These notes were prepared to accompany the first lecture in a seminar on logarithmic geometry. As we shall see in later lectures, logarithmic geometry offers a natural approach to study semistable schemes. By this, we mean proper, flat morphisms $f : X \to S$ whose fibres over geometric points $\bar{s} \in S$ are such that the singular points of $X_{\bar{s}}$ have local rings isomorphic to $\bar{k}[x_1, \ldots, x_n]/(x_1 \cdots x_s)$ for some $s \leq n$.

More generally, logarithmic geometry is useful for studying problems of compactification in algebraic geometry. Suppose we have a morphism $g : U \to T$ that is smooth, but not proper, and whose fibres satisfy certain desirable properties (such as being semistable varieties). As properness is often desirable, we might hope for a morphism $f : X \to S$ extending g, but such that f is proper and flat. To study g we are naturally lead to consider functions on X that restrict to invertible functions on U. This is not a sheaf of rings (consider f and $-f$), but it is a sheaf of monoids.

Logarithmic geometry will allow us to generalise this. We begin by studying monoids and their spectra. Note that these notes closely follow those of Ogus [1].

0.1 Spec of a monoid

We recall the definition of a monoid:

Definition 1. *A monoid is a triple $(M, *, e)$ where M is a set, $*$ is an associative binary operation on M, and e is a 2-sided identity element of M.*

Hence monoids are "almost" groups, albeit we cannot always invert elements. As with groups, we shall often write the operation $*$ as additive of multiplicative, depending on which is more convenient. All monoids considered will be commutative. We state without proof the following, though a proof can be found in Chapter 1.1 of Ogus [1].

Fact 1. *Arbitrary limits and colimits exist in the category of commutative monoids.*

In particular we may form fibred products and pushout diagrams.

Example 1. *Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers under addition. Then it is immediately verified that $\mathbb{N}$ is a monoid.*

There is an obvious way to embed $\mathbb{N}$ in the group $\mathbb{Z}$. It can be shown that this embedding is universal in the following sense: Given any morphism of monoids $f : \mathbb{N} \to M$ where M is a group, f factors uniquely as $\mathbb{N} \to \mathbb{Z} \to M$.

This generalises to arbitrary monoids.

Definition 2. *Let N be a monoid. The groupification of N, denoted N^{gp}, is the unique group, along with a morphism from $N \to N^{gp}$, such that any monoid morphism $f : N \to G$ where G is a group factors through N^{gp}.*

Note that the morphism from N to N^{gp} need not be injective as the example below from Gillam [2] illustrates.

Example 2. *Let $P_N = \{0, \cdots, N\}$, and define an operation on P_N via $a + b = max\{a + b, N\}$. Note that this makes P_N into a commutative monoid. We claim that $P_N^{gp} = \{0\}$.*

Indeed, note that given any monoid M and elements $a, b, m \in M$ such that $a + m = b + m$, we must have that the images of a and b in M^{gp} coincide. To see this, note that the image of m is invertible in M^{gp} by definition, and so the result follows by adding the inverse of m to both sides of the equality.

Thus, for any $a, b \in P_N$ we find that $a + N = b + N$, whence the image of of P in P^{gp} is the trivial subgroup. By the universal property of P_N^{gp} it follows that $P_N^{gp} = \{0\}$. In particular, the map from $P_N \to P_N^{gp}$ is not injective.

We can avoid such unintuitive scenarios by requiring that our monoids are integral, which we now define.

Definition 3. *Let M be a monoid. We say that M is integral if the cancellation law holds in M That is, given $m, m', m" \in M$, whenever*

$$m + m' = m + m",$$

we have that $m' = m"$.

Integral monoids shall play a large role in defining log structures on schemes. Of similar importance is that of the group of units of a monoid. Given a monoid M, the group of units of M, denoted by M^*, is the set $\{m \in M : \exists n \in M \text{ such that } m + n = 0\}$. The non-units are denoted by M^+, and the quotient M/M^* is denoted by $\bar{M}$. Note that there are no non-trivial units in $\bar{M}$.

