the set of prime ideals of O_L that divide the prime ideals from S or $ma_0bD(f)$. Using Lemma 3.2, it follows that

$$\begin{aligned} \mathfrak{d}_{M_1/L} &\supseteq \prod_{\mathfrak{P}\in U} \mathfrak{P}^{m(1+\mathrm{ord}_{\mathfrak{P}}(u(m)))} \\ &\supseteq \prod_{\mathfrak{P}\in U} \mathfrak{P}^m \prod_{\mathfrak{P}} \mathfrak{P}^{m\,\mathrm{ord}_{\mathfrak{P}}(u(m))} \supseteq u(m)^m \prod_{\mathfrak{P}\in U} \mathfrak{P}^m. \end{aligned} \tag{4.20}$$

First, by prime number theory, $u(m) \leq m^{\pi(m)} \leq 4^m$ (see Rosser and Schoenfeld [20, Corollary 1]). Hence $|N_{L/\mathbb{Q}}(u(m)^m)| \leq 4^{m^2n^2d}$. Second, by an argument similar to the proof of (3.5), defining V to be the set of prime ideals of O_L which are contained in S or divide $ma_0bD(f)$,

$$\begin{aligned} N_L(\prod_{\mathfrak{P}\in U}\mathfrak{P}) &\leq N_K(\prod_{\mathfrak{p}\in V}\mathfrak{p})^{[L:K]} \leq N_K(\prod_{\mathfrak{p}\in V}\mathfrak{p})^{n^2} \\ &\leq (Q_S N_S(ma_0bD(f))^{n^2} \leq (Q_S e^{d\cdot h(ma_0bD(f))})^{n^2} \\ &\leq Q_S^{n^2} m^{n^2d} e^{2n^3d(\log n+\widehat{h})} \leq Q_S^{n^2} m^{n^2d} n^{2n^3d} e^{2n^3d\widehat{h}} \end{aligned}$$

where in the last estimate we have used Lemma 3.7. By combining this estimate and that for $|N_{L/\mathbb{Q}}(u(m)^m)|$ with (4.20), we obtain

$$N_L\mathfrak{d}_{M_1/L} \leq 6^{m^2n^2d} n^{2mn^3d} Q_S^{mn^2} e^{2mn^3d\widehat{h}}. \tag{4.21}$$

Finally, by inserting this estimate and the one arising from Lemma 4.1,

$$|D_L| \leq n^{4n^2d} \cdot e^{4n^2d\widehat{h}} \cdot |D_K|^{n^2} \tag{4.22}$$

into (4.19), after some computations, we obtain (4.16).

We now prove (ii). Let $m = 2$. Take $L = K(\alpha_1, \alpha_2, \alpha_3)$, $M_1 = L(\sqrt{\gamma_1/\gamma_2})$, $M_2 = L(\sqrt{\gamma_1/\gamma_3})$, so that $M = M_1M_2$. Completely similarly to (4.21), but now using $[L : K] \leq n^3$ instead of $\leq n^2$, we get

$$N_L\mathfrak{d}_{M_1/L} \leq 6^{4n^3d} n^{4n^4d} Q_S^{2n^3} e^{4n^4d\widehat{h}}.$$

For $N_L\mathfrak{d}_{M_2/L}$ we have the same estimate. So by Lemma 3.3 (ii),

$$N_L\mathfrak{d}_{M/L} \leq (N_L\mathfrak{d}_{M_1/L})^2 (N_L\mathfrak{d}_{M_2/L})^2 \leq 6^{16n^3d} n^{16n^4d} Q_S^{8n^3} e^{16n^4d\widehat{h}}.$$

By inserting this inequality and the one arising from Lemma 4.1,

$$|D_L| \leq n^{6n^3d} \cdot e^{6n^3d\widehat{h}} \cdot |D_K|^{n^3}$$

into $|D_M| = N_L\mathfrak{d}_{M/L}|D_L|^{[M:K]}$, after some computations we obtain (4.17). □

Proof of Theorem 2.1. Let $m \geq 3$ and let $x, y \in O_S$ be a solution to $by^m = f(x)$ with $y \neq 0$. We have $x - \alpha_i = \gamma_i\xi_i^m$ $(i = 1, \ldots, n)$ with the γ_i, ξ_i as in Lemma 4.2. Let $M := K(\alpha_1, \alpha_2, \sqrt[m]{\gamma_1/\gamma_2}, \rho)$, where ρ is a primitive m-th root of unity, and let T be the set of places of M lying above the places from S. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the prime ideals (finite places) in S, and $\mathfrak{P}_1, \ldots, \mathfrak{P}_{t'}$ the prime ideals in T. Then $t' \leq [M : K]t \leq m^2n^2t$. Further, let $P_T := \max_{i=1}^{t'} N_M\mathfrak{P}_i$, $Q_T := \prod_{i=1}^{t'} N_M\mathfrak{P}_i$.

