Néron models: Introduction

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September 22, 2017

1 Basic definitions

In this seminar we shall study Néron models, a special type of model. Much of this section is from Chapter 1 of [1].

Definition 1.1. Let S be a Dedekind scheme of dimension 1 with fraction field K (that is, a normal Noetherian scheme of dimension 1). if X/K is a scheme, then a scheme $\mathcal{X}/S$ along with a fixed isomorphism $f : \mathcal{X}_K \to X$ is a *model* of X over S.

Suppose X_K is a smooth and separarted K-scheme of finite type. A *Néron model* is an S-model X which is smooth, separated, and of finite type over S and which satisfies the Néron mapping property (or NMP for short):

For each smooth S-scheme Y and morphism $u_K : Y_K \to X_K$, there exists a unique S-morphism $u : Y \to X$ extending u_K.

We can generalise this definition to arbitrary irreducible schemes S, not just Dedekind schemes. Later in the seminar we'll see examples of this.

The NMP implies an important property called the extension property for étale points.

Definition 1.2. Let X/S be a scheme, where S is a Dedekind scheme. Then X is said to satisfy the *extension property for étale points at a point s* if for each étale local $\mathcal{O}_{S,s}$ algebra R' with field of fractions K', the map $X(R') \to X_K(K')$ is surjective.

The map $X(R') \to X_K(K')$ is injective whenever X is separated over S, and it is injective whenever X/S is proper by the valuative-criterion for properness.

Remark 1.3. Étale local $\mathcal{O}_{S,s}$ algebras R' are faithfully flat extensions of discrete valuation rings $\mathcal{O}_{S,s} \subset R'$ such that the extension of fraction fields is finite and separable and the extension of discrete valuation rings is unramified.

Rather than work with all local étale $\mathcal{O}_{S,s}$ algebras it suffices by a limit argument to show the property for $\mathcal{O}^{sh}_{S,s}$, the strict henselisation of $\mathcal{O}_{S,s}$. This is defined to be the direct limit $\lim_{\rightarrow}(R, r)$ where R is an étale local $\mathcal{O}_{S,s}$ algebra and r is a homomorphism of the residue field of R into a fixed separable closure of the residue field of $\mathcal{O}_{S,s}$.

Proposition 1.4. *Let X/S be a Néron model of its generic fibre X_K, where S is a Dedekind scheme as before.*

- *X is uniquely determined by X_K up to canonical isomorphism.*
- *X satisfies the extension property for étale points.*
- *If S'/S is étale, $X \times_S S'$ is a Néron model for $X_{K'} = X_K \times_K K'$.*

Proof. The first property follows from the NMP. Indeed, if Y is another Néron model then we have an isomorphism $u_K : Y_K \to X_K$ extending to $u : Y \to X$ and u_K^{-1} extending to $u' : X \to Y$. The compositions of u and u' are the identities on their respective generic fibres. Morphisms from a reduced scheme to a separated scheme that generically agree are necessarily equivalent, whence the result.

We show the second property in the case S is a discrete valuation ring, with the general result following by a limit argument (see Lemma 1.2.5 of [?]). Let s be the closed point, R' an étale local $\mathcal{O}_{S,s}$ scheme. Then given a morphism Spec$K' \to X_K$, i.e. an element of $X_K(K')$, this extends uniquely by the NMP to a morphism Spec$R' \to X$. The result follows. □

Néron models are mostly used to study group schemes. We'll look at abelian varieties and group schemes in more depth later in the seminar. For now we note the following definitions:

Definition 1.5. A *group scheme* X over a scheme S is a representable functor X : (Sch/S) → (Groups), i.e. a scheme X such that for all S-schemes T, $X(T)$ has the structure of a group.

A group scheme X over a scheme S is said to be an *abelian scheme* if it is proper and smooth over S with connected fibres.

Abelian schemes are particularly "well-behaved" for constructing Néron models as the following lemma shows:

Lemma 1.6 (Proposition 1.2.8 of [?]). *Let S be a Dedekind scheme, X/S an abelian scheme. Then X is the Néron model of its generic fibre.*

Conversely, suppose we begin with an abelian variety over K. Then we have the following:

Theorem 1.7 (Theorem 1.4.3 of [?]). *Let S be a Dedekind scheme with field of fractions K. Let A_K be an abelian variety over K. Then A_K admits a Néron model over S.*

For the proof see [?].

Remark 1.8. The group structure on A_K extends via the NMP to a group structure on the Néron model of A_K over S. The Néron model is not in general an abelian scheme as it may not be proper. This leads us to the notion of an abelian variety having good reduction.