Example 3. *Let k be any field, and let $R = k[[t]]$ be the ring of power series in variable t over k. Then R is a discrete valuation ring with maximal ideal t, and its multiplicative monoid has group of units isomorphic to $\{a + tf(t) : a \in k^*\}$. Furthermore, we have that $R/R^* \simeq \mathbb{N}$ as monoids.*

Definition 4. *Let M be a monoid. An ideal of M is a submonoid I such that whenever $n \in I$ and $m \in M$, we have $n + m \in I$. I is a prime if $I \neq M$ and whenever $n + m \in I$ we must have either $n \in I$ or $m \in I$. A face of a monoid is a submonoid F such that whenever $n + m \in F$ we have both n and m are in F.*

Remark 1. *Every prime ideal P defines a face $F_P = M \setminus P$. Indeed, suppose $m + n \in F_P$. If either or both of n and m are in P, then so too must their sum be in P. Finally, F_P is closed under the binary operation. If both n and m are in F, but $n + m \in P$, then as P is prime we would have either n or m is in P. Hence F_P is a face.*

Conversely, any face determines a unique prime ideal of M, taken to be the compliment of the face. Hence there is a bijection between the set of prime ideals and the set of faces of a given monoid.

Example 4. *Let $M = (\mathbb{Z}, *)$, and let p be any prime number. Then $p\mathbb{Z}$ is a prime ideal of the monoid M.*

Furthermore, the compliment in M of $p\mathbb{Z}$ is the set $(p\mathbb{Z})^c = \{a \in \mathbb{Z} : p \nmid a\}$. By unique factorisation we see that $ab \in (p\mathbb{Z})^c$ if and only if both a and b belong to the compliment, confirming that $(p\mathbb{Z})^c$ is a face of the monoid M.

Remark 2. *The group of units M^* is contained in every face of M, and is in fact the smallest face of M. Likewise, the compliment M^+ of the group of units is the unique maximal ideal of M.*

Definition 5. *Let M be a monoid. We denote by Spec(M) the set of prime ideals of M. For any ideal I of M, we denote by $Z(I)$ the set of prime ideals containing I.*

Lemma 1. *Given a monoid M, there is a natural topology on Spec(M) whose closed subsets are of the form $Z(I)$ for an ideal I.*

The proof of this is omitted. The resulting topological space is referred to as the Zariski topology. As in the case with spectra of rings, we find that the irreducible closed subsets are in bijection with prime ideals of M. Furthermore, given a homomorphism between two monoids, the inverse image of prime ideals and faces are prime ideals and faces, respectively, and so there is to any such homomorphism a resulting morphism of topological spaces.

Spectra shall be relevant later when we define charts of sheaves of monoids on a topological space.

Example 5. *Let $M = \mathbb{N}^2$. Let P be a non-empty prime ideal, so that there exists some element $(a, b) \in P$. Note we cannot have $a = b = 0$ as then $P = M$. Then $(a, b) = a(1, 0) + b(0, 1)$, and as P is prime we conclude that P must contain either $(1, 0)$ or $(0, 1)$ or both. We find that the unique closed point of Spec(M) is $M^+ =< (1, 0), (0, 1) >$ that is the intersection of the closed subsets associated to the prime ideals $< (0, 1) >$ and $< (1, 0) >$, respectively. The other closed point is defined by the trivial prime.*

Example 6. *Let P be any monoid, and let R be a ring. Define the ring $R[P]$ as the R-algebra whose underlying R-module is the free R-module with basis elements given by P, and whose multiplication law is the R-linear extension of $[a][b] = [a + b]$. $R[P]$ is called the R-monoid algebra of P.*

In particular, we have that $\mathbb{Z}[\mathbb{N}] \simeq \mathbb{Z}[t]$, where $t = [1]$. More generally, given any ring R we have that $R[\mathbb{N}^k] \simeq R[t_1, \cdots, t_k]$

0.2 Properties of morphisms of monoids

Let us state some properties of morphisms of monoids.

Definition 6. *A morphism of monoids $f : M \to N$ is*

- *local if $f^{-1}(N^*) = M^*$;*
- *sharp if the induced homomorphism $M^* \to N^*$ is an isomorphism;*
- *logarithmic if the induced homomorphism $f^{-1}(N^*) \to N^*$ is an isomorphism;*
- *strict if the induced homomorphism $\bar{f} : \bar{M} \to \bar{N}$ is an isomorphism.*

As suggested by the name, logarithmic homomorphisms are of special importance in logarithmic geometry. The following lemma helps paint a more intuitive picture of what it means to be logarithmic.

