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Tesi di Laurea

Differential Operators on the Noncommutative Torus

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Introduction

L'algèbre n'est qu'une géométrie écrite; la géométrie n'est qu'une algèbre figurée.

S. Germain

Noncommutative geometry finds its origin in one of the deepest ideas of mathematics: the one of duality between algebra and geometry. Even though NCG, as we intend it, is a pretty new theory, the concept of duality has been present in mathematics for a long time: one just needs to think about Cartesian geometry: every point in the space is uniquely determined by its coordinates, geometric objects are described using (algebraic) equations, and their properties are studied using techniques taken from both algebra and analysis.

However, it is in the XX century that the biggest progresses in the direction of NCG are done. One of the turning points is the mathematical formulation of quantum mechanics, done by Heisenberg in 1925. From a mathematical point of view, transition from classical to quantum mechanics is done, among other things, by passing from the *commutative algebra of classical observables* to the *noncommutative algebra of quantum mechanical observables.* In classical mechanics, an observable is described in terms of a real valued function f on a manifold (the phase space of the system). In quantum mechanics one replaces the algebra of real valued functions by the (noncommutative) algebra of self adjoint operators on a Hilbert space $\mathcal{H}$.

	classical mechanics	**quantum mechanics**
OBSERVABLE	$f : X \to \mathbb{R}$, continuous, X phase space	s.a. operator $A = A^*$ on a Hilbert Space $\mathcal{H}$
	commutative	non commutative

Moreover, by a celebrated theorem due to Gelfand and Naimark (cf. Theorem 3.3) one knows that the category of locally compact Hausdorff spaces is equivalent to the dual of the category of commutative C* algebras. So one could choose to consider noncommutative C* algebras as the dual of the category of some *noncommutative locally compact Hausdorff* spaces. This proposal turns out to be successful.

What one tipically does in noncommutative geometry is treating certain classes of noncommutative algebras as noncommutative spaces, and trying to extend tools of geometry, topology and analysis to this new setting.

The Noncommutative Torus. The noncommutative torus is one of the earliest and probably the most accessible and best studied example of noncommutative space. In the two dimensional case it is defined as the universal C* algebra generated by two unitaries subject to a certain commutation relation. It can be viewed as a deformation of the algebra of smooth functions on the 2 dimensional torus. Actually we are not dealing with a single algebra, but with a family of algebras parametrized by an irrational number $\theta \in [0, 1]$.

One can obviously study this space from different points of view. The starting point of our study is a paper published by Johnathan Roesemberg in 2008 ([**20**]), where the construction of a Laplace operator is pursued.

The outline of the work is as follows.

In Chapter 1 we introduce the 2-torus, which we mainly study from an analytical point of view in the sense of Fourier analysis and operator theory.

In Chapter 2 we develop some of the theory of C*-algebras, which are the main tools we will use to study noncommutative spaces.

In Chapter 3 we construct the noncommutative torus as a C*-algebra, starting from a certain dynamical system (the irrational rotations on the circle) and constructing the *crossed product* associated to this situation. We also prove and state results about the properties of this algebra.

In Chapter 4 we show how to construct differential operators on the noncommutative torus. We are interested in particular in analogues of the Laplace and Dirac operators.

Finally, Chapter 5 is devoted to conclusions and possible further developments.

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Francesca

Totally mystery in concept, however, for there was no way of describing what hyperspace was, unless one made use of mathematical symbols which could, in any case, not be translated into anything comprehensible.

I.Asimov

CHAPTER 1

Tori

Donuts. Is there anything they can't do?

Homer J. Simpson

There are plenty of motivations for the study of spaces like tori, coming from different areas of mathematics. From a geometrical point of view, the n-torus is one of the easiest examples of compact manifolds one could think of. Moreover, tori happen to have a group structure, which turns them into compact Lie groups. They are also deeply connected with geometric complex analysis and algebraic geometry, since they arise as elliptic curves on the complex plane.

In this section, we will investigate them from the point of view of harmonic analysis, which goes back to the work of Fourier on periodic functions. From a physical point of view, periodic functions are deeply related to waves and periodic signals. In particular, Fourier was interested in decomposing periodic vibrations in terms of sinusoidal functions. What is highly remarkable about this theory is that it is equivalent to consider periodic functions on whole of the plane, or just functions on the torus, which is a compact space: if we have a periodic function on the real line, we can equivalently consider our function to be defined on a circle of length equal to the period of the function we started from.

More precisely, one starts from the following equivalence relation on the real numbers: for every $x \in \mathbb{R}$,

$$x \sim x + 1. \tag{1}$$

This is equivalent to requiring that $x \sim x + n$ for every $n \in \mathbb{Z}$.

The additive structure on $\mathbb{R}$ descends to a well defined abelian group structure on the quotient set $\mathbb{R}/\sim = \mathbb{R}/\mathbb{Z}$. This set can be viewed as a circle by bending the segment $[0, 1]$, so that the end points coincide. This identification is actually an isomorphism between the additive group $\mathbb{R}/\mathbb{Z}$ and the circle group S^1, i.e. the multiplicative group of complex numbers of modulus one, which is realized via exponential function:

$$\mathbb{R}/\mathbb{Z} \ni [x] \to e^{2\pi i x} \in S^1, \tag{2}$$

where the correspondence is evidently well defined. Actually, this map is not only a group isomorphism, but rather a Lie group homomorphism.

A function on $\mathbb{R}$ is said to be *1-periodic* if $f(x+1) = f(x)$ for all $x \in \mathbb{R}$.

