The Picard functor for curves over discrete valuation rings

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1. Introduction

In the previous talk, we have seen that if $f\colon X \to S$ is a flat and strongly projective morphism with geometrically integral fibres, then the Picard sheaf

$$\mathrm{Pic}_{X/S} = \mathrm{R}^1 f_* \mathbf{G}_{\mathrm{m},X}$$

for the fppf topology on S is representable by a scheme. In particular, we have seen the decomposition by Hilbert polynomials:

$$\mathrm{Pic}_{X/S} = \coprod_{\Phi \in \mathbf{Q}[x]} \mathrm{Pic}^{\Phi}_{X/S},$$

where each $\mathrm{Pic}^{\Phi}_{X/S}$ is strongly quasi-projective.

Over a field, we have a stronger representability result that does not require the condition that the scheme be geometrically integral.

Theorem 1.1 (Grothendieck, Murre and Oort; see [1, § 8.2, Theorem 3]). *Let X be a proper scheme over a field K. Then $\mathrm{Pic}_{X/K}$ is represented by a K-scheme that is locally of finite type.*

Unfortunately, there is no common generalisation that shows the representability of $\mathrm{Pic}_{X/S}$ for a strongly projective scheme X over a general scheme S whose fibres are not geometrically integral.

Let R be a discrete valuation ring, and let $S = \operatorname{Spec} R$. We write K for the field of fractions of R and k for the residue field.

Let X be a proper flat curve over S. We assume that the generic fibre X_K is normal and geometrically irreducible.

Theorem 1.2. *Assume that X is regular, and either that k is perfect or X admits an étale quasi-section. Let $\mathrm{Pic}^{[0]}_{X/S}$ be the subfunctor of $\mathrm{Pic}_{X/S}$ given by line bundles of total degree 0, and let $E_{X/S}$ be the schematic closure in $\mathrm{Pic}_{X/S}$ of the unit section $\operatorname{Spec} K \to \mathrm{Pic}_{X_K/K}$. Then $E_{X/S}$ is a subsheaf of $\mathrm{Pic}^{[0]}_{X/S}$, the quotient sheaf $Q_{X/S} = \mathrm{Pic}^{[0]}_{X/S}/E_{X/S}$ is represented by a group scheme over S, and this is a Néron model of $\mathrm{Pic}^0_{X_K/K}$.*

Theorem 1.3. *In the setting of Theorem 1.2, assume in addition that the greatest common divisor of the geometric multiplicities of the the irreducible components of the special fibre X_k equals 1. Then $\mathrm{Pic}^0_{X/S}$ is separated over S, and the projection $\mathrm{Pic}^{[0]}_{X/S} \to Q_{X/S}$ induces an isomorphism from $\mathrm{Pic}^0_{X/S}$ to the identity component of $Q_{X/S}$.*

2. Preliminaries

We begin with a criterion for the formal smoothness of the Picard functor.

Proposition 2.1 ([1, § 8.4, Proposition 2]). *Let $f\colon X \to S$ be a finitely presented proper flat morphism. Let s be a point of S such that $\mathrm{H}^2(X_s, \mathcal{O}_{X_s}) = 0$. Then there is an open neighbourhood U of s such that $\mathrm{Pic}_{X_U/U}$ is formally smooth over U.*

Corollary 2.2. *Let $f\colon X \to S$ be finitely presented, proper, flat, and with fibres of dimension ≤ 1. Then $\mathrm{Pic}_{X/S}$ is formally smooth over S.*

Proof. This follows because $\mathrm{H}^2(X_s, \mathcal{O}_{X_s}) = 0$ for all $s \in S$ by Grothendieck's vanishing theorem [2, Theorem III.2.7]. □

Corollary 2.3. *Let X be a proper curve over a field K. Then $\mathrm{Pic}^0_{X/K}$ is representable by a smooth K-scheme.*

Proof. This follows from the fact that $\mathrm{Pic}^0_{X/K}$ is representable, locally of finite type and formally smooth. □

Next, we collect some results on Picard schemes of proper smooth schemes over a field.