Lemma 2. *Let $f : M \to N$ be a homomorphism of monoids. Then f is logarithmic if and only if it is sharp and local.*

Proof. Suppose that f is logarithmic. Then $f^{-1}(N^*)$ is a group contained in M that necessarily contains $M*$. Hence it is equal to M^*, and so f is sharp and local.

Conversely, if f is sharp and local, then $f^{-1}(N^*) = M^*$ is isomorphic to $N*$, and so f is also logarithmic. □

Example 7. *Let A be a ring, and let $S \subset A$ be a multiplicative subset not containing 0. Let $M = \{a \in A : a \in (S^{-1}A)^*\}$. Then the natural inclusion $M \to A$, considered as a monoid homomorphism to the multiplicative monoid of A, is logarithmic.*

0.3 Sheaves of monoids and charts

Logarithmic schemes shall turn out to be monoidal spaces.

Definition 7. *A monoidal space* $(X, \mathcal{M}_X)$ *is a pair with* X *a topological space and* $\mathcal{M}_X$ *a sheaf of monoids on* X*. A morphism of monoidal spaces*

$$(f, f^\flat) : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$$

is a pair $(f, f^\flat)$ *where* $f : X \to Y$ *is a continuous map of topological spaces and*

$$f^\flat : f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$$

is a homomorphism of sheaves of monoids such that the induced morphism on stalks $f^\flat_x : \mathcal{M}_{Y,f(x)} \to \mathcal{M}_{X,x}$ *is a local morphism of monoids.*

Remark 3. *By functoriality, it is equivalent in the above definition to consider* $\mathcal{M}_Y \to f^\flat : f_*\mathcal{M}_X$*.*

Example 8. *Let* X *and* Y *be any two schemes with a morphism* $f : X \to Y$ *between them. Consider the sheaf of monoids* $\mathcal{M}_X = \mathcal{O}_X$ *on* X *and* $\mathcal{M}_Y = \mathcal{O}_Y$ *on* Y*. That is, we consider the sheaf of rings but only with the multiplicative structure. The assumption that* f *is a morphism of schemes ensures that the induced ring homomorphism*

$$f^\flat : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,X}$$

is a local homomorphism, i.e. $f^\flat(m_{Y,f(x)}) \subset m_{X,x}$*, and hence that the induced morphism of monoids is a local homomorphism of monoids.*

Example 9. *Recall that given any monoid* Q *we may define a topological space Spec*(Q)*. This has a natural structure of a monoidal space. A base of open sets can be taken to be of the form Spec*(Q_f)*, where* $f \in Q$ *is some element and* Q_f *is the localisation of* Q *at* f*. That is,* Q_f*, along with the natural homomorphism* $Q \to Q_f$*, satisfies the universal property that given any homomorphism from* $Q \to M$ *with the image of* f *invertible, this must factor uniquely through* $Q \to Q_f$*. When* Q *is integral,* Q_f *meets our intuition in that it's elements are of the form* $q - f^r$*, with* r *some integer and* $q \in Q$*.*

A monoidal space isomorphic to Spec(Q) *for some monoid* Q *is said to be affine. If a monoidal space admits a cover by such affine open sets, we say that it is a monoscheme.*

As with locally ringed spaces, many properties of sheaves of monoids can be checked at the stalks. For example, we say a sheaf of monoids is integral if for every open set $U \subset X$ the sheaf $\mathcal{M}_X(U)$ is an integral monoid, and this is equivalent to the stalk $\mathcal{M}_{X,x}$ being an integral monoid for every $x \in X$. Similarly, morphisms of monoidal spaces are said to have property P of homomorphisms of monoids if the induced monoid homomorphism between stalks has property P for all points $x \in X$.

Definition 8. *A homomorphism of sheaves of monoids* $f : \mathcal{M} \to \mathcal{N}$ *on a topological space* X *is said to be logarithmic if the induced homomorphism* $f^{-1}(\mathcal{N}^*) \to \mathcal{N}^*$ *is an isomorphism. Here,* $f^{-1}(\mathcal{N}^*)(U) = f(U)^{-1}(\mathcal{N}(U)^*)$*.*

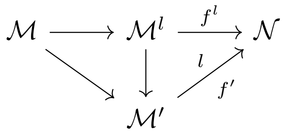
Remark 4. *That* $f^{-1}(\mathcal{N}^*)$ *is a sheaf of monoids follows from the assumption* $\mathcal{M}$ *is a sheaf of monoids and the fact* f *is a homomorphism of sheaves.*

In general, the morphisms between monoidal spaces in logarithmic geometry do not result in logarithmic homomorphisms of sheaves of monoids, and so we seek a way to factor an arbitrary homomorphism of sheaves of monoids through a logarithmic one. This is the content of the following lemma.