If F is a complex valued function on the unit circle S^1 and $f : \mathbb{R} \to \mathbb{C}$ is defined as

$$f(x) = F(e^{2\pi i x}) \tag{3}$$

for all $x \in \mathbb{R}$, then f is a 1-periodic function. Conversely, if f is a 1-periodic function on the real line, then there exists a function F on S^1 such that (3) holds.

1. The 2-torus

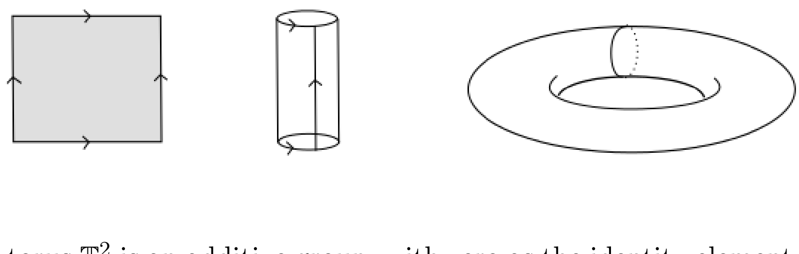
In complete analogy with the construction of the circle using the equivalence relation (1), we endow the space of vectors $(x, y) \in \mathbb{R}^2$ with the following equivalence relation:

$$(x, y) \sim (x+1, y) \sim (x, y+1). \tag{4}$$

The proof that $\sim$ is indeed an equivalence relation on $\mathbb{R}^2$ is straightforward. Moreover, we recall that every equivalence relation partitions the set on which it is defined into equivalence classes.

Definition 1.1. The 2-torus $\mathbb{T}^2$ is defined as the quotient set $\mathbb{R}^2/\sim$ of equivalence classes with respect to the equivalence relation (4).

From a geometrical point of view, this identification brings together the left and right side of the square $[0, 1]^2$, and the top and bottom sides as well. The resulting figure is a 2-dimensional manifold which, if embedded in $\mathbb{R}^3$, looks like a donut.



The 2-torus $\mathbb{T}^2$ is an additive group, with zero as the identity element. If we look at it as the square $[0, 1]^2$, we have multiple elements belonging to the equivalence class of the the zero element ($\mathbf{0} = (0,0) \sim (0,1) \sim (1,0) \sim (1,1)$). If one wants to avoid this appearance of multiple identity elements, one can also think of the torus as the square $\left[-\frac{1}{2}, \frac{1}{2}\right]^2$, where, again, the endpoints are identified.

However, in some situations it is convenient to use exponential notation, and think of the 2-torus as the following subset of $\mathbb{C}^2$

$$\{(e^{2\pi i x}, e^{2\pi i y}) | (x,y) \in [0,1]^n\} \tag{5}$$

in the same way as the unit interval $[0,1]$ can be thought of as the unit circle in $\mathbb{C}$ once 1 and 0 are identified.

The definition of the torus as a quotient set can be easily generalized to arbitrary dimensions.

2. Fourier Analysis

Roughly speaking, the purpose of this section is to find a way of representing a periodic function as a sum of simpler periodic functions, namely sinusoidal ones. From a physical point of view, this is deeply related to the possibility of representing a tone in terms of harmonics.

In the same way as seen in equation (3), one can identify every function F on $\mathbb{T}^2$ with a function f on $\mathbb{R}^2$:

$$f(x,y) = F(e^{2\pi i x}, e^{2\pi i y}) \tag{6}$$

with the property that $f(x,y) = f(x+1,y) = f(x,y+1) \quad \forall (x,y) \in \mathbb{R}^2$. Such a function f is said to be *1-periodic* in every coordinate. The converse is also true.

On the 2-torus one can naturally define a measure by restriction of the 2 dimensional Lebesgue measure. Translation invariance of the Lebesgue measure and the periodicity of functions on $\mathbb{T}^2$ imply that for all $f : \mathbb{T}^2 \to \mathbb{C}$ we have

$$\int_{\mathbb{T}^2} f(x,y)dxdy = \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^2} f(x,y)dxdy = \int_{[a,a+1]\times[b,b+1]} f(x,y)dxdy \tag{7}$$

for any $(a,b) \in \mathbb{R}^2$. The integral is well defined.

With these conventions in mind, we define for $1 \leq p < \infty$ the linear space $L^p(\mathbb{T}^2)$ of complex, Lebesgue measurable 1-periodic (in each variable) functions on $\mathbb{R}^2$ for which the norm

$$\|f\|_p = \left(\int_{\mathbb{T}^2} |f(x,y)|^p dxdy\right)^{\frac{1}{p}}. \tag{8}$$

is finite.

All $(L^p(\mathbb{T}^2), \|\cdot\|)$ are Banach spaces (see [**21**][chapter 3], for all the details).

We are mainly interested in two particular cases: $L^1(\mathbb{T}^2)$, the class of absolutely integrable functions, and $L^2(\mathbb{T}^2)$, the class of square integrable functions.

On $L^2(\mathbb{T}^2)$ we can define an inner product

$$\langle f, g\rangle = \int_{\mathbb{T}^2} f(x,y)\overline{g(x,y)}dxdy, \tag{9}$$

which turns $L^2(\mathbb{T}^2)$ into a Hilbert space.

Definition 2.1. A *trigonometric polynomial* on $\mathbb{T}^2$ is an element of the complex linear span of

$$\{\exp(2\pi i(mx+ny)) : (m,n) \in \mathbb{Z}^2, (x,y) \in \mathbb{R}^2\}, \tag{10}$$

hence a function of the form

$$P(x,y) = \sum_{m,n\in\mathbb{Z}} a_{m,n} e^{2\pi i(mx+ny)} \tag{11}$$

where $\{a_{m,n}\}_{(m,n)\in\mathbb{Z}^2}$ is a finite supported sequence, i.e. $a_{m,n} \neq 0$ for a finite number of $(m,n) \in \mathbb{Z}^2$. The degree of P is the largest number $|p| + |q|$ such that $a_{p,q}$ is non zero.