We clearly have

$$\gamma_1\xi_1^m - \gamma_2\xi_2^m = \alpha_2 - \alpha_1, \quad \xi_1, \xi_2 \in O_T, \tag{4.23}$$

and the left-hand side is a binary form of non-zero discriminant which splits into linear factors over M. By Proposition 3.11, we have

$$h(\xi_1) \le c_1' m^6 P_T R_T \Big(1 + \frac{\log^* R_T}{\log^* P_T}\Big) \times \times \Big(R_M + h_M \cdot d_M^{-1} \log Q_T + m d_M A + B), \tag{4.24}$$

where $A = \max(h(\gamma_1), h(\gamma_2))$, $B = h(\alpha_1 - \alpha_2)$, $d_M = [M : \mathbb{Q}]$ and c_1' is the constant c_1 from Proposition 3.11, but with s, d replaced by the upper bounds m^2n^2s, m^2n^2d for the cardinality of T and $[M : \mathbb{Q}]$, respectively, and R_T is the T-regulator.

Using $d \le 2s$ we can estimate c_1' by the larger but less complicated bound,

$$c_1' \le 2^{50}(4m^2n^2s)^{7m^2n^2s}. \tag{4.25}$$

Next, by (3.5),

$$P_T \le Q_T \le Q_S^{[M:K]} \le Q_S^{m^2n^2}. \tag{4.26}$$

Let C be the upper bound for $|D_M|$ from (4.16). Thus, by Lemma 3.5 and (3.9),

$$\max(h_M, R_M) \le 5C(\log^* C)^{m^2n^2d-1}.$$

Further, A can be estimated from above by the bound from (4.8), and B by

$$h(\alpha_1) + h(\alpha_2) + \log 2 \le h(f) + (n+1)\log 2 \le \widehat{h} + (n+1)\log 2$$

in view of Lemma 3.6. Together with (4.26), this implies

$$\begin{aligned} R_M + h_M \cdot d_M^{-1} \log Q_T + m d_M A + B \\ \le 7C(\log^* C)^{m^2n^2d-1} \cdot d^{-1} \log Q_S \le 7C(\log^* C)^{m^2n^2d}. \end{aligned} \tag{4.27}$$

Next, by (3.9), the inequality $d + t \le 2s$, and (4.26), we have

$$\begin{aligned} R_T &\le C^{1/2}(\log^* C)^{m^2n^2d-1}(\log^* P_T)^{t'} \\ &\le C^{1/2}(\log^* C)^{m^2n^2d-1}(m^2n^2 \log^* Q_S)^{m^2n^2t} \\ &\le (m^2n^2)^{m^2n^2s} C^{1/2}(\log^* C)^{2m^2n^2s-1} \end{aligned}$$

and

$$1 + \frac{\log^* R_T}{\log^* P_T} \le 4m^2n^2s \log^* C,$$

hence

$$P_T R_T \Big(1 + \frac{\log^* R_T}{\log^* P_T}\Big) \le (4m^2n^2)^{m^2n^2s} Q_S^{m^2n^2} C^{1/2} (\log^* C)^{2m^2n^2s}. \tag{4.28}$$

Combining (4.27), (4.28) with (4.24) gives

$$\begin{aligned} h(\xi_1) &\leq 7m^6 c_1'(4m^2n^2)^{m^2n^2s} Q_S^{m^2n^2} C(\log^* C)^{4m^2n^2s} \\ &\leq 2^{50}(4m^2n^2s)^{13m^2n^2s} Q_S^{m^2n^2} C^2. \end{aligned}$$