Lemma 3 (Proposition 2.1.1.5 of Ogus [1])**.** *Let* $f : \mathcal{M} \to \mathcal{N}$ *be a homomorphism of sheaves on monoids on a topological space* X*, and let* l *be one of the following properties: local, sharp, logarithmic. Then there exists a factorisation of* f

$$f = \mathcal{M} \to \mathcal{M}^l \xrightarrow{f^l} \mathcal{N},$$

where f^l *has property* l *and satisfies the following universal property: Given any factorisation of* f *as* $f = \mathcal{M} \to \mathcal{M}' \xrightarrow{f'} \mathcal{N}$ *where* f' *has property* l*, there exists a unique homomorphism* $\mathcal{M}^l \to \mathcal{M}'$ *that makes all triangles in the following diagram commute:*



The final concept needed to define logarithmic structures is that of charts and coherence. Given a sheaf of monoids $\mathcal{M}$ on a topological space X, a chart for $\mathcal{M}$ is a way of describing $\mathcal{M}$ via a constant sheaf of monoids. If Q is any monoid, let $\mathcal{Q}$ denote the constant sheaf of monoids whose sections on any open set are equal to Q. The following lemma has a scheme-theoretic analogue, and we state it without proof.

Lemma 4. *Let $(X, \mathcal{M}_X)$ be a monoidal space, and let Q be a monoid. Then there is a natural map*

$$Mor(X, Spec(Q)) \to Hom(Q, \Gamma(X, \mathcal{M}_X)),$$

and in fact this is an isomorphism.

Hence to give a homomorphism from Q to $\mathcal{M}_X$ (or equivalently from $\mathcal{Q} \to \mathcal{M}$) is equivalent to giving a morphism of monoidal spaces from $X \to \mathrm{Spec}(Q)$.

Definition 9. *Let $\mathcal{M}$ be a sheaf of monoids on a topological space X and let Q be a monoid. A chart for $\mathcal{M}$ subordinate to Q is a monoid homomorphism $f : Q \to \Gamma(X, \mathcal{M})$ and such that the associated logarithmic map $f^{log} : \mathcal{Q}^{log} \to \mathcal{M}$ is an isomorphism. The chart is coherent if Q is a finitely generated monoid.*

At times it is convenient to replace $\mathrm{Spec}(Q)$ with a *monoscheme*. The following lemma allows us to make sense of this.

Lemma 5 (Proposition 2.1.1.2 of Ogus [1])**.** *Let Q be a monoid, $(X, \mathcal{M})$ a monoidal space, and suppose we have a homomorphism of monoids $f^\flat : Q \to \mathcal{M}$ corresponding to a morphism of monoidal spaces*

$$f : X \to S = Spec(Q).$$

Then every point $x \in X$ gives a natural isomorphism

$$\mathcal{Q}_x^{log} \to f_{log}^*(\mathcal{M}_{S,f(x)}),$$

where $f_{log}^*(\mathcal{M}_{S,f(x)})$ is the unique monoid as in Lemma 3 associated to the homomorphism $f^{-1}(\mathcal{M}_{S,f(x)}) \to \mathcal{M}_x$. Furthermore, the following are equivalent:

- *$f^\flat$ is a chart for $\mathcal{M}$.*
- *For every $x \in X$, the homomorphism $\mathcal{Q}_x^{log} \to \mathcal{M}_x$ induced from the inclusion $\mathcal{M}_x^* \to \mathcal{M}$ and $f_x^\flat : Q \to \mathcal{M}_x$ is an isomorphism.*
- *The homomorphism $f_{log}^\flat : f_{log}^*(\mathcal{M}_S) \to \mathcal{M}_X$ is an isomorphism.*

Example 10. *Let Q be any monoid. Form the monoidal space $Spec(Q)$, where $Q \to \mathcal{O}_{Spec(Q)}$ corresponds to the identity morphism on $Spec(Q)$. This trivially satisfies the condition on the induced homomorphism of stalks, and so we also have a chart.*

Now consider $Spec(\mathbb{Z}[Q])$. There is a natural injection of Q into the underlying ring that sends an element $q \to 1[q]$. Once again we find that Q is a global chart for this monoidal space.