We point out that, since $\exp(a+b) = \exp(a)\exp(b)$, the product of two trigonometric polynomials is still a trigonometric polynomial.

Proposition 2.1. *The set of trigonometric polynomials is dense in* $L^2(\mathbb{T}^n)$ [1].

Proof Sketch: We sketch the proof in dimension 1. The proof in higher dimension is a straightforward generalization. Since $\mathbb{T}$ is compact, $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$. Therefore the proof reduces to showing that to every function $f \in C(\mathbb{T})$, and to every $\varepsilon > 0$ there exists a trigonometric polynomial P such that $\|f - P\|_2 < \varepsilon$. In order to do this, we need to construct a sequence Q_k of trigonometric polynomials with the following properties:

(1) $Q_k(t) \geq 0 \quad$ for all $t \in \mathbb{T}$,
(2) $\int_{\mathbb{T}} Q_h(t)dt = 1$
(3) for every $\delta > 0$, $Q_k(t) \to 0$ uniformly on $[-\pi, -\delta] \cup [\delta, \pi]$ as $k \to \infty$.

The construction of the Q_k's can be done in many ways. One possible choice is defining them as

$$Q_k(t) = c_k \left(\frac{1+\cos t}{2}\right)^k \tag{12}$$

[1] The theorem actually holds for every $L^p(\mathbb{T}^n)$, with $1 \leq p < \infty$.

where c_k are chosen so that the normalization in (2) holds. It can be shown by explicit computations, that the properties (1) – (3) hold.
Once we have constructed the Q_k's, we can associate to each $f \in C(\mathbb{T})$ the functions P_k defined by

$$P_k(t) = \int_{\mathbb{T}} f(t)Q_k(z-t)dt. \tag{13}$$

From translation invariance of the measure and from the fact that Q_k is a trigonometric polynomial, we get that P_k is actually a trigonometric polynomial as well. Using the defining properties of the Q_k's, one can be shown that the P_k's approximate the function f in the L^2 norm, which ends the proof.
For all the details, we refer to [**21**], 4.24. □

2.1. Fourier Coefficients.

Definition 2.2. For a complex valued function $f \in L^1(\mathbb{T}^2)$ and for any $\mathbf{k} = (m,n) \in \mathbb{Z}^2$, we define the $\mathbf{k}$–th *Fourier coefficient* of f as

$$\widehat{f}(\mathbf{k}) = \int_{\mathbb{T}^2} f(x,y)e^{-2\pi i(mx+ny)}dxdy. \tag{14}$$

Definition 2.3. The *Fourier Series* of f at $x \in \mathbb{T}^2$ is the formal series

$$\sum_{\mathbf{k}=(m,n)\in\mathbb{Z}^2} \widehat{f}(\mathbf{k})e^{2\pi i(mx+ny)} \tag{15}$$

A fundamental problem in harmonic analysis is whether the partial sums of the Fourier series of a function f converge back to the function as $N \to \infty$.
We have a first result for continuously differentiable functions:

Theorem 2.2. *Let f be a continuously differentiable function on $\mathbb{T}^2$. Then the partial sums of the Fourier series of f converge uniformly to f.*

Proof. We refer to [**16**], theorem 2.8, where the result is stated in the one dimensional case. The generalization to dimension two is straightforward. □

This theorem gives us a sufficient condition for the Fourier series of a function to converge back uniformly to the function we started from, but finding the exact class of functions whose Fourier series converge uniformly is rather complicated. The same also if we look for functions whose Fourier series converge point-wise.

However, we can get an answer to this question by changing the notion of convergence and turn our attention to the L^2 case. In order to do this, we need some facts about Hilbert spaces.

Definition 2.4 (Complete Orthonormal System). Let $\mathcal{H}$ be a separable Hilbert space with complex inner product $\langle \cdot | \cdot \rangle$. An *orthonormal system* for $\mathcal{H}$ is a subset $\{\varphi_k\}_{k\in K}$ of $\mathcal{H}$ such that

(1) $\langle \varphi_k | \varphi_j \rangle = 0$ for all $k \neq j$;
(2) $\langle \varphi_k | \varphi_k \rangle = 1$ for all k.

Moreover, if the linear span of the family φ_k is dense in $\mathcal{H}$, the family $\{\varphi_k\}_{k\in K}$ is called a *complete* orthonormal system.

Definition 2.5 (Countable direct sum). Suppose $\{Y_h\}_{h=0}^{\infty}$ is a sequence of Hilbert spaces. Let $\mathcal{H}$ denote the set of sequences $\{y_h\}_{h=0}^{\infty}$, with $y_h \in Y_h$, which satisfy

$$\sum_{h=0}^{\infty} \|y_h\|_{Y_h}^2 < \infty. \tag{16}$$

Then $\mathcal{H}$ is a Hilbert space under the natural inner product and it is denoted by

$$\mathcal{H} = \bigoplus_{h=0}^{\infty} Y_h. \tag{17}$$

Proposition 2.3 ([**16**], 2.6). *Let $\mathcal{H}$ be a separable Hilbert space and let $\{\varphi_k\}_{k\in\mathbb{Z}}$ be a complete orthonormal system for $\mathcal{H}$. Then:*

(1) *For every $f \in \mathcal{H}$ we have*

$$\|f\|_{\mathcal{H}}^2 = \sum_{k\in\mathbb{Z}} |\langle f | \varphi_k \rangle|^2. \tag{18}$$

(2) *For every $f \in \mathcal{H}$, f is equal to the limit, with respect to the $\mathcal{H}$-norm, of the formal series:*

$$\sum_k \langle f | \varphi_k \rangle \varphi_k, \tag{19}$$

whenever the series converges.