Definition 1.9. Let S be a Dedekind scheme with field of fractions K, and let A_K be an abelian variety. Then A_K is said to *have good reduction at a closed point* $s \in S$ if A_K extends to a smooth, proper scheme $\overline{A}$ over $\mathcal{O}_{S,s}$.

2 Example: Elliptic curves

The construction of Néron models for elliptic curves is very "hands-on". We also care about this particular case, as the Jacobian is particularly nice and the Picard scheme will naturally generalise the Jacobian of a curve. The purpose of this example is to illustrate that the construction of the Néron model is potentially very concrete.

Assume that R is a strictly Henselian discrete valuation ring with the characteristic of the residue field different from 2 or 3.

Let E_K/K be an elliptic curve in $\mathbb{P}^2_K$ defined by its Weierstrass form

$$y^2z = x^3 + \beta xz^2 + \gamma z^3.$$

The discriminant of E_K is then $\Delta = 4\beta^3 + 27\gamma^2$ and the j-invariant is $j = 2^6 3^3 4\beta^3/\Delta$.

E_K is a group scheme, where without loss of generality the point $(0 : 1 : 0)$ is the unit section. Let π denote a uniformizer of R with valuation $v(\pi) = 1$.

Lemma 2.1 (Lemma 1.5.2 of [?]). *Given $n \in \mathbb{Z}$, the change of coordinates*

$$(x : y : z) \to (\pi^{-2n}x : \pi^{-3n}y : z)$$

induces on the equation for E_K the following changes

$$\Delta \to \pi^{12n}\Delta, \beta \to \pi^{4n}\beta, \gamma \to \pi^{6n}\gamma.$$

Using this lemma we may assume that β and γ are in R and that $\min\{v(\beta^3), v(\gamma^2)\}$ is minimal for all choices of n. This is then the minimal Weierstrass equation, and it defines the minimal Weierstrass model $\overline{E}$ of E_K over Spec$R$ in $\mathbb{P}^2_R$.

In good situations (namely, $v(\Delta) = 0$) $\overline{E}$ is smooth and it is then an abelian scheme extending E_K. Hence it is a Néron model of E_K. In general we must work harder to get the Néron model from the minimal Weierstrass model using blow-ups. This is done explicitly in blr.

2.1 Where this is going

Let C be a (smooth, proper) curve over K, where K is as before. Then the Jacobian $J(C)$ of C is the set of line-bundles of degree 0. It can be viewed as an abelian variety over K, and hence we may construct its Néron model. The case of elliptic curves is particularly nice because the elliptic curve is isomorphic to its Jacobian.

It turns out the Jacobian is the connected component of the identity of a larger group scheme called the Picard variety, which arises from the Picard functor Pic. This connected component Pic^0 can be defined for more general cases than curves, and thus serves as a generalisation of the Jacobian. One can ask whether a Néron model exists for Pic^0 of a variety X/K. This leads us to study the Picard functor in subsequent lectures. For the remainder of this lecture we'll look at a paper of Serre and Tate [2] for a motivation of the usefulness of Néron models.

3 The paper of Serre and Tate (1968)

The paper of Serre and Tate [2] says when an abelian variety defined over a field $K = \text{Frac}(R)$, R a DVR, has good reduction. The notation for this is as follows:

R is a DVR, $K = \text{Frac}(R)$ its fraction field, v the valuation defining R in K, and k the residue field of R. We assume that k is perfect. K_s will denote a fixed separable closure of K, with $\bar{v}$ an extension of v to K_s. We shall let $I(\bar{v})$ or simply I denote the inertia subgroup of $\text{Gal}(\bar{K}/K)$, and $D(\bar{v})$ the decomposition group.

Here are some facts we shall need from algebraic number theory:

- $D(\bar{v})/I(\bar{v}) \cong \text{Gal}(\bar{k}/k)$, where $\bar{k}$ is the algebraic closure of k. This is the separable closure of k as k is perfect.
- Given an extension $K \subset L \subset K_s$, L/K is unramified at v if and only if l is fixed by $I(\bar{v})$.
- Let T be a set on which $\text{Gal}(K_s/K)$ acts. We say that T is *unramified at v* if $I(\bar{v})$ acts trivially on T.

Definition 3.1. Let m be an integer prime to $\text{char}(K)$. Then for an abelian variety A/K, we define

$$A_m = \text{Hom}(\mathbb{Z}/m\mathbb{Z}, A(K_s)).$$

Equivalently, A_m is the set of K_s-rational points of A with order dividing m. It is a free $\mathbb{Z}/m\mathbb{Z}$-module of rank $2\dim(A)$.