This lemma allows us to generalise the notion of a chart by replacing $\mathrm{Spec}(Q)$ with an arbitrary monoscheme, viz. a chart for $\mathcal{M}_X$ is a morphism $f : X \to S$ with S a monoscheme such that the associated homomorphism $f_{log}^\flat : f_{log}^*(\mathcal{M}_S) \to \mathcal{M}_X$ is an isomorphism.

Definition 10. *Let $\mathcal{M}$ be a sheaf of monoids on a topological space X. Then $\mathcal{M}$ is coherent (resp. quasi-coherent) if there exists an open cover $\mathcal{U}$ of X such that the restriction of $\mathcal{M}$ to each $U \in \mathcal{U}$ admits a chart (resp. a chart subordinate to a finitely generated monoid). If $\mathcal{M}$ is both coherent and integral, we say it is fine.*

0.4 Logarithmic schemes

We arrive at last at the definition of logarithmic structures.

Definition 11. *Let $(X, \mathcal{O}_X)$ be a scheme. A prelogarithmic structure on X is a homomorphism of sheaves of monoids $\alpha : \mathcal{P} \to \mathcal{O}_X$ on either $X_{ét}$ or X_{zar}. A logarithmic structure is a prelogarithmic structure such that $\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$ is an isomorphism. Given two logarithmic structures $\alpha : \mathcal{M} \to \mathcal{O}_X$ and $\alpha' : \mathcal{M}' \to \mathcal{O}_X$, a morphism from α to α' is a homomorphism $\theta : \mathcal{M} \to \mathcal{M}'$ such that $\alpha' \circ \theta = \alpha$.*

Example 11. *Let us return to looking at a previous example, where we considered a ring A, a multiplicative subset S of A, and let $M = \{a \in A : a \in (S^{-1}A)^*\}$. Suppose that T is a multiplicative subset of A containing S, and let $N = \{a \in A : a \in (T^{-1}A)^*\}$. Then the inclusions of M and N into A are logarithmic homomorphisms of monoids and so provide a logarithmic structures on Spec(A). Furthermore, the inclusion of $S \to T$ is then a morphism of logarithmic structures.*

Remark 5. *The "logarithmic" part of a logarithmic structure can be seen as follows: Given a logarithmic structure $\alpha : \mathcal{M} \to \mathcal{O}_X$, we have an isomorphism $\mathcal{M}^*$ with $\mathcal{O}_X^*$. Given units u, v of $\mathcal{O}$, we then have $\alpha^{-1}(uv) = \alpha^{-1}(u) + \alpha^{-1}(v)$ (here we are using the additive notation for $\mathcal{M}$). Hence $\alpha^{-1}(u)$ can be thought of as the logarithm of u, and given any element $f \in \mathcal{O}_X$, $\alpha^{-1}(f)$ is the possible empty set of logarithms of f.*

Remark 6. *We have seen in an earlier example that the inclusion $\mathcal{O}_X^* \to \mathcal{O}$ makes any scheme a monoidal space. In fact, it is easily checked that this equips any scheme with a logarithmic structure. It is referred to as the trivial log structure, and is in fact the initial object in **Log**$_X$, the category of logarithmic structures on X.*

Given a morphism of schemes, we may form the direct image and inverse image log structures, respectively.

Lemma 6. *Let $f : X \to Y$ be a morphism of schemes.*

1. *Let $\alpha : \mathcal{M} \to \mathcal{O}_X$ be a log structure on X, and form the Cartesian diagram:*

$$\begin{array}{ccc} f_*^{log}(\mathcal{M}) & \xrightarrow{f_*^{log}(\alpha)} & \mathcal{O}_Y \\ \downarrow & & \downarrow \\ f_*(\mathcal{M}) & \xrightarrow{f_*(\alpha)} & f_*(\mathcal{O}_X) \end{array}$$

   Then $f_*^{log}(\alpha)$ is a log structure on Y, and

$$f_*^{log} : \textbf{Log}_X \to \textbf{Log}_Y$$

   is a functor from the category of log structures on Y to those on X.