Proposition 2.4. *Let X be a smooth proper scheme over a field K. Then $\mathrm{Pic}^0_{X/K}$ is a proper group scheme over K.*

Proof. By Theorem 1.1, $\mathrm{Pic}_{X/K}$ is representable by a K-scheme that is locally of finite type. By a general result on connected components of group schemes, $\mathrm{Pic}^0_{X/K}$ is of finite type. The properness follows by the valuative criterion of properness, using the fact that X is smooth and proper. □

Corollary 2.5. *Let X be a smooth projective curve over a field K. Then $\mathrm{Pic}^0_{X/K}$ is an Abelian variety.*

Proof. This follows from the above proposition and the formal smoothness of $\mathrm{Pic}_{X/K}$ (Corollary 2.2). □

We now study the connected component $\mathrm{Pic}^0_{X/K}$ in more detail.

Lemma 2.6. *Let X be a proper, smooth and geometrically connected curve over a field K, and let L be a line bundle of degree 0 on X. Then the point of $\mathrm{Pic}_{X/K}(K)$ defined by L lies on the connected component $\mathrm{Pic}^0_{X/K}(K)$.*

Proof. We may assume that K is algebraically closed, so every line bundle of degree 0 has the form $\mathcal{O}_X(\sum_{i=1}^n (x_i - y_i))$ with $n \geq 0$ and $x_i, y_i \in X(K)$. Since $\mathrm{Pic}^0_{X/K}(K)$ is a subgroup of $\mathrm{Pic}_{X/K}(K)$, it suffices to show that for all $x, y \in X(K)$, the line bundle $\mathcal{O}_X(x - y)$ defines an element of $\mathrm{Pic}^0_{X/K}(K)$. Since X is smooth, every $y \in X(K)$ defines a map

$$\begin{aligned} X &\longrightarrow \mathrm{Pic}_{X/K} \\ x &\longmapsto [\mathcal{O}_X(x - y)]. \end{aligned}$$

Since X is connected, the image of this map is contained in $\mathrm{Pic}^0_{X/K}$, which proves the claim. □

Corollary 2.7 (cf. [1, § 9.2, Corollary 13]). *Let X be a proper curve over a field K, and let $\bar{K}$ be an algebraic closure of K. Then $\mathrm{Pic}^0_{X/K}$ classifies those line bundles whose partial degree on each irreducible component of $X_{\bar{K}}$ is zero.*

Proof. This is proved by reducing to the case where K is algebraically closed and constructing a surjective homomorphism

$$\mathrm{Pic}_{X/K} \longrightarrow \prod_C \mathrm{Pic}_{\tilde{C}/K}$$

with connected kernel, where C runs over the irreducible components of X and $\tilde{C}$ is the normalisation of C. □

Finally, we will need a criterion for separatedness of group schemes.

Proposition 2.8 (SGA 3, tome 1, exposé $\mathrm{VI_B}$, proposition 5.1). *Let G be a group scheme over a scheme S. Then G is separated over S if and only if the unit section $e\colon S \to G$ is a closed immersion.*

Proof. In general, any section of a separated morphism is a closed immersion (EGA I 5.4.6), so if G is separated over S, then e is a closed immersion. Conversely, if e is a closed immersion, then we see from the Cartesian diagram

$$\begin{array}{ccc} G & \longrightarrow & S \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle e} \\ G \times_S G & \xrightarrow{\delta} & G \end{array}$$

(where $\delta(gh) = gh^{-1}$) that the diagonal morphism $\Delta\colon G \times_S G \to G$ is a closed immersion, so G is separated over S. □

3. The Picard scheme of a family of curves with geometrically integral fibres

We now specialise these to the case of curves. If X is a projective geometrically integral curve over a field K, and let g be the arithmetic genus of X, defined by

$$g = \dim_K \mathrm{H}^1(X, \mathcal{O}_X).$$

If L is a line bundle of degree d on X, then the Riemann–Roch formula implies that for all $n \in \mathbf{Z}$, the Euler characteristic of $L(n)$ equals $1 - g + d + n \deg \mathcal{O}(1)$. This means that the Hilbert polynomial of L equals $\Phi_L = 1 - g + d + (\deg \mathcal{O}(1))x \in \mathbf{Q}[x]$. In particular, classifying line bundles by Hilbert polynomial is equivalent to classifying them by degree.