Definition 3.2. Let l be a prime different from $\mathrm{char}(K)$. Then for an abelian variety A/K, we define

$$T_l(A) = \lim_{\leftarrow n} A_{l^n} = \mathrm{Hom}(\mathbb{Q}_l/mathbbZ_l, A(K_S)).$$

Theorem 3.3 (Theorem 1 of [2])**.** *Let A/K be an abelian variety. Let l be a prime number different from char(k). The following are equivalent:*

1. *A has good reduction at v;*
2. *A_m is unramified at v for all m prime to char(k);*
3. *A_m is unramified at v for infinitely many m prime to char(k).*
4. *$T_l(A)$ is unramified at v.*

Proof. By a limit argument we note that (4) holds if and only if A_{l^n} is unramified at v for all values of n. Hence we have that (2) $\Rightarrow$ (4) $\Rightarrow$ (3), and it remains only to show that (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1). We'll start by showing (1) $\Rightarrow$ (2).

Now, A has good reduction if and only if there exists a smooth, proper model $\bar{A}$ extending A over S. One can show using the NMP that this implies the Néron model N of A over S is then proper. Conversely, if N is proper over S then clearly A has good reduction. Thus we have that N is an abelian scheme over S, and so in particular its special fibre $\tilde{N}$ is an abelian variety over k.

Our goal is to show that if m is prime to $\mathrm{char}(k)$, then $A_m = A_m^I$, where A_m^I is the set of elements of A_m fixed by $I = I(\bar{v})$. We shall accomplish this by showing A_M^I is isomorphic to a group $\tilde{N}_m$, which is a free $\mathbb{Z}/m\mathbb{Z}$-module of rank $2\mathrm{dim}(A)$. As A_m has the same structure and these are finite groups, we shall be done.

We define $\tilde{N}_m = \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \tilde{N}(\bar{k}))$ (recall that k is perfect, so $\bar{k} = k^{sep}$. As $\tilde{N}$ is an abelian variety, $\tilde{N}_m$ is a free $\mathbb{Z}/m\mathbb{Z}$-module of rank $2\mathrm{dim}(\tilde{N}) = 2\mathrm{dim}(A)$. The following lemma will imply the result:

Lemma 3.4. *We have that $A_m^I \cong \tilde{N}_m$.*

Proof of Lemma. Let $L = K_s^I$, and note that

$$\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, A(L)) = \mathrm{Hom}_I(\mathbb{Z}/m\mathbb{Z}, A(K_s)) = A_m^I.$$

If $\mathcal{O}_L$ is the ring of integers of L with respect to $\bar{v}$, its residue field is $\bar{k}$, and $\mathcal{O}_L$ is a henselian local ring. By the NMP we have $A(L) = N(\mathcal{O}_L)$. Furthermore, as $\mathcal{O}_L$ is henselian and N is smooth, Corollary 6.2.13 of [3] shows that we have a surjection $r : N(\mathcal{O}_L) \to \tilde{N}(\bar{k})$. This gives a short exact sequence

$$0 \to \ker(r) \to N(\mathcal{O}_L) \to \tilde{N}(\bar{k}) \to 0,$$

and applying the left-exact functor $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, -)$ we get

$$0 \to \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z} \to A_m^I \to \tilde{N}_m.$$

Surjectivity follows from the fact N_m is smooth over $\mathcal{O}_L$ as multiplication by m is an étale endomorphism of N, m being prime to the characteristic of k. One then applies Corollary 6.2.13 of [3] to get surjectivity. That $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z} = 0$ uses the fact that, as multiplication by m is an étale endomorphism of N and $\mathcal{O}_L$ is henselian, the kernel of r has no torsion points of order dividing m. See, for example, Lemma 3.28 of [5]. □

Let us now consider the implication (3) ⇒ (1). For this we need the following fact: $\tilde{N}^0$, the connected component of $\tilde{N}$ containing the identity, is an extension of an abelian variety B by a linear gorup H, where $H = S \times U$ for S a torus and U unipotent. We need another lemma to proceed:

Lemma 3.5. *Let $c = [\tilde{N} : \tilde{N}^0]$. Then $\tilde{N}_m$ is an extension of a group of order dividing c by a free $\mathbb{Z}/m\mathbb{Z}$-module of rank $dim(S) + 2dim(B)$.*

Proof of Lemma. Clearly $[\tilde{N}_m : \tilde{N}_m^0]$ divides $[\tilde{N} : \tilde{N}^0] = c$. But $H(\bar{k})$ is m-divisible, H being a linear group and m being prime to $\mathrm{char}(k)$, and so the sequence

$$0 \to H_m \to \tilde{N}_m^0 \to B_m$$

is also right-exact. To see this, let $b \in B_m$, so that $mb = 0$. There exists an element $a \in \tilde{N}^0$ mapping to b, and ma necessarily maps to 0. Hence we can find some element $h \in H$ mapping to ma. By the divisibility of H, there exists h' with $mh' = -h$, and hence the element $h' + a$ of $\tilde{N}_m^0$ maps to b.