Using

$$h(x) \leq \log 2 + h(\alpha_1) + h(\gamma_1) + mh(\xi_1), \quad h(y) \leq m^{-1}(h(b) + h(f) + nh(x)),$$

and the upper bound for $h(\gamma_1)$ from (4.8), we get

$$h(x), h(y) \leq 2^{51} mn(4m^2n^2s)^{13m^2n^2s} Q_S^{m^2n^2} C^2. \tag{4.29}$$

Now substituting C, i.e., the upper bound for $|D_M|$ from (4.16), and some algebra gives the upper bound (2.4) from Theorem 2.1. □

Proof of Theorem 2.2. Let $x, y \in O_S$ be a solution to $by^2 = f(x)$ with $y \neq 0$. We have $x - \alpha_i = \gamma_i \xi_i^m$ $(i = 1, \dots, n)$ with the γ_i, ξ_i as in Lemma 4.2. Let

$$M := K(\alpha_1, \alpha_2, \alpha_3, \sqrt{\gamma_1/\gamma_3}, \sqrt{\gamma_2/\gamma_3}),$$

and let T be the set of places of M lying above the places from S. Notice that $[M : K] \leq 4n^3$. Then

$$\gamma_1\xi_1^2 - \gamma_2\xi_2^2 = \alpha_2 - \alpha_1, \quad \gamma_1\xi_1^2 - \gamma_3\xi_3^2 = \alpha_3 - \alpha_1, \quad \xi_1, \xi_2 \in O_T. \tag{4.30}$$

By applying Proposition 3.12 to (4.30), and doing the same computations as above, we obtain the same bound as in (4.29), but with $m = 2$ and m^2n^2 replaced by $4n^3$, and with C the upper bound for $|D_M|$ from (4.17). After some computation, we obtain the bound (2.6) from Theorem 2.2. □

5. Proof of Theorem 2.3

We assume that in some finite extension G of K, the polynomial f factorizes as $a_0(X - \alpha_1)\cdots(X - \alpha_n)$. For $i = 1, \dots, n$, let $L_i = \mathbb{Q}(\alpha_i)$, let $d_{L_i}, h_{L_i}, R_{L_i}$ denote the degree, class number and regulator of L_i, and let T_i be the set of places of L_i lying above the places in S. Further, denote by R_{T_i} the T_i-regulator of L_i, and denote by t_i the cardinality of T_i. Let $Q_{T_i} := \prod_{\mathfrak{P} \in T_i} N_{L_i}\mathfrak{P}$, where the product is over all prime ideals in T_i. The group of T_i-units $O_{T_i^*}$ is finitely generated and by Lemma 2 of [14] (see also

[8], [9] and [7]) we may choose a fundamental system of T_i-units, i.e., basis of $O_{T_i}^*$ modulo torsion $\eta_{i1},\dots,\eta_{i,t_i-1}$ such that

$$\text{(5.1)}\qquad \begin{cases} \prod_{j=1}^{t_i-1} h(\eta_{ij}) \le c_{1i}R_{T_i}, \\ \max_{1\le j\le t_i-1} h(\eta_{ij}) \le c_{2i}R_{T_i}, \end{cases}$$

where

$$c_{1i} = \frac{((t_i-1)!)^2}{2^{t_i-2}d_L^{t_i-1}}, \quad c_{2i} = 29e\sqrt{t_i-2}d_{L_i}^{t_i-1}\log^* d_{L_i}c_{i1}.$$

We estimate these upper bounds from above. First noting $t_i \le [L_i:K]s \le ns$ we have the generous estimate

$$\text{(5.2)}\qquad c_{i1}, c_{i2} \le 1200t_i^{2t_i} \le 1200(ns)^{2ns}.$$

For the class number and regulator h_{L_i}, R_{L_i}, we have similarly to (4.15):

$$\text{(5.3)}\qquad \begin{aligned} \max(h_{L_i}, R_{L_i}, h_{L_i}R_{L_i}) &\le 5|D_{L_i}|^{1/2}(\log^*|D_{L_i}|)^{nd-1} \\ &\le (n^3d)^{nd}e^{(2n-2)d\widehat{h}}|D_K|^n. \end{aligned}$$