Theorem 3.1 ([1, § 9.3, Theorem 1]). *Let $f\colon X \to S$ be a strongly projective flat morphism with geometrically integral fibres of dimension* 1*. Then $\mathrm{Pic}_{X/S}$ is smooth and separated over S. Furthermore, there is a decomposition*

$$\mathrm{Pic}_{X/S} = \coprod_{n \in \mathbf{Z}} (\mathrm{Pic}_{X/S})^n,$$

where $(\mathrm{Pic}_{X/S})^n$ denotes the open and closed subscheme of $\mathrm{Pic}_{X/S}$ classifying line bundles of degree n. We have $(\mathrm{Pic}_{X/S})^0 = \mathrm{Pic}^0_{X/S}$. Moreover, for all $n \in \mathbf{Z}$ the S-scheme $(\mathrm{Pic}_{X/S})^n$ is quasi-projective and is a torsor under $\mathrm{Pic}^0_{X/S}$.

Proof. We already know that each $(\mathrm{Pic}_{X/S})^n$ is quasi-projective, and in particular separated. The smoothness follows from Corollary 2.2. It follows from Corollary 2.7 that $\mathrm{Pic}^0_{X/S}$ is equal to $(\mathrm{Pic}_{X/S})^0$. The claim that each $(\mathrm{Pic}_{X/S})^n$ is a torsor under $\mathrm{Pic}^0_{X/S}$ now follows from the fact that they become isomorphic once X has a section over S. □

Let R be a discrete valuation ring with field of fractions K and residue field k. We write $S = \operatorname{Spec} R$. Using the above results, we will now show that $\mathrm{Pic}^0_{X/S}$ is a Néron model of its generic fibre.

Theorem 3.2 ([1, § 9.5, Theorem 1]). *Let $f\colon X \to S$ be a projective flat curve with geometrically integral fibres such that X is regular. Then $\mathrm{Pic}^0_{X/S}$ is a Néron model of $\mathrm{Pic}^0_{X_K/K}$.*

Proof. We know that $\mathrm{Pic}^0_{X/S}$ is smooth, separated and of finite type by Theorem 3.1. We have to show that it satisfies the Néron mapping property. For this we may replace R by its strict Henselisation, so that $f\colon X \to S$ admits a section (here we use that the special fibre has a smooth point). Now let $T \to S$ be a smooth morphism, and let $u\colon T_K \to \mathrm{Pic}_{X_K/K}$ be a morphism of K-schemes. Because f admits a section, u defines a line bundle L on $X_K \times_K T_K$. Because $X \times_S T$ is regular, L extends to a line bundle on $X \times_S T$. This in turn defines a morphism $u'\colon T \to \mathrm{Pic}_{X/S}$ of S-schemes. Hence the natural group homomorphism

$$\mathrm{Hom}(T, \mathrm{Pic}_{X/S}) \longrightarrow \mathrm{Hom}(T_K, \mathrm{Pic}_{X_K/K})$$

is surjective. Since $\mathrm{Pic}_{X/S}$ is separated, this homomorphism is also injective, so it is an isomorphism. For projective flat curves with geometrically integral fibres, Pic^0 classifies line bundles of degree 0 by Theorem 3.1, so the above homomorphism induces an isomorphism

$$\mathrm{Hom}(T, \mathrm{Pic}^0_{X/S}) \xrightarrow{\sim} \mathrm{Hom}(T_K, \mathrm{Pic}^0_{X_K/K}).$$

This shows that $\mathrm{Pic}^0_{X/S}$ is a Néron model of $\mathrm{Pic}^0_{X_K/K}$. □

4. A representable quotient of $\mathrm{Pic}^{[0]}_{X/S}$

As before, let R be a discrete valuation ring with field of fractions K and residue field k. We write $S = \operatorname{Spec} R$.

Let $f\colon X \to S$ be a proper flat morphism with geometric fibres purely of dimension 1. Since we no longer assume that the fibres of f are geometrically integral, $\mathrm{Pic}_{X/S}$ is not necessarily representable; we view it as a sheaf for the fppf topology on $\operatorname{Spec} S$.

Let $\mathrm{Pic}^{[0]}_{X/S}$ be the subsheaf of $\mathrm{Pic}_{X/S}$ classifiying line bundles of total degree 0. (In other words, there is a degree morphism from $\mathrm{Pic}_{X/S}$ to the constant sheaf $\mathbf{Z}$, and $\mathrm{Pic}^{[0]}_{X/S}$ is its kernel.) Then we have inclusions

$$\mathrm{Pic}^{0}_{X/S} \subseteq \mathrm{Pic}^{[0]}_{X/S} \subseteq \mathrm{Pic}_{X/S}\,.$$

Because the degree of a line bundle is locally constant, the inclusion $\mathrm{Pic}^{[0]}_{X/S}$ into $\mathrm{Pic}_{X/S}$ is relatively representable by open and closed immersions.