The number $\langle f | \varphi_k \rangle$ is called the *k-th Fourier coefficient* of the function f with respect to the basis $\{\varphi_k\}$.

In particular, we want to apply this whole setting to the case of the Hilbert space $\mathcal{H} = L^2(\mathbb{T}^2)$, with inner product the one defined in equation (9).

Theorem 2.4. *The set*

$$\{(x, y) \mapsto \exp(2\pi i(mx + ny)) \mid k = (m, n) \in \mathbb{Z}^2\} \tag{20}$$

is a countable complete orthonormal system for $L^2(\mathbb{T}^2)$

PROOF. Orthonormality is a consequence of the following identity:

$$\int_{\mathbb{T}^2} e^{2\pi i(mx+ny)}\overline{e^{2\pi i(m'x+n'y)}} = \delta_m^{m'}\delta_n^{n'}. \tag{21}$$

Completeness is a consequence of (2.1). □

Using (2.4) we can restate proposition (2.3) in the case of L^2 periodic functions.

PROPOSITION 2.5. *Following facts hold for $f, g \in L^2(\mathbb{T}^2)$:*

(1) ***(Plancherel's identity):***

$$\|f\|_{L^2}^2 = \sum_{k\in\mathbb{Z}^2} |\widehat{f}(k)|^2. \tag{22}$$

(2) *The function $f(x,y)$ is almost everywhere equal to the L^2 limit of the series*

$$\sum_{|k|\leq N} \widehat{f}(k)e^{2\pi i(mx+ny)}. \tag{23}$$

(3) ***(Parseval's relation)***

$$\int_{\mathbb{T}^2} f(x,y)\overline{g(x,y)}dxdy = \sum_{k\in\mathbb{Z}^2} \widehat{f}(k)\overline{\widehat{g}(k)}. \tag{24}$$

(4) *The map*

$$f \mapsto \{\widehat{f}(k)\}_{k\in\mathbb{Z}^2}$$

is an isometry from $L^2(\mathbb{T}^2)$ onto ℓ^2.

This theorem shows us that the *natural* notion of convergence for Fourier series is the L^2 convergence: we have found a space which is appropriate to solve our problem. By choosing this space, we avoid many problems, however, we have both advantages and disadvantages, and there are other possible choices one could make.

If we consider C^∞ functions, we have a result about convergence. The topology is the so-called *Fréchet topology* i.e. the topology of point-wise convergence in all derivatives.

THEOREM 2.6 (Inversion for $C^\infty(\mathbb{T}^2)$, [**19**]5.4). *For $f \in C^\infty(\mathbb{T}^2)$, the Fourier series of f converges in the Fréchet C^∞ topology to f. The Fourier coefficients are* rapidly decreasing*: for any k there is a constant C_k such that*

$$\sup_{m,n\in\mathbb{Z}} (1+m^2+n^2)^k |f_{mn}|^2 < \infty. \tag{25}$$

2.2. Linear Operators on a Hilbert space. Now that we have a decomposition of L^2 functions in terms of Fourier series, we are ready to define differential operators on the 2-torus, more precisely on the Hilbert space $\mathcal{H} = L^2(\mathbb{T}^2)$. In particular we are interested in studying the behavior of first and second order differential operators. In order to do this, we start from the more general setting of linear operators on a Hilbert space.

Definition 2.6. Let $\mathcal{H}_1, \mathcal{H}_2$ be complex Hilbert spaces. A *linear operator* from $\mathcal{H}_1$ into $\mathcal{H}_2$ is a linear map $T : D(T) \to \mathcal{H}_2$ defined on a linear subspace $D(T)$ of $\mathcal{H}_1$. The set $D(T)$ is called the *domain* of the operator T.

Unless otherwise specified, we will always suppose $D(T)$ to be dense in $\mathcal{H}_1$.

In the case of finite dimensional Hilbert spaces, every linear operator is continuous. In the infinite dimensional case, this fact is, in general, not true.

Definition 2.7. A bounded linear operator between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is a linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$, for which there exists some $C \geq 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in X$.

The smallest such C is called the *operator norm* of T and it is given by

$$\|T\| = \sup_{x\in\mathcal{H}_1, x\neq 0} \frac{\|Tx\|}{\|x\|}. \tag{26}$$

The following geometric picture is helpful: a bounded linear operator maps the closed unit ball in $\mathcal{H}_1$ to the closed ball of radius $\|T\|$ in $\mathcal{H}_2$.

Theorem 2.7. *Let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear operator. The following statements are equivalent.*

1. *T is bounded.*
2. *T is continuous* [2].
3. *T is continuous in a point $x \in \mathcal{H}$.*

Proof. See [**21**][Theorem 5.4]. □

In the following, we will denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the vector space of densely defined linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ and by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. $B(\mathcal{H}_1, \mathcal{H}_2)$ is a normed vector space with respect to the operator norm (26). Moreover, since every Hilbert space is complete, $(B(\mathcal{H}_1, \mathcal{H}_2), \|\cdot\|)$ is a Banach space (see [**16**], Theorem 3.2).

[2] We mean continuous in the norm topology.

If we are dealing with a single Hilbert space $\mathcal{H}$, we can define the spaces $\mathcal{L}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$.

PROPOSITION 2.8. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined linear operator. If T is continuous, then T can be defined on all of $\mathcal{H}$.*

PROOF. By a density argument, we can extend $T : D(T) \to \mathcal{H}$ by continuity to an operator $T : \mathcal{H} \to \mathcal{H}$. □

This simple proposition implies, combined with (2.7), that every bounded operator is defined on all of $\mathcal{H}$.