Further, from (3.9), $d \le 2s$, we deduce

$$\text{(5.4)}\qquad \begin{aligned} R_{T_i} &\le (n^3d)^{nd}e^{(2n-2)d\widehat{h}}|D_K|^n(\log^* P_{T_i})^{ns-1} \\ &\le (n^3d)^{nd}e^{(2n-2)d\widehat{h}}|D_K|^n(n\log^* P_S)^{ns-1} \\ &\le (4n^7s^2)^{ns}e^{(2n-2)d\widehat{h}}|D_K|^n(\log^* P_S)^{ns-1}. \end{aligned}$$

By inserting this and (5.2) into (5.1), we obtain

$$\text{(5.5)}\qquad \prod_{j=1}^{t_i-1} h(\eta_{ij}) \le C_1 := 1200(4n^9s^4)^{ns}e^{2nd\widehat{h}}|D_K|^n(\log^* P_S)^{ns-1},$$

$$\text{(5.6)}\qquad \max_{1\le j\le t_i-1} h(\eta_{ij}) \le C_1.$$

Now let x, y and m satisfy

$$\text{(5.7)}\qquad by^m = f(x),\quad m\in\mathbb{Z}_{\ge 3},\ x,y\in O_S,\ y\ne 0,\ y \text{ not a root of unity},$$

Lemma 5.1. *For $i = 1,2$ there are $\gamma_i, \xi_i \in L_i^*$, and integers $b_{i1}\cdots b_{i,t_i}$ of absolute value at most $m/2$, such that*

$$\text{(5.8)}\qquad \begin{cases} (x-\alpha_i)^{h_{L_1}h_{L_2}} = \eta_{i1}^{b_{i1}}\cdots\eta_{i,t_i-1}^{b_{i,t_i-1}}\gamma_i\xi_i^m, \\ h(\gamma_i) \le C_2 := (2n^3s)^{6ns}|D_K|^{2n}e^{4nd\widehat{h}}(\widehat{h}+\log^* P_S). \end{cases}$$

Proof. For convenience, we put $r := h_{L_1}h_{L_2}$. By symmetry, it suffices to prove the lemma for $i = 1$. For notational convenience, in the proof of this lemma only, we suppress the index $i = 1$ (so $L = L_1, T = T_1, t = t_1$, etc.). We use the same notation as in the proof of Lemma 4.2. Similar to (4.9), (4.10), we have

$$\frac{[x-\alpha]}{[1,\alpha]} + \frac{[g(x)]}{[g]} \supseteq \frac{[D(f)]}{[f]^{2n-2}}, \quad [b][f]^{-1}[y]^m = \frac{[x-\alpha]}{[1,\alpha]} \cdot \frac{[g(x)]}{[g]},$$

where $[\cdot]$ denote fractional ideals with respect to O_T. From these relations, it follows that there are integral ideals $\mathfrak{B}_1, \mathfrak{B}_2$ of O_T and a fractional ideal $\mathfrak{A}$ of O_T, such that

$$\frac{[x-\alpha]}{[1,\alpha]} = \mathfrak{B}_1\mathfrak{B}_2^{-1}\mathfrak{A}^m,$$

where

$$\mathfrak{B}_1 \supseteq [b] \cdot \frac{[D(f)]}{[f]^{2n-2}}, \quad \mathfrak{B}_2 \supseteq [f] \cdot \frac{[D(f)]}{[f]^{2n-2}}.$$

Since

$$[a_0][1,\alpha] \subseteq [a_0]\prod_{j=1}^{n}[1,\alpha_j] \subseteq [f] \subseteq [1],$$

it follows that $[1,\alpha]^{-1} \supseteq [a_0]$. Hence

$$[x-\alpha] = \mathfrak{C}_1\mathfrak{C}_2^{-1}\mathfrak{A}^m,$$

where $\mathfrak{C}_1, \mathfrak{C}_2$ are ideals of O_T such that

$$\mathfrak{C}_1, \mathfrak{C}_2 \supseteq [a_0bD(f)].$$

Raising to the power r, we get

$$(x-\alpha)^r = \gamma_1\gamma_2^{-1}\lambda^m, \tag{5.9}$$

for some non-zero $\gamma_1, \gamma_2 \in O_T$ and $\lambda \in L^*$ with

$$[\gamma_k] \supseteq [a_0bD(f)]^r \quad \text{for } k = 1, 2.$$

By Lemma 3.8, there exist $\varepsilon_1, \varepsilon_2 \in O_T^*$ such that for $k = 1, 2$,

