Néron models seminar, Fall 2017, Sept. 28.

Bas Edixhoven, The Hilbert functor and its representability.

Reference: Ch. 5, by Nitin Nitsure, in Fundamental Algebraic Geometry, Grothendieck's FGA explained, AMS Math. Surveys & Mon., 123.

With less detail: see ~~&~~ § 8.2 of Bosch-Lütkebohmert-Raynaud.

§0. Motivation. Needed for repr. of Picard functor; must parametrise divisors on a given X/S. Also useful for many other things, for example $\underline{\mathrm{Hom}}_{\mathrm{Sch}/S}(X,Y)$.

I follow BLR here bec. they got rid of the noetherianness.

§1. Def. For S any scheme, $X \to S$ any S-scheme, let

$\mathrm{Hilb}_{X/S} : \mathrm{Sch}/S \to \mathrm{Set}$ be the contravariant functor

$(T \to S) \mapsto \{$closed subsch. Z of X_T with $Z \to T$ finitely presented, proper and flat$\}$.

$$\begin{tikzcd} T' \arrow[r] \arrow[dr] & T \arrow[d] \\ & S \end{tikzcd} \quad \mapsto \quad (Z \mapsto Z_{T'}).$$

Def. Same situation, plus $\mathcal{F}$ an $\mathcal{O}_X$-module that is locally fin. presented. Let $\mathrm{Quot}_{\mathcal{F}/X/S} : \mathrm{Sch}/S \to \mathrm{Set}$ be the contravariant functor

$(T \to S) \mapsto \{q : \mathcal{F}_T \twoheadrightarrow \mathcal{G}$ surjection of $\mathcal{O}_{X_T}$-modules, with $\mathcal{G}$ loc. fin. presentation, $\mathcal{O}_T$-flat, with support proper over $T\}$ / isom.

$$\begin{tikzcd} \mathcal{F}_T \arrow[r, two heads, "q_1"] \arrow[dr, two heads, "q_2"'] & \mathcal{G}_1 \arrow[d, "\cong"] \\ & \mathcal{G}_2 \end{tikzcd}$$

Note: $\mathrm{Hilb}_{X/S} = \mathrm{Quot}_{\mathcal{O}_X/X/S}$.

Two very simple examples. $\mathrm{Hilb}_{\emptyset/\mathrm{Spec}\,\mathbb{Z}} = \mathrm{Spec}\,\mathbb{Z}$, $\mathrm{Hilb}_{\mathrm{Spec}\,\mathbb{Z}/\mathrm{Spec}\,\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}\times\mathbb{Z})$.

And 1 more: $\mathrm{Hilb}_{\mathbb{P}^1/\mathbb{Z}} = \left(\coprod_{n\in\mathbb{N}} \mathbb{P}^n\right) \sqcup \mathrm{Spec}\,\mathbb{Z}$.

For the representability thm. we need an extra definition.

Def. A morph. of schemes $X \xrightarrow{f} S$ is strongly (quasi) projective if it is of fin. pres., and if $\exists\, \mathcal{E}$ loc. free $\mathcal{O}_S$-module of const. fin. rank s.t. X has ~~a~~ a closed immersion (immersion) into $\mathbb{P}_S(\mathcal{E}) := \mathrm{Proj}_S(\mathrm{Sym}_{\mathcal{O}_S}\mathcal{E})$, $\begin{smallmatrix} T \\ \downarrow g \\ S \end{smallmatrix} \mapsto \{q : g^*\mathcal{E} \twoheadrightarrow \mathcal{L},\ \mathcal{L}$ inv. $\mathcal{O}_T$-mod.$\}/\cong$

Given such an immersion, let $\mathcal{O}_X(1)$ be the induced very ample $\mathcal{O}_X$-module.

For X/S a smooth proper curve, ~~with geom. conn. fibres~~,

$\mathrm{Hilb}_{X/S} = \left(\coprod_{n\in\mathbb{N}} X^{(n)}\right) \sqcup S$, where $X^{(n)} = X^n/S_n$, fibr. pr. $/S$.

Thm. (Grothendieck, Altman-Kleiman). Let $f: X \to S$ in Sch be strongly quasi-projective and let $\mathcal{O}_X(1)$ be an induced very ample inv. $\mathcal{O}_X$-module. Let $\mathcal{F}$ be a quotient of some $(f^*B)(\nu)$ with B an $\mathcal{O}_S$-module that is loc. free of finite constant rank, and $\nu \in \mathbb{Z}$. Then $\mathrm{Quot}_{\mathcal{F}/X/S}$ is represented by a disjoint union of strongly quasi proj. S-schemes. If moreover f is proper, (i.e., f is strongly projective) then $\mathrm{Quot}_{\mathcal{F}/X/S}$ is repr. by a disj. union of strongly projective S-schemes.