Let $E_{X/S} \subseteq \mathrm{Pic}_{X/S}$ be the schematic closure of the unit section $e_K\colon \mathrm{Spec}\,K \to \mathrm{Pic}_{X_K/K}$. This $E_{X/S}$ is the subsheaf of $\mathrm{Pic}_{X/S}$ generated by the images of all morphisms of sheaves $Z \to \mathrm{Pic}_{X/S}$ where Z is a flat S-scheme and where the induced morphism $Z_K \to \mathrm{Pic}_{X_K/K}$ on the generic fibre factors as $Z_K \to \mathrm{Spec}\,K \xrightarrow{e_K} \mathrm{Pic}_{X_K/K}$, where the first map is the structure morphism.

If Z is a flat S-scheme, then every morphism $Z \to \mathrm{Pic}_{X/S}$ such that $Z_K \to \mathrm{Pic}_{X_K/K}$ factors through the unit section has image contained in $\mathrm{Pic}^{[0]}_{X/S}$. This implies that $E_{X/S}$ is contained in $\mathrm{Pic}^{[0]}_{X/S}$. Therefore we can form the quotient

$$Q_{X/S} = \mathrm{Pic}^{[0]}_{X/S}/E_{X/S}$$

in the category of sheaves for the fppf topology on $\mathrm{Spec}\,S$.

Definition. Let $f\colon X \to S$ be a proper flat finitely presented morphism of schemes. A *rigidificator* for f is a subscheme $Y \subseteq X$ that is finite, flat and finitely presented over S and such that the functor

$$\begin{aligned} \mathbf{Sch}_S^{\mathrm{op}} &\to \mathbf{Sets} \\ T &\mapsto \Gamma(X_T, \mathcal{O}_{X_T}) \end{aligned}$$

is a subfunctor of

$$\begin{aligned} \mathbf{Sch}_S^{\mathrm{op}} &\to \mathbf{Sets} \\ T &\mapsto \Gamma(X_T, \mathcal{O}_{X_T}) \end{aligned}$$

i.e. if for every S-scheme T the natural map $\Gamma(X_T, \mathcal{O}_{X_T}) \to \Gamma(Y_T, \mathcal{O}_{Y_T})$ is injective.

Proposition 4.1 ([1, § 9.5, Proposition 3]). *Let $f\colon X \to S$ be a proper flat curve such that X_K is normal and geometrically irreducible. Then the sheaf $Q_{X/S}$ (for the fppf topology on $\mathrm{Spec}\,S$) is represented by a smooth separated S-group scheme. The quotient map $\mathrm{Pic}^{[0]}_{X/S} \to Q_{X/S}$ is an isomorphism on the generic fibres.*

Proof. We fix a rigidificator $i\colon Y \rightarrowtail X$. Let $\mathbf{G}_{\mathrm{m},X}[Y]$ denote the kernel of the natural surjection $\mathbf{G}_{\mathrm{m},X} \to i_*\mathbf{G}_{\mathrm{m},Y}$. We consider the short exact sequence

$$1 \longrightarrow \mathbf{G}_{\mathrm{m},X}[Y] \longrightarrow \mathbf{G}_{\mathrm{m},X} \longrightarrow i_*\mathbf{G}_{\mathrm{m},Y} \longrightarrow 1.$$

of sheaves for the étale topology on X. Taking higher direct images under the structure morphism f and using the fact that push-forward of étale sheaves by a finite morphism is an exact functor, we obtain a long exact sequence

$$1 \longrightarrow f_*(\mathbf{G}_{\mathrm{m},X}[Y]) \longrightarrow f_*\mathbf{G}_{\mathrm{m},X} \longrightarrow (f \circ i)_*\mathbf{G}_{\mathrm{m},Y} \longrightarrow \mathrm{R}^1 f_*(\mathbf{G}_{\mathrm{m},X}[Y]) \longrightarrow \mathrm{R}^1 f_*\mathbf{G}_{\mathrm{m},X} \longrightarrow 1$$

of sheaves for the étale topology on S. We recall that

$$\mathrm{R}^1 f_*\mathbf{G}_{\mathrm{m},X} = \mathrm{Pic}_{X/S}\,.$$

Furthermore, we write

$$\mathrm{Pic}_{X/S}[Y] = \mathrm{R}^1 f_*(\mathbf{G}_{\mathrm{m},X}[Y]).$$

The definition of rigidificators implies $f_*(\mathbf{G}_{\mathrm{m},X}[Y]) = 1$. We obtain an exact sequence

$$1 \longrightarrow f_*\mathbf{G}_{\mathrm{m},X} \longrightarrow (f \circ i)_*\mathbf{G}_{\mathrm{m},Y} \longrightarrow \mathrm{Pic}_{X/S}[Y] \longrightarrow \mathrm{Pic}_{X/S} \longrightarrow 1$$

of sheaves for the étale topology on S.