$$h(\varepsilon_k\gamma_k) \leq \frac{r}{d_L}\log N_T(a_0bD(f)) + cR_L + \frac{h_L}{d_L}\log Q_T,$$

where $c \leq 39d_L^{d_L+2} \leq 39(2ns)^{2ns+2}$. There are $\varepsilon \in O_T^*$, a root of unity ζ of L, and integers $b_1, \ldots, b_{t-1}$ of absolute value at most $m/2$, such that

$$\varepsilon_2\varepsilon_1^{-1} = \zeta\varepsilon^m\eta_1^{b_1}\cdots\eta_{t-1}^{b_{t-1}}.$$

Writing

$$\gamma := \zeta^{-1}\frac{\varepsilon_1\gamma_1}{\varepsilon_2\gamma_2}, \quad \xi := \varepsilon\lambda$$

where $\eta_1, \dots, \eta_{t-1}$ are the fundamental units of O_T^* satisfying (5.5), (5.6), we get

$$x - \alpha = \eta_1^{b_1} \cdots \eta_{t-1}^{b_{t-1}}\gamma\xi^m,$$

where

$$h(\gamma) \le \frac{2r}{d_L}\log N_T(a_0 bD(f)) + 2cR_L + 2\frac{h_L}{d_L}\log Q_T. \tag{5.10}$$

By (5.3), $d \le 2s$, (4.13), (3.5) we have

$$h_L, R_L \le (2n^3 s)^{2ns} e^{2nd\widehat{h}}|D_K|^n, \quad r = h_{L_1}h_{L_2} \le (2n^3 s)^{4ns} e^{4nd\widehat{h}}|D_K|^{2n},$$
$$d_L^{-1}\log N_T(a_0 bD(f)) \le (2n-1)\log n + 2n\widehat{h},$$
$$d_L^{-1}\log Q_T \le d^{-1}\log Q_S \le s\log^* P_S.$$

By inserting these bounds into (5.10) and using $n \ge 2$, after some algebra we obtain the upper bound C_2. □

Completion of the proof of Theorem 2.3. In what follows, let $L := K(\alpha_1, \alpha_2)$, $d_L := [L : \mathbb{Q}]$, T the set of places of L lying above the places from S, and t the cardinality of T. Let again $x, y \in O_S$ and m an integer ≥ 3 with $by^m = f(x)$, $y \ne 0$ and y not a root of unity. Put

$$X := \max_{i=1,\dots,n} h(x - \alpha_i).$$

Without loss of generality we assume

$$m \ge (10n^2 s)^{38ns}|D_K|^{6n} P_S^{n^2} e^{11nd\widehat{h}}. \tag{5.11}$$

Then

$$X \ge \max(C_3, m(4d)^{-1}(\log 3d)^{-3}), \tag{5.12}$$
$$\text{with } C_3 := (10n^2 s)^{37ns}|D_K|^{6n} P_S^{n^2} e^{11nd\widehat{h}}.$$

Indeed, by Lemma 3.9 we have

$$m \le \frac{n \cdot X + h(a_0) + h(b)}{h(y)} \le (2d(\log(3d))^3(nX + 2\widehat{h}).$$

If $X < C_3$ this contradicts (5.11). If $X \ge C_3$ the other lower bound for X in the maximum easily follows.

We assume without loss of generality, that

$$X = h(x - \alpha_2).$$

If $|x - \alpha_2|_v \leq 1$ for $v \in T$, then using $x \in O_S$ we have

$$\begin{aligned} X &\leq \frac{1}{d_L} \log \left(\prod_{v \notin T} \max(1, |x - \alpha_2|_v) \right) \\ &\leq \frac{1}{d_L} \log \left(\prod_{v \notin T} \max(1, |\alpha_2|_v) \right) \leq h(\alpha_2) \leq \frac{\log^*(n+1)}{2} + h(f), \end{aligned}$$

which is impossible by (5.12). Hence $\max_{v \in T} |x - \alpha_2|_v > 1$. Choose $v_0 \in T$ such that

$$|x - \alpha_2|_{v_0} = \max_{v \in T} |x - \alpha_2|_v. \tag{5.13}$$

Then we have

$$\begin{aligned} X &\leq \frac{1}{d_L} \left(\log \left(|x - \alpha_2|_{v_0}^t \prod_{v \notin T} \max(1, |x - \alpha_2|_v) \right) \right) \\ &\leq \frac{1}{d_L} \left(\log \left(|x - \alpha_2|_{v_0}^t \prod_{v \notin T} \max(1, |\alpha_2|_v) \right) \right). \end{aligned}$$

which gives

$$|x - \alpha_2|_{v_0} \geq \frac{e^{X d_L / t}}{\prod_{v \notin T} \max(1, |\alpha_2|_v)^{1/t}}.$$