§2. Decomposition by Hilbert polynomials.

For k a field, X a k-scheme, projective, with given very ample $\mathcal{O}_X(1)$, $\mathcal{F}$ a coherent $\mathcal{O}_X$-module we have the Hilbert polynomial $\chi(\mathcal{F}) \in \mathbb{Q}[\lambda]$, $\underset{\in\mathbb{Z}}{n} \mapsto \chi(X, \mathcal{F}(n)) = \sum_i (-1)^i \dim_k H^i(X, \mathcal{F}(n))$.

For X/S strongly proj, with fixed $\mathcal{O}_X(1)$, and $\mathcal{F}$ of fin. pres and $\mathcal{O}_S$-flat,

$s \mapsto \chi(\mathcal{F}_s)$ is loc. constant, and then $S = \coprod_{\Phi \in \mathbb{Q}[\lambda]} S_\Phi$.

$$\text{So then } \mathrm{Quot}_{\mathcal{F}/X/S} \overset{\text{as functors!}}{=} \coprod_{\Phi \in \mathbb{Q}[\lambda]} \mathrm{Quot}^{\Phi}_{\mathcal{F}/X/S}.$$ Same for $\mathrm{Hilb}_{X/S}$.

these are projective.

__Example.__ for $n \geq 0$ $\mathrm{Hilb}^1_{\mathbb{P}^n/\mathbb{Z}} = \mathbb{P}^n$, 1 point in $\mathbb{P}^n$.

For $n \geq 1$: 2 points in $\mathbb{P}^n$, and $\mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$: $\mathrm{Hilb}^2_{\mathbb{P}^n} = L^{(2)}$

$$\begin{array}{c} L \\ \downarrow \\ \mathrm{Gr}(n+1, 2) \end{array}$$

lines in $\mathbb{P}^n$.

For $n \geq 2$: 3 points in $\mathbb{P}^n$, $\mathrm{Hilb}^3_{\mathbb{P}^n/\mathbb{Z}}$, is harder!

Why?

Because $\mathrm{Spec}(k[x,y]/(x^2, xy, y^2)) \hookrightarrow \mathbb{P}^2_k$ is not a complete intersection: at $(0,0)$, say, the ideal needs 3 generators.

The proof of repres. proceeds in 2 main steps.

For simplicity, we only look at the case where $f: X \to S$ is strongly projective.

$$\mathrm{Hilb}^3_{\mathbb{P}^2_{\mathbb{Z}}} \hookrightarrow \mathrm{Gr}(6,3), \quad \begin{pmatrix} Z \rightarrowtail \mathbb{P}^2_k \text{ closed} \\ \text{subsch. length } 3 \end{pmatrix} \longmapsto \ker\left(\mathcal{O}_{\mathbb{P}^2_k}(2)(\mathbb{P}^2_k) \to \mathcal{O}_Z(2)(Z)\right)$$

$$= \Gamma(\mathbb{P}^2_k, I_Z(2)) \subset k[x,y,z]_2.$$

This is an immersion, that is, a locally closed embedding.

§ 3. Reduction to $\mathrm{Quot}^{\Phi}_{\pi^*B/\mathbb{P}_S(\mathcal{E})/S}$ (to get rid of the geometry; exit X/S)

Here $\mathcal{B}$ and $\mathcal{E}$ are loc. free $\mathcal{O}_S$-modules of constant finite rank, $\pi : \mathbb{P}_S(\mathcal{E}) \to S$. Let X be a closed subscheme of $\mathbb{P}_S(\mathcal{E})$, of finite presentation over S, and let $\mathcal{F}$ be a quotient of some $(\pi^*\mathcal{B})(\nu)|_X$, $\mathcal{O}_S$-flat.

Thm. (Lemma 5.17 in Nitsure). Let $\Phi \in \mathbb{Q}[\lambda]$. The inclusion morphism

$$\mathrm{Quot}^{\Phi}_{\mathcal{F}/X/S} \longrightarrow \mathrm{Quot}^{\Phi}_{\pi^*(\mathcal{B})(\nu)/\mathbb{P}(\mathcal{E})/S}$$

is represented by closed immersions.

Rem. What $\boxed{\text{does}}$ this mean? $\forall\, T \to S$, $\forall\, q : (\pi^*\mathcal{B})(\nu)_T \twoheadrightarrow \mathcal{G}$ on $\mathbb{P}(\mathcal{E})_T$

$$\begin{tikzcd} \mathrm{Quot}^{\Phi}_{\mathcal{F}/X/S} \arrow[r] & \mathrm{Quot}^{\Phi}_{\pi^*(\mathcal{B})(\nu)/\mathbb{P}(\mathcal{E})/S} \\ P \arrow[u] \arrow[r] & T \arrow[u, "q"'] \end{tikzcd}$$

in Functors$((Sch/S)^{op}, Set)$

P " a scheme, " a closed immersion.