2. *Let $\alpha : \mathcal{N} \to \mathcal{O}_Y$ be a log structure on Y, and let*

$$f_{log}^*(\alpha) : f_{log}^*(\mathcal{N}) \to \mathcal{O}_X$$

be the log structure associated to the homomorphism of sheaves of monoids from $f^{-1}(\mathcal{N}) \to \mathcal{O}_X$. Then

$$f^*_{log} : \textbf{\textit{Log}}_Y \to \textbf{\textit{Log}}_X$$

is a functor from the category of log structures on Y to those on X, and is the left adjoint to f^{log}_*.

The section following this features an extended example of log structures in the form of DF structures.

Definition 12. *A log scheme $(X, \mathcal{O}_X)$ is a scheme together with a log structure $\alpha : \mathcal{M} \to \mathcal{O}_X$. A morphism of log schemes $f : X \to Y$ is a morphism of the underling schemes along with a homomorphism $f^\flat : \mathcal{M}_Y \to f_*(\mathcal{M}_X)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M}_Y & \xrightarrow{f^\flat} & f_*(\mathcal{M}_X) \\ \downarrow{\scriptstyle \alpha_Y} & & \downarrow{\scriptstyle f_*(\alpha_X)} \\ \mathcal{O}_Y & \xrightarrow{f^\sharp} & f_*(\mathcal{O}_X) \end{array}$$

Example 12. *The previous example of the union of two axis in affine space is a logarithmic scheme, with the log structure as given.*

Example 13. *Let $P \to A$ be a homomorphism from a monoid P to the multiplicative monoid of a ring A. Such a homomorphism is called a log ring, and we can form a log scheme Spec$(P \to A)$ whose underlying scheme is Spec(A) and whose log structure is that associated to the prelog structure $P \to \mathcal{O}_A$.*

We conclude by relating locally monoidal spaces to log schemes, as well as looking at how a chart on a log scheme is the same as a morphism of locally monoidal spaces.

Lemma 7 (Proposition 3.1.2.4 of Ogus [1])**.** *Let X be a log scheme, Q a monoid, and $S =$ Spec(Q) a monoidal space. The following are equivalent:*

- *A monoid homomorphism $\alpha : Q \to \Gamma(X, \mathcal{M}_X)$.*
- *A morphism of locally monoidal spaces $a : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$*
- *A morphism of log schemes $f : X \to A_Q$, where $A_Q =$ Spec$(P \to \mathbb{Z}[P]$*

Furthermore, α is a chart for $\mathcal{M}_X$ if and only if f satisfies $f^*_{log}(\mathcal{M}_{A_Q}) \to \mathcal{M}_X$ is an isomorphism. In this case we say that f is strict, and any of the above 3 conditions is said to be a chart for $\mathcal{M}_X$.

0.5 Extended example: DF Structures and Normal-crossing divisors

We conclude by looking at a family of examples of log structures. In this section, X will be used to denote a scheme. DF structures, once defined, will be shown to yield log structures on an arbitrary scheme X that admit particularly nice charts. Normal crossing divisors will be shown to be a specific case of DF structures.

Definition 13. *A DF structure on X is a finite sequence $\gamma_.$ of homomorphisms $\gamma_i : \mathcal{L}_i \to \mathcal{O}_X$, where $\mathcal{L}_i$ is an invertible sheaf on X.*

To a DF structure we now associate a sheaf of monoids $\mathcal{P}$. On an open set U of X, the sections of $\mathcal{P}$ are of the form (a, I), where $I = (I_1, \ldots, I_n) \in \mathbb{N}^n$ is a multi-index and $a \in \mathcal{L}_1^{I_1} \otimes \cdots \otimes \mathcal{L}_n^{I_n}(U)$. The monoid structure is defined via tensor product. We then define a map $\gamma : \mathcal{P} \to \mathcal{O}_X$ by sending $(a, I) \to \gamma^I(a)$, where $\gamma^I = \gamma_1^{I_1} \otimes \cdots \otimes \gamma_n^{I_n} : \mathcal{L}^I \to \mathcal{O}_X$. This gives a prelog structure whose associated log structure is $\mathcal{M}$.