The sheaf $\mathrm{Pic}_{X/S}[Y]$ can be interpreted as classifying line bundles on X that are rigidified along Y. Using this, one can show that $\mathrm{Pic}_{X/S}[Y]$ is represented by an algebraic space that is smooth over S [1, § 8.2; § 8.3, Theorem 3; § 8.4, Proposition 2].

Passing to line bundles of total degree 0, we obtain an exact sequence

$$1 \longrightarrow f_*\mathbf{G}_{\mathrm{m},X} \longrightarrow (f\circ i)_*\mathbf{G}_{\mathrm{m},Y} \longrightarrow \mathrm{Pic}^{[0]}_{X/S}[Y] \longrightarrow \mathrm{Pic}^{[0]}_{X/S} \longrightarrow 0$$

and a commutative diagram

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
H & \longrightarrow & E_{X/S} & & \\
\downarrow & & \downarrow & & \\
\mathrm{Pic}^{[0]}_{X/S}[Y] & \xrightarrow{r} & \mathrm{Pic}^{[0]}_{X/S} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathrm{Pic}^{[0]}_{X/S}[Y]/H & \xrightarrow{\bar r} & Q_{X/S} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

where H is the kernel of the composed map $\mathrm{Pic}^{[0]}_{X/S}[Y] \to Q_{X/S}$. A diagram chase (or the snake lemma) shows that $\bar r$ is an isomorphism. We now note that H equals the closure of the kernel of r. This kernel is flat over S because it is a quotient of $(f\circ i)_*\mathbf{G}_{\mathrm{m},Y}$; it follows that its closure H is flat over S. From this it follows that $\mathrm{Pic}^{[0]}_{X/S}/H$, and hence $Q_{X/S}$, is representable by an algebraic space [1, § 8.4, Proposition 9]. This algebraic space is separated because the unit section is a closed immersion (pass to an étale covering to make $Q_{X/S}$ into a scheme and apply Proposition 2.8). Finally, one uses the fact that a smooth separated group object in the category of algebraic spaces is representable by a scheme [1, § 6.6, Corollary 3]. □

5. The weak Néron criterion

We will give a criterion for a smooth separated group scheme of finite type to be a Néron model of its generic fibre. For this we need *Weil's extension theorem.*

Theorem 5.1 (Weil; see [1, § 4.4, Theorem 1]). *Let S be a normal Noetherian scheme, let Z be a smooth S-scheme, and let G be a smooth and separated group scheme over S. Let u be an S-rational map from Z to G. If u is defined in codimension ≤ 1 (i.e. is defined on some open subset whose complement has codimension ≥ 2), then u is defined everywhere.*

Now let R be a discrete valuation ring with field of fractions K, let R^{sh} be a strict Henselisation of R, and let K^{sh} be the field of fractions of R^{sh}.

Definition. If X is a smooth separated R-scheme, we say that X satisfies the *weak Néron property* if the canonical map

$$X(R^{\mathrm{sh}}) \longrightarrow X_K(K^{\mathrm{sh}})$$

is surjective.

Proposition 5.2 (cf. [1, § 3.5, Proposition 3]). *Let X be a smooth separated R-scheme of finite type having the weak Néron property. Let Z be a smooth R-scheme, and let u be a K-rational map from Z_K to X_K. Then u extends to an R-rational map from Z to X.*

Corollary 5.3. *Let G be a smooth separated group scheme of finite type over R. Then G is a Néron model of its generic fibre if and only if G has the weak Néron property.*

Proof. The "only if" direction is trivial. The "if" direction follows from Proposition 5.2, Theorem 5.1 and the separatedness of G.