Concretely: $\exists$ closed subscheme Z of T s.t. $\forall\, T' \to T$ s.t. ($q_{T'}$ factors through $\mathcal{F}_{T'}$) $\Leftrightarrow$ ($T' \to T$ fact. through $Z \hookrightarrow T$).

Sketch of proof. $\mathcal{I} \overset{i}{\hookrightarrow} (\pi^*\mathcal{B})(\nu)_T \twoheadrightarrow \mathcal{F}_T$ exact, on $\mathbb{P}(\mathcal{E})_T$

$$\begin{tikzcd} \mathcal{I} \arrow[r, hook, "i"] \arrow[dr, "q\circ i"'] & (\pi^*\mathcal{B})(\nu)_T \arrow[r, two heads] \arrow[d, "q"] & \mathcal{F}_T \\ & \mathcal{G} & \end{tikzcd}$$

We need to understand for which $T' \to T$ $(q\circ i)_{T'} = 0$. So, consider the functor $(T' \to T) \mapsto \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{E})_{T'}}}(\mathcal{I}_{T'}, \mathcal{G}_{T'})$, let's call it H.

Nitsure, Thm 5.8.

Claim: H is represented by a scheme$/T'$ of the form $\mathrm{Spec}_{T'}(\mathrm{Sym}_{\mathcal{O}_{T'}} \mathcal{Q}) =: V$ with $\mathcal{Q}$ an $\mathcal{O}_{T'}$-module of fin. pres.

Consequence:

$$\begin{tikzcd} T' \arrow[r, "q\circ i"] & V \\ Z \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & T' \arrow[u, "0"'] \end{tikzcd}$$

Behind the proof of repr. of H: cohomology and base change basics. See Nitsure Thm 5.6, Mumford A.V. §5, BLR right after Thm. 7 of Ch. 8.2. This is not a difficult result at all, and it looks as the start of the theory of derived categories.

§4. Embedding $\mathrm{Quot}^{\Phi}_{\pi^*B/\mathbb{P}(\mathcal{E})/S}$ into Grass.

Nibure 5.5.5, BLR §8.2 Thm 8'.

Situation: S a scheme, B & $\mathcal{E}$ loc free $\mathcal{O}_S$-mod. of constant finite rank, $\pi : \mathbb{P}(\mathcal{E}) \to S$, $\Phi \in \mathbb{Q}[\lambda]$.

<u>Thm</u>. $\exists\, m \in \mathbb{Z}$, depending only on $\mathrm{rk}(\mathcal{E})$, $\mathrm{rk}(B)$ and Φ, such that $\forall\, r \geq m$ $\forall$ field k,
$\forall s \in S(k)$, $\forall$ $q : (\pi^*B)_s \twoheadrightarrow \mathcal{F}$ with $\chi(\mathbb{P}(\mathcal{E})_s, \mathcal{F}) = \Phi$;
$\forall i \geq 1$ $H^i((\pi^*B)_s(r)) = 0$, $H^i(\mathcal{F}_s(r)) = 0$, $H^i((\ker q)(r)) = 0$,
$\pi^*B(r)_s$, $\mathcal{F}_s(r)$, $(\ker q)(r)$ are generated by global sections.

This uses "Castelnuovo–Mumford regularity" ---
We take such an m.

<u>Thm</u>. (BLR. §8.2 Thm 8').

$$\mathrm{Quot}^{\Phi}_{\pi^*B/\mathbb{P}(\mathcal{E})/S} \longrightarrow \mathrm{Grass}\left(B \otimes_{\mathcal{O}_S} \mathrm{Sym}^m(\mathcal{E})\right)$$

$$\left(T \to S,\ (\pi^*B)_T \xrightarrow[\cancel{}]{q} \mathcal{F}\right) \longmapsto \pi_{T*}\left((\ker q)(m)\right) \hookrightarrow \pi_{T*}\left((\pi^*B)(m)_T\right) = \left(B \otimes_{\mathcal{O}_S} \mathrm{Sym}^m \mathcal{E}\right)_T.$$

on $\mathbb{P}(\mathcal{E})_T \xrightarrow{\pi_T} T$

is a closed immersion, and $\mathrm{Quot}^{\Phi}_{\pi^*B/\mathbb{P}(\mathcal{E})/S}$ is strongly projective over S.

The proof has 2 steps.

1. The map is an isomorphism to the maximal (loc. closed) subscheme of Grass over which the ~~closed subscheme~~ quotient of $(\pi^*B)(m)$ is flat and has Hilbert polynomial Φ. (existence of such a subscheme: Nibure Thm. 5.13).

2. Use valuative criterion of properness to show that the image is closed. (easy...)