As H_m and B_m are free $\mathbb{Z}/m\mathbb{Z}$-modules of ranks $\dim(S)$ and $2\dim(B)$ respectively, we are done. □

Let us now assume that (3) holds, so that there exists infinitely many m prime to $\mathrm{char}(k)$ with $A_m = A_m^I$. In particular, we may find such an m where $m > c$. For such an m, we have $A_m = A_m^I = \tilde{N}_m$, where A_m is free of rank $2\dim(A)$. By considering the cardinalities of these $\mathbb{Z}/m\mathbb{Z}$-modules, we have

$$|A_m| = m^{2\dim(A)} \leq cm^{\dim(S)+2\dim(B)} \leq m^{\dim(S)+2\dim(B)+1},$$

so in particular we find that $2\dim(A) \leq 2\dim(B) + \dim(S)$. But as $\dim(A) = \dim(U) + \dim(S) + \dim(B)$, we find that $U = S = 0$. Thus $\tilde{N}$ is an abelian variety (note the index of $\tilde{N}^0$ in $\tilde{N}$ is finite), hence proper over k. It is a fact from algebraic geometry that given a smooth scheme X over $\mathrm{Spec}(R)$, where R is a DVR, such that the generic fibre is geometrically connected and the special fibre is proper, then X is proper over $\mathrm{Spec}(R)$. Hence we conclude that N, the Néron model of A, is proper, and hence that A has good reduction at v.

□

We see from the above proof that Néron models provide a useful tool for studying abelian varieties over the fraction field of a DVR. Their use is also found in the proof of the second theorem of Serre and Tate in [2]. For the statement of this theorem we need a definition.

Definition 3.6. Let A/K be an abelian variety as in the first theorem. We say that A has *potential good reduction at* v if there exists a finite field extension K'/K along with an extension v'/v of v to K' such that $A \times_K K'$ has good reduction at v'.

The second theorem deals with an abelian variety having potential good reduction. Given a prime l different from $\mathrm{char}(k)$, define $\rho_l : \mathrm{Gal}(K_s/K) \to \mathrm{Aut}(T_l)$ to be the l-adic representation corresponding to T_l.

Theorem 3.7 (Theorem 2 of [2]). *Let A/K be an abelian variety as in the first theorem.*

- *A has potential good reduction if and only if the image of $I(\bar{v})$ under ρ_l is finite.*
- *When this holds, the restriction of ρ_l to $I(\bar{v})$ is independent of l in that its kernel is independent of l ands it character has values in $\mathbb{Z}$ independent of l.*

Proof. The proof of the first statement largely follows from the first theorem. See [5] for a proof. We do not include it here as it does not directly illustrate the usefulness of Néron models.

For the second statement, note that we only care about the action of $I(\bar{v})$, so we may assume without loss of generality that $K = (K_s)^{I(\bar{v})}$, i.e. that the Galois group of K_s/K is equal to $I(\bar{v})$.

Consider a finite extension $K \subset K' \subset \bar{K}$, where $\bar{K}$ is a fixed algebraic closure of K. Let $G_{K'} = \mathrm{Gal}(K'/K) = \mathrm{Gal}(K_s/K_s \cap K') \subset I(\bar{v})$. If an abelian variety $A_{K'} = A \times_K K'$ has good reduction, then necessarily $G_{K'} \subset \ker(\rho_l)$. Thus $G_{K'}$ is in the kernel regardless of l, and so the kernel of ρ_l is defined by the finite extensions at which $A_{K'}$ has good reduction. Hence $\ker(\rho_l)$ is independent of l.

Now assume that there exists a finite K'/K for which $A_{K'}$ has good reduction, and let N' be the Néron model of $A_{K'}$. This is an abelian scheme over $\mathcal{O}'_v$ by assumption. Note that $G_{K'}$ acts on $A_{K'}$ be its action on K'. By the Néron Mapping Property, this extends to an action of $G_{K'}$ on N' that commutes with the action of $G_{K'}$ on $\mathcal{O}'_v$. (There is a slight subtlety here: the NMP requires that we have a K'-morphism. This holds in this case, but only after viewing $A_{K'}$ as a K'-scheme after composing with the morphism from $\mathrm{Spec}(K') \to \mathrm{Spec}(K')$ induced by the action of $G_{K'}$.) As $G_{K'}$ acts trivially on $k = \bar{k}$, it acts on $\tilde{N}'$. A theorem of Weil (see Chapter VII of [4]) then informs us that the action of $G_{K'}$ on $T_l(A_{K'})$ has an integral character independent of l. As $T_l(A) \cong T_l(A') \cong T_l(\tilde{N}')$, we are done. ∎

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