6. Sketch of proof of the main results

Proof of Theorem 1.2 (sketch). By Proposition 4.1, $Q_{X/S}$ is a smooth separated S-group scheme with generic fibre $\mathrm{Pic}_{X_K/K}$. We need to prove that $Q_{X/S}$ is of finite type over S. We sketch a proof of how to do this using intersection theory of divisors on the special fibre of X.

Let X be a proper flat curve over $S = \mathrm{Spec}\, R$, where R is a strictly Henselian discrete valuation ring, such that X is normal and X_K is geometrically irreducible. Let D be the group of Cartier divisors on X with support on X_k, and let D_0 be the subgroup of D consisting of principal divisors. Then there is a canonical complex

$$0 \longrightarrow D_0 \longrightarrow D \xrightarrow{\alpha} \mathbf{Z}^I \xrightarrow{\beta} \mathbf{Z} \longrightarrow 0,$$

where α is given by taking intersection numbers with each irreducible component (suitably normalised), and β is a "weighted" degree function.

Lemma 6.1 ([1, § 9.5, Lemma 9]). *There is a canonical surjective group homomorphism*

$$\sigma\colon (\ker\beta)/(\mathrm{im}\,\alpha) \longrightarrow Q_{X/S}(S)/Q^0_{X/S}(S),$$

which is an isomorphism if the canonical map $\mathrm{Pic}^{[0]}_{X/S}(S) \longrightarrow Q^0_{X/S}(S)$ *is surjective.*

If X is regular, then intersection theory shows that the group $(\ker\alpha)/(\mathrm{im}\,\beta)$ is finite, and hence $Q_{X/S}(S)/Q^0_{X/S}(S)$ is finite. Since $Q_{X/S}$ is smooth, and in particular locally of finite type, it follows that $Q_{X/S}$ is of finite type over S [SGA 3, tome 1, exposé $\mathrm{VI_B}$, 3.6].

By Corollary 5.3, it remains to prove that the natural map

$$Q_{X/S}(R^{\mathrm{sh}}) \longrightarrow Q_{X/S}(K^{\mathrm{sh}})$$

is surjective. We consider the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Pic}^{[0]}(X_{R^{\mathrm{sh}}}) & \longrightarrow & \mathrm{Pic}^{[0]}_{X/S}(R^{\mathrm{sh}}) & \longrightarrow & Q_{X/S}(R^{\mathrm{sh}}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Pic}^{[0]}(X_{K^{\mathrm{sh}}}) & \longrightarrow & \mathrm{Pic}^{[0]}_{X_K/K}(K^{\mathrm{sh}}) & \longrightarrow & Q_{X/S}(K^{\mathrm{sh}}).
\end{array}$$

The bottom left horizontal map is surjective thanks to our assumption that either k is perfect or X admits an étale quasi-section. Furthermore, by the regularity of X, we can extend line bundles on $X_{K^{\mathrm{sh}}}$ to $X_{R^{\mathrm{sh}}}$, so the left vertical map is surjective. It follows that the middle vertical map is surjective. Finally, the bottom right horizontal map is an isomorphism by the definition of $Q_{X/S}$. This finishes the proof of Theorem 1.2. □

Proof of Theorem 1.3 (sketch). For this we will use the following result.

Theorem 6.2 (Raynaud; see [1, § 9.4, Theorem 2]). *Let S be the spectrum of a discrete valuation ring, and let $f\colon X \to S$ be a proper flat curve such that $f_*\mathcal{O}_X = \mathcal{O}_S$ and such that X is normal. If the greatest common divisor of the geometric multiplicities of the irreducible components of the special fibre of X equals 1, then $\mathrm{Pic}_{X/S}$ is an algebraic space over S and $\mathrm{Pic}^0_{X/S}$ is representable by a separated S-scheme.*

Since $\mathrm{Pic}^0_{X/S}$ is a separated S-scheme, the intersection of $\mathrm{Pic}^0_{X/S}$ with $E_{X/S}$ is trivial, so the quotient map $\mathrm{Pic}^{[0]}_{X/S} \to Q_{X/S}$ induces an open immersion $\mathrm{Pic}^0_{X/S} \to Q_{X/S}$. It follows that this is an isomorphism from $\mathrm{Pic}^0_{X/S}$ to the identity component of $Q_{X/S}$. □

References

[1] S. Bosch, W. Lütkebohmert and M. Raynaud, *Néron Models*. Springer-Verlag, Berlin/Heidelberg, 1990.

[2] R. Hartshorne, *Algebraic Geometry*. Springer-Verlag, New York, 1977.