2.3. Adjoint Operators.

2.3.1. *Bounded Operators.* To construct the adjoint of a bounded operator, we will use the following result.

THEOREM 2.9. *If $\lambda : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is sesquilinar and bounded, in the sense that*

$$M = \sup\{|\lambda(x, y)| : \|x\| = 1 = \|y\|\} < \infty, \tag{27}$$

then there exists a unique $S \in \mathcal{B}(\mathcal{H})$ such that

$$\lambda(x, y) = \langle x|Sy\rangle \tag{28}$$

for all $x, y \in \mathcal{H}$. Moreover, $\|S\| = M$.

PROOF. The proof uses Riesz' representation theorem. See [**22**], Theorem 12.8. □

THEOREM 2.10. *Let $T \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator. Then there exists a unique $T^* \in \mathcal{B}(\mathcal{H})$ that satisfies*

$$\langle T^*x, y\rangle = \langle x, Ty\rangle. \tag{29}$$

PROOF. Let $T \in \mathcal{B}(\mathcal{H})$. The map $\mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that sends

$$(x, y) \mapsto \langle Tx|y\rangle \tag{30}$$

is linear in the first argument, adjoint linear in the second, and bounded, since T is bounded. Then by (2.9) there exists a unique $T^* \in \mathcal{B}(\mathcal{H})$ such that

$$\langle x|T^*y\rangle = \langle Tx|y\rangle \tag{31}$$

for all $x, y \in \mathcal{H}$. Moreover we have $\|T^*\| = \|T\|$.

DEFINITION 2.8. (1) T^* is called the *Hilbert space adjoint of T*.

(2) A bounded operator $T \in \mathcal{B}(\mathcal{H})$ is called *self-adjoint* if $T^* = T$.
(3) It is called *normal* if $T^*T = TT^*$.

2.4. Unbounded Operators. Many important operators which occur in mathematical physics are not bounded, the Laplace operator on $L^2(\mathbb{T}^2)$ belongs to this class. Since unbounded operators are only defined on a dense subset of a Hilbert space, one needs to be very careful in defining the domain. Indeed, the construction of the adjoint of an operator is not so straightforward as in the bounded case.

DEFINITION 2.9. The *graph* of a linear operator $T : D(T) \to \mathcal{H}$ is the set of pairs

$$\{(\varphi, T\varphi) \mid \varphi \in D(T)\}, \tag{32}$$

and it is denoted by $\Gamma(T)$. $\Gamma(T)$ is a subset of $\mathcal{H} \oplus \mathcal{H}$, which is a Hilbert space with inner product

$$\langle(\varphi_1, \psi_1), (\varphi_2, \psi_2)\rangle = \langle\varphi_1, \varphi_2\rangle + \langle\psi_1, \psi_2\rangle. \tag{33}$$

T is called a *closed* operator if $\Gamma(T)$ is a closed subset of $\mathcal{H} \times \mathcal{H}$.

DEFINITION 2.10. Let T be a densely defined operator on a Hilbert space $\mathcal{H}$. Let $D(T^*)$ be the set of $y \in \mathcal{H}$ for which there is a $x \in \mathcal{H}$ with

$$\langle Tv, y\rangle = \langle v, x\rangle \quad \text{for all} \quad v \in D(T). \tag{34}$$

For any such $y \in \mathcal{H}$ we define $T^*y = x$. T^* is called the *adjoint of* T.

DEFINITION 2.11. (1) A densely defined operator T on a Hilbert space $\mathcal{H}$ is called *symmetric* if and only if

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \text{for all} \quad x, y \in D(T). \tag{35}$$

(2) T is called *self-adjoint* if $T = T^*$, that is, if and only if T is symmetric and $D(T) = D(T^*)$.
(3) T is called *normal* if T has closed graph, and $L^*L = LL^*$-which clearly implies that $D(L^*L) = D(LL^*)$.

EXAMPLE 2.1. Let $\mathcal{H} = L^2([0, 1])$, and $T : D(T) \to \mathcal{H}$ the operator defined by

$$(Tf)(x) = -i\frac{df}{dx}(x) = -if'(x), \tag{36}$$

with $D(T) = \{f \in \mathcal{H}, f \text{ is differentiable}, f' \in \mathcal{H} \text{ and } f(0) = 0 = f(1)\} \subset \mathcal{H}$.

We claim that this operator is unbounded. Indeed, let us consider the sequence $f_n(x) = \sin(2\pi nx)$ in $\mathcal{H}$. Then

$$\|f_n\|^2 = \int_0^1 |\sin(2\pi nx)|^2 dx = \frac{1}{2} \tag{37}$$

and

$$\|Df_n\|^2 = \int_0^1 |Df_n|^2 = (2\pi n)^2 \int_0^1 |\cos(2\pi nx)|^2 dx = 2(\pi n)^2. \tag{38}$$

Therefore

$$\frac{\|Df_n\|}{\|f_n\|} \to \infty \quad \text{as} \quad n \to \infty. \tag{39}$$

Moreover, the operator is symmetric: for every $f, g \in D(T)$ we have

$$\begin{aligned} \langle Tf, g\rangle = -i \int \overline{f'(x)} g(x) dx = -i \left[\overline{f(x)} g(x)\right]_0^1 + i \int \overline{f(x)} g'(x) dx = \\ -i \int f(x) \overline{g'(x)} dx = \langle f, T^* g\rangle, \end{aligned} \tag{40}$$

but not selfadjoint, since

$$D(T^*) = \{f \in \mathcal{H} \text{ is differentiable}, f' \in \mathcal{H}\} \supset D(T). \tag{41}$$

However, we can define a new operator $\mathcal{T}$ imposing periodicity boundary conditions on the domain:

$$D(\mathcal{T}) = \{f \in L^2(\mathbb{T}), f \text{ is differentiable and } \; f' \in L^2(\mathbb{T})\}, \tag{42}$$

so that $\mathcal{T}$ is not only symmetric, but also self-adjoint.