Remark 7. *Note that the injection $\mathcal{O}_X^* \to \mathcal{P} : u \to (u, 0)$ identifies $\mathcal{O}_X^*$ with the sheaf of units of $\mathcal{P}$, essentially by the definition of $\mathcal{P}$.*

We'll see an example shortly in the form of normal crossing divisors, but first let us define a chart for this log structure. For each i, the image $\mathcal{I}_i$ of γ_i is a quasi-coherent sheaf of ideals on X, hence it defines a closed subscheme Y_i of X. Let $p : \mathcal{P} \to \mathbb{N}^n$ denote the projection onto the second factor. Because the $\mathcal{L}_i$ are invertible, on any small enough open set U we can choose a choice of generators a_i of $\mathcal{L}_i$ and define a splitting of p by sending $I \to (a_1^{I_1} \cdots a_n^{I_n}, I)$. Hence we have an exact sequence

$$0 \to \mathcal{O}_X^* \to \mathcal{P} \to \mathbb{N}^n \to 0,$$

whence $\overline{\mathcal{P}} \cong \mathcal{N}^n$ and so the chart $\beta : \mathcal{P} \to \mathcal{M}$ defines a homomorphism $\bar{\beta} : \mathbb{N}^n \to \overline{\mathcal{M}}$.

Lemma 8. *The map $\bar{\beta} : \mathcal{N}^n \to \overline{\mathcal{M}}$ lifts locally to a chart $\mathbb{N}^n \to \mathcal{M}$.*

Proof. Let U be any open set on which the $\mathcal{L}_i$ are free, and let $\sigma : \mathcal{N}^n \to \mathcal{P}$ denote the splitting defined above. The composition $\beta \circ \sigma : \mathbb{N}^n \to \mathcal{M}$ lifts $\bar{\beta}$ and is the chart we desire. □

We turn our attention now to divisors with normal crossings.

Definition 14. *Suppose X is a regular scheme. A divisor with normal crossings is a closed subscheme $Y \subset X$ such that the intersection of any set of irreducible components of Y is also regular. A scheme Y is a normal crossings scheme if for every $y \in Y$, there exists an étale neighbourhood U of y and a closed immersion identifying U with a divisor with strict normal crossings in a regular scheme X.*

Remark 8. *If the neighbourhood can be taken as a Zariski open neighbourhood, we say Y is a strict normal crossings scheme. we shall work with this case in this example, but many results hold true in the étale topology. See Chapter III.1.8 of Ogus [1] for details.*

Remark 9. *Y is a strict divisor with normal crossings in X if and only if for every point x of X, there exists a regular sequence $(t_1, \ldots, t_m)$ generating the maximal ideal of $\mathcal{O}_{X,x}$ and $r \in \mathbb{N}$ such that $t_1 \ldots t_r$ generates the ideal of Y in $\mathcal{O}_{X,x}$.*

Example 14. *Perhaps the simplest example of a strict normal crossing scheme is $Y = Spec(\mathbb{C}[x, y]/(xy))$, the union of the coordinate axes in the affine plane.*

Let Y be a strict divisor with normal crossings in X, and suppose its irreducible components Y_i are defined by ideals $\mathcal{I}_i$. The inclusions $\mathcal{I}_i \to \mathcal{O}_X$ defines a DF-structure on X. By combining the discussion of DF structures with this, we obtain

Lemma 9. *Let $Y \to X$ be a strict divisor with normal crossings in a regular scheme X, and γ the DF structure defined by the ideals of the irreducible components of Y in X. Let $\alpha : \mathcal{M} \to \mathcal{O}_X$ denote the corresponding log structure. Given a point $x \in X$, let $\{Y_1, \cdots, Y_r\}$ the set of irreducible components of Y containing x. Then in a neighbourhood of x we have a chart $\beta : \mathbb{N}^r \to \mathcal{M}$ such that $\alpha(\beta(e_i))$ is a generator of $\mathcal{I}_i$ for all i.*

Thus logarithmic geometry naturally lends itself to the study of such divisors. Later in the seminar we shall look at a paper by Kato [3] where logarithmic geometry applied in such a manner.

Bibliography

[1] Arthur Ogus *Lectures on Algebraic Logarithmic Geometry.* Preprint, 2017

[2] William Gillam *Log Geometry.* Online lecture notes, 2009

[3] Kazuya Kato *Logarithmic Structures of Fontaine-Illusie.* Algebraic Analysis, Geometry and Number Theory, 1989