Thus we have

$$\left| 1 - \frac{x - \alpha_1}{x - \alpha_2} \right|_{v_0} = \frac{|\alpha_2 - \alpha_1|_{v_0}}{|x - \alpha_2|_{v_0}} \leq \frac{|\alpha_2 - \alpha_1|_{v_0} \prod_{v \notin T} \max(1, |\alpha_2|_v)^{1/t}}{e^{X d_L / t}}. \tag{5.14}$$

Put $s(v_0) = 1$ if v_0 is real, $s(v_0) = 2$ if v is complex, and $s(v_0) = 0$ if v_0 is finite. Since by Lemma 3.6 we have

$$\begin{aligned} &|\alpha_2 - \alpha_1|_{v_0} \prod_{v \notin T} \max(1, |\alpha_2|_v)^{1/t} \\ &\qquad \leq 2^{s(v_0)} \max(1, |\alpha_2|_{v_0}) \max(1, |\alpha_1|_{v_0}) \prod_{v \notin T} \max(1, |\alpha_2|_v) \\ &\qquad \leq 2^{s(v_0)} \exp(d_L(h(\alpha_1) + h(\alpha_2))) \\ &\qquad \leq 2^{(n+1)s(v_0)} \exp((d_L h(f)), \end{aligned}$$

(5.14) gives us

$$\left|1-\frac{x-\alpha_1}{x-\alpha_2}\right|_{v_0} \leq \exp\Big((n+1)s(v_0)\log 2 + d_L h(f) - X d_L/t\Big). \tag{5.15}$$

Notice that by (5.12) we have

$$\left|1-\frac{x-\alpha_1}{x-\alpha_2}\right|_{v_0} < 1. \tag{5.16}$$

In general, we have for $y \in L$ with $|1-y|_{v_0} < 1$ and any positive integer r,

$$|1-y^r|_{v_0} \leq 2^{r\cdot s(v_0)}|1-y|_{v_0}.$$

Hence

$$\left|1-\left(\frac{x-\alpha_1}{x-\alpha_2}\right)^{h_{L_1}h_{L_2}}\right|_{v_0} \leq \exp\Big((h_{L_1}h_{L_2}+n+1)s(v_0)\log 2 + d_L h(f) - X d_L/t\Big).$$

Using (5.12) and the estimates (5.3), $h(f) \leq \widehat{h}$, $d_L \leq nd$, $s \leq t \leq ns$, this can be simplified to

$$\left|1-\left(\frac{x-\alpha_1}{x-\alpha_2}\right)^{h_{L_1}h_{L_2}}\right|_{v_0} \leq \exp(-X d_L/2t). \tag{5.17}$$

On the other hand using Proposition 3.10 and Lemma 5.1 we get a Baker type lower bound

$$\begin{aligned} &\left|1-\left(\frac{x-\alpha_1}{x-\alpha_2}\right)^{h_{L_1}h_{L_2}}\right|_{v_0} \\ &= \left|1-\frac{\gamma_1}{\gamma_2}\cdot\eta_{11}^{b_{11}}\cdots\eta_{1,t_1-1}^{b_{1,t_1-1}}\cdot\eta_{21}^{-b_{21}}\cdots\eta_{2,t_2-1}^{-b_{2,t_2-1}}\cdot\left(\frac{\xi_1}{\xi_2}\right)^m\right|_{v_0} \\ &\geq \exp\Big(-c_1(t_1+t_2,d_L)\cdot\frac{N(v_0)}{\log N(v_0)}\Theta\log B\Big) \end{aligned} \tag{5.18}$$