3. Spectral Theorems

Roughly speaking, spectral theorems establish conditions under which a linear operator can be decomposed in the sum of simpler operators. This allows us to generalize the problem of diagonalizing a finite dimensional matrix to the case of Hilbert spaces. For this subject we refer to [**16**], Chapters 7, 8 and [**22**], Chapters 12, 13. The formulation of the spectral theorems as presented here can be found in [**6**].

DEFINITION 3.1. Let $T \in \mathcal{B}(\mathcal{H})$. A complex number λ is said to be in the *resolvent set* $\rho(T)$ of T if $(\lambda\mathbb{I} - T)$ is a bijection. If $\lambda \notin \rho(T)$ then λ is said to be in the *spectrum* $\sigma(T)$ of T.

Observe that if $T \in \mathcal{B}(\mathcal{H})$, then automatically $(\lambda\mathbb{I} - T)^{-1} \in \mathcal{B}(\mathcal{H})$ [3].

DEFINITION 3.2. Let $T \in \mathcal{B}(\mathcal{H})$.

(1) An $x \neq 0$ which satisfies $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$ is called an *eigenvector* of T; λ is called the corresponding *eigenvalue*.
If λ is an eigenvalue, then $(\lambda I - T)$ is not injective, so λ is in the spectrum of T.
The set of all eigenvalues of T is called the *point spectrum of* T and it is denoted by $\sigma_p(T)$.
(2) If $\lambda \in \sigma(T)$ is not an eigenvalue and if the image of $\lambda\mathbb{I} - T$ is not dense, then λ is said to be in the *residual spectrum of* T, and we write $\lambda \in \sigma_r(T)$.

DEFINITION 3.3. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. An operator $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is said to be *compact* if for any bounded sequence $\{v_i\} \in X$ $\{Tv_i\}$ has a convergent subsequence.

THEOREM 3.1 (Spectral Theorem for Bounded Operators). *Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and $T \in \mathcal{B}(\mathcal{H})$ a compact and normal operator. Then*

(1) *If $\sigma_p(T)$ is finite and $Y_h = \mathrm{Ker}(\lambda_h\mathbb{I} - T)$, where $\sigma_p(T) = \{\lambda_1, \ldots, \lambda_k\}$, then*

$$\mathcal{H} = \bigoplus_{h=0}^{k} Y_h \tag{43}$$

(2) *If $\sigma_p(T)$ is countable and $Y_h = \mathrm{Ker}(\lambda_h\mathbb{I} - T)$, where $\lambda_k \in \sigma_p(T)$, then $\mathcal{H}$ is countable direct sum of the Y_h.*
(3) *If $\mathcal{H}$ is separable, then $\mathcal{H}$ admits an at most countable complete orthonormal system of eigenvectors of T.*
(4) *If T is injective, then $\mathcal{H}$ admits an at most countable complete orthonormal system of eigenvectors of T.*

3.1. Unbounded operators. We have learned that when constructing an unbounded operator, one must also specify its domain. Therefore, one will also need to be careful in defining its resolvent set:

DEFINITION 3.4. Let T be a closed operator on a Hilber space $\mathcal{H}$. A complex number λ is in the *resolvent set* $\rho(T)$ if $\lambda\mathbb{I} - T$ is a linear isomorphism of $D(T)$ onto $\mathcal{H}$ with bounded inverse.

[3] This follows from the fact for every invertible bounded operator, its inverse is bounded as well.

Therefore, unless otherwise specified, we will always suppose our unbounded operators to have closed graph. In this case, the definitions of *spectrum*, *point spectrum* and *residual spectrum* are the same for unbounded operators as they are for bounded ones. Before restating the spectral theorem for unbounded operators, we need one last extra definition.

DEFINITION 3.5. Let T be a linear operator in $\mathcal{H}$. T is said to have *compact resolvent* if for every $\lambda \in \rho(T)$, the operator $(\lambda\mathbb{I} - T)^{-1}$ is compact.

THEOREM 3.2 (Spectral Theorem for Unbounded Operators). *Let $\mathcal{H}$ a Hilbert space over $\mathbb{C}$ and T a normal operator in $\mathcal{H}$ with compact resolvent. Then $\sigma(T)$ is at most countable and $\mathcal{H}$ admits a complete othonormal system of eigenvectors of T. More precisely, following facts holds:*

(1) *If $\sigma_p(T)$ is finite and $Y_h = \mathrm{Ker}(\lambda_h\mathbb{I} - T)$, where $\sigma_p(T) = \{\lambda_1, \ldots, \lambda_k\}$, then*

$$\mathcal{H} = \bigoplus_{h=0}^{k} Y_h \tag{44}$$

(2) *If $\sigma_p(T)$ is countable and $Y_h = \mathrm{Ker}(\lambda_h\mathbb{I} - T)$, where $\lambda_k \in \sigma_p(T)$, then $\mathcal{H}$ is countable direct sum of the Y_h.*

3.2. The Laplacian on $\mathbb{T}^1$. We are going to consider again the Hilbert space $\mathcal{H} = L^2(\mathbb{T})$ of 1-periodic complex valued L^2 functions on the circle. The simplest differential operator on the circle is of course $-i\frac{d}{dx}$, which we presented in example (2.1).

We are actually interested in a second order differential operator, namely the Laplace operator. To be able to differentiate twice, we recall that, given two linear operators $T_1 : D(T_1) \to \mathcal{H}$, $T_2 : D(T_2) \to \mathcal{H}$, the domain of the composition $T_2 \circ T_1$ is given by

$$D(T_2 \circ T_1) = \{v \in D(T_1) | T_1 v \in D(T_2)\}. \tag{45}$$

In the particular case we are considering, the domain is made of twice differentiable functions with square integrable second derivative:

$$D(\Delta) = \{f \text{ twice differentiable and } f'' \in \mathcal{H}\}. \tag{46}$$

We get the following operator

$$\Delta = \frac{d^2}{dx^2} : D(\Delta) \subset \mathcal{H} \to \mathcal{H} \tag{47}$$

Since every function in $\mathcal{H}$ can be developed in Fourier series, we can look at what happens to the basis elements, and we get that

$$\Delta e^{2\pi i n x} = -4\pi^2 n^2 e^{2\pi i n x}. \tag{48}$$

Thus we get that the complete orthonormal system for $L^2(\mathbb{T})$ presented in (2.4) is actually a complete orthonormal system of eigenvectors of the Laplace operator. Therefore we can decompose $L^2(\mathbb{T})$ into the countable direct sum of eigenspaces with eigenvalues $\{-4\pi^2 n^2 : n = 0, 1, \dots\}$. Each eigenspace has multiplicity two, except for the eigenspace of zero, the kernel of the operator, which has multiplicity one.

Remark 3.1. Note that, again,

$$\left\| \frac{e^{inx}}{n} \right\| \to 0 \text{ but } \left\| \Delta \left(\frac{e^{inx}}{n} \right) \right\| \to \infty \tag{49}$$

as $n \to \infty$, so Δ is unbounded.

Remark 3.2. The eigenspace decomposition shown above is a particular case of the spectral theorem for unbounded operators we previously stated (3.2).

3.3. The Laplacian on $\mathbb{T}^2$. By similar considerations, one gets that for the 2 dimensional case the Laplacian on $L^2(\mathbb{T}^2)$ has the form

$$\Delta = +\frac{d^2}{dx^2} + \frac{d^2}{dy^2}. \tag{50}$$

Again, recalling that an orthonormal basis for $L^2(\mathbb{T}^2)$ is given by $\{e^{2\pi(mx+ny)}\}$, for $n, m \in \mathbb{Z}$, we investigate what happens when we apply the Laplacian to the basis elements. One gets

$$\begin{aligned} \frac{d^2}{dx^2} e^{2\pi i(mx+ny)} &= -4\pi^2 m^2 e^{2\pi i(mx+ny)} \\ \frac{d^2}{dy^2} e^{2\pi i(mx+ny)} &= -4\pi^2 n^2 e^{2\pi i(mx+ny)}, \end{aligned}$$

which implies

$$\Delta e^{2\pi i(mx+ny)} = -4\pi^2 (m^2 + n^2) e^{2\pi i(mx+ny)}. \tag{51}$$

Thus $L^2(\mathbb{T}^2)$ decomposes into eigenspaces with eigenvalues

$$\{-4\pi^2 (n^2 + m^2) : n, m = 0, 1, \dots\}.$$

By the same consideration as above (49), the 2-dimensional Laplacian is unbounded.

The construction of the Laplace operator on higher dimensional tori is a straightforward generalization of the constructions presented here.

CHAPTER 2

C*-Algebras and Their Representations

This chapter contains a brief introduction to some mathematical methods in noncommutative geometry. In particular, it is devoted to a major result in functional analysis, a theorem which we owe to Gelfand and Naimark, which is the starting point for the study of C*-algebras as abstract entities.

1. C* Algebras

Unless otherwise specified, our ground field will be $\mathbb{C}$.

DEFINITION 1.1. Let $\mathcal{A}$ be a complex vector space. We endow $\mathcal{A}$ with a bilinear $\mathcal{A}$-valued function, which we call a *product*, that is a map

$$\begin{aligned} \cdot : \mathcal{A} \times \mathcal{A} &\to \mathcal{A} \\ (a, b) &\mapsto a \cdot b. \end{aligned} \tag{52}$$

with the properties that

$$c \cdot (\alpha a + \beta b) = \alpha(c \cdot a) + \beta(c \cdot b) \qquad (\alpha a + \beta b) \cdot c = \alpha(a \cdot b) + \beta(b \cdot c). \tag{53}$$

for all $a, b, c \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C}$.

The couple $(\mathcal{A}, \cdot)$ is called an *algebra*. Algebras are, *a priori*, noncommutative, i.e.

$$a \cdot b \neq b \cdot a. \tag{54}$$

For the cases we are interested in, we will suppose all algebras to be associative, i.e.

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \tag{55}$$

for all $a, b, c \in \mathcal{A}$. $\mathcal{A}$ is called unital if it possesses a multiplicative unit, which we denote by e.

In the following we will write ab instead of $a \cdot b$.

DEFINITION 1.2. A *Banach algebra* is an algebra $(\mathcal{A}, \|\cdot\|)$ equipped with a norm with respect to which it is a Banach space (complete normed space) and which satisfies $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathcal{A}$.

If the algebra is unital, it can be easily seen that there is at most one such e and that $\|e\| \geq 1$.

DEFINITION 1.3. An *involution* on $\mathcal{A}$ is an involutive anti-automorphism of $\mathcal{A}$, i.e. an involution

$$\begin{aligned} * : \mathcal{A} &\to \mathcal{A} \\ x &\longmapsto x^* \end{aligned}$$

which is compatible with the algebra structure. In formulas:

$$(x+y)^* = x^* + y^* \tag{56}$$

$$(\lambda x)^* = \overline{\lambda} x^* \tag{57}$$

$$(xy)^* = y^* x^* \tag{58}$$

$$x^{**} = x \tag{59}$$

for all $x, y \in \mathcal{A}, \lambda \in \mathbb{C}$.

An algebra equipped with such an involution is called a **-algebra.*

DEFINITION 1.4 (C* algebra). A Banach *-algebra $(\mathcal{A}, \|\cdot\|)$ that satisfies the so-called *C*-property*

$$\|a^* a\| = \|a\|^2 \quad \forall a \in \mathcal{A} \tag{60}$$

is called a *C*-algebra.*

REMARK 1.1. For a C*-algebra it follows automatically from (60) that the unit element has norm 1.

1.1. Examples of C^*-algebras. Even though the definition sounds complicated, C*-algebras are a very common mathematical object and they naturally arise in many fields. Therefore we are going to provide some typical and well known examples of C*-algebras.

EXAMPLE 1.1. Let $\mathcal{A} = \mathrm{Mat}_n(\mathbb{C})$ be the algebra of $n \times n$ matrices with complex coefficients. With the identity matrix as unit, $\mathcal{A}$ is a unital C*-algebra with respect to the operator norm

$$\|T\| = \sup_{x \in \mathbb{C}^n, x \neq 0} \frac{\|Tx\|}{\|x\|}. \tag{61}$$

and the involution defined by the adjoint matrix: $^* : T \mapsto T^*$.

It is possible to generalize the construction to the direct sum of complex matrix algebras, and get that

$$\mathcal{A} = \mathrm{Mat}_{n_1}(\mathbb{C}) \oplus \mathrm{Mat}_{n_2}(\mathbb{C}) \oplus \cdots \oplus \mathrm{Mat}_{n_k}(\mathbb{C}) \tag{62}$$

is a finite dimensional unital C*-algebra.

The converse also holds: it is possible to show that every finite dimensional C*-algebra can be represented as a direct sum of matrix algebras. For the proof of this fact we refer to [**5**].

EXAMPLE 1.2. Let $\mathcal{H}$ be a separable Hilbert space. We have previously mentioned (cf. Subsection 2.2) that the vector space $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ is a Banach space, with respect to the operator norm

$$\|T\| = \sup_{x\in\mathcal{H}, x\neq 0} \frac{\|Tx\|}{\|x\|}. \tag{63}$$

Moreover, with product the composition of operators and with the identity operator as unit element, $B(\mathcal{H})$ is a Banach algebra.

PROPOSITION 1.1. *The involution defined by the adjoint operator $T \mapsto T^*$ turns $\mathcal{B}(\mathcal{H})$ into a C*-algebra.*

PROOF. It is straightforward to show that $*$ satisfies all the axioms for an involution.
Property (60) follows from the following identity:

$$\|A^*A\| = \sup_{\|x\|=1=\|y\|} |\langle A^*Ax|y\rangle| = \sup_{\|x\|=1=\|y\|} |\langle Ax|Ay\rangle| = \|A\|^2. \tag{64}$$

The proof of this equation implicitly uses Schwartz inequality: $|\langle Ax|Ay\rangle| \leq \|Ax\|\|Ay\|$ for every $x, y \in \mathcal{H}$. ☐

Every norm closed subalgebra $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$, which is closed under adjoints is a C*-algebra and it is called a *concrete C*-algebra*. The converse is also true, and is due to Gelfand Naimark and Segal: every C*-algebra can be realized as a closde subalgebra of $\mathcal{B}(\mathcal{H})$ for a suitable $\mathcal{H}$. We will develop it in more detail in an upcoming section (cf. Section 4).

The previous two examples we gave are, in general, noncommutative. The next example is the key example for the commutative case.

EXAMPLE 1.3. Let X be a compact and Hausdorff topological space. We define in a natural way the commutative $\mathbb{C}$-algebra $C(X)$ of continuous functions $f : X \to$

$\mathbb{C}$ with pointwise sum and product:

$$(f+g)(x) = f(x) + g(x) \tag{65}$$

$$(f \cdot g)(x) = f(x)g(x) \tag{66}$$

for all $f, g : X \to \mathbb{C}$.

Since X is compact, the *supremum norm*

$$\|f\| = \sup_{x \in X} |f(x)|$$

is well defined. Moreover, since the limit of a uniformly convergent sequence of continuous functions is continuous as well, $C(X)$ is complete with respect to that norm.

We can endow $C(X)$ with an isometric involution

$$* : C(X) \to C(X) \quad f^*(x) = \overline{f(x)} \quad \forall x \in X,$$

with respect to which $C(X)$ is a commutative unital C*-algebra, with unit the function $f \equiv 1$.

The construction can be generalized to the case of a locally compact (but not compact) Hausdorff space Y. In that case we will lose unitality.

Starting from a (locally) compact Hausdorff space we have constructed a commutative (unital) C*-algebra.
Actually one can prove that we have a *functorial* map from the category of locally compact Hausdorff spaces with proper continuous maps to the category of C*-algebras with *-homeomorphisms. However, we are not interested in going further in this direction.

At this point, a natural question arises: given a commutative C*-algebra, can we always construct a (locally) compact Hausdorff space, whose algebra of continuous functions is exactly the algebra we started from? The answer is positive, and this is exactly the content of the theorem of Gelfand-Naimark.

2. Spectra, Ideals, Representations

Gelfand's theory is based on the notion of *spectrum* of an element in a Banach algebra and from the fact that, as we will see, the spectrum is not empty.

DEFINITION 2.1. Let $\mathcal{A}$ be a unital Banach algebra. We say that an element $x \in \mathcal{A}$ is *invertible* if it has