where

$$\Theta := \max(h(\xi_1/\xi_2), m(d)) \cdot \max(h(\gamma_1/\gamma_2), m(d)) \cdot \prod_{j=1}^{t_1-1} h(\eta_{1j}) \cdot \prod_{j=1}^{t_2-1} h(\eta_{2j}),$$

$$B := \max\{3, m, |b_{11}|, \ldots, |b_{1,t_1-1}|,\ |b_{21}|, \ldots, |b_{2,t_2-1}|),$$

$$N(v_0) := \begin{cases} 2 & \text{if } v_0 \text{ is infinite} \\ N_L\mathfrak{P} & \text{if } v_0 = \mathfrak{P} \text{ is a prime ideal } \mathfrak{P}, \end{cases}$$

$$c_1(t_1+t_2, d_L) := 12(16ed_L)^{3t_1+3t_2+2}(\log^* d_L)^2.$$

We estimate the above parameters. First, by (5.8), we have $h(\gamma_i) \le C_2$ for $i = 1, 2$. Moreover, the exponents b_{ij} in (5.8) have absolute values at most $m/2$. Together with (5.6) and (5.12), these imply

$$\begin{aligned} (5.19)\, h(\xi_1/\xi_2) &\le \max h(\xi_1) + h(\xi_2) \\ &\le \frac{2}{m}(X + C_2) + \tfrac{1}{2}(t_1 + t_2 - 2)C_1 \le \frac{3}{m} \cdot X + 2nsC_1 \\ &\le (3 + 4d(\log 3d)^3 \cdot 2nsC_1) \cdot \frac{X}{m} \le 4^{ns+2}C_1 \cdot \frac{X}{m}, \end{aligned}$$

where we have used $t_1, t_2 \le ns$, $d \le 2s$, $n \ge 2$. Further, using (5.5) and $h(\gamma_1/\gamma_2) \le 2C_2$, we get

$$\Theta \le C_1^2 \cdot 4^{ns+2}C_1 \cdot \frac{X}{m} \cdot 2C_2 \le C_4 \cdot \frac{X}{m}, \tag{5.20}$$

where

$$C_4 := 2 \times 10^7 \big(4^{10} n^{45} s^{18}\big)^{ns} |D_K|^{5n} e^{10nd\widehat{h}} (\widehat{h} + 1)(\log^* P_S)^{3ns-2}.$$

Next, using $d_L \le n(n-1)d \le 2n(n-1)s$, $t_1, t_2 \le ns$, we have

$$c_1(t_1 + t_2, d_L) \le C_5 := (32en^2 s)^{6ns+3}. \tag{5.21}$$

Finally, by (3.5), (5.11) we have

$$N(v_0) \le P_T \le P_S^{[L:K]} \le P_S^{n(n-1)}$$

and $B = m$ since the exponents b_{ij} in (5.8) have absolute values at most $m/2$. Inserting these and (5.20), (5.21) into (5.18), we arrive at the lower bound

$$\left|1 - \left(\frac{x - \alpha_1}{x - \alpha_2}\right)^{h_{L_1} h_{L_2}}\right|_{v_0} \ge \exp\Big(- C_4 C_5 P_S^{n(n-1)} \frac{X}{m} \log m \Big).$$

A comparison with the upper bound (5.17) gives

$$\exp\Big(- C_4 C_5 P_S^{n(n-1)} \frac{X}{m} \log m\Big) \le \exp(-d_L X/2t).$$

By dividing out X and inserting $t \le n^2 s$, $d \le 2s$, we arrive at

$$\begin{aligned}\frac{m}{\log m} &\le 2n^2 s C_4 C_5 P_S^{n(n-1)}\\ &< (10n^2 s)^{35ns} |D_K|^{5n} e^{10nd\widehat{h}} (\widehat{h}+1) \cdot P_S^{n(n-1)} (\log^* P_S)^{3ns-1}.\end{aligned}$$

Applying the inequalities $(\log X)^B \le (B/2\epsilon)^B X^\epsilon$ for $X > 1$, $B > 0$, $\epsilon > 0$ and $X + 1 \le (e^{c-1}/c)e^{cX}$ for $X > 0$, $c \ge 1$, we arrive at our final estimate

$$m < (10n^2 s)^{40ns} |D_K|^{6n} P_S^{n^2} e^{11nd\widehat{h}}.$$

This completes our proof of Theorem 2.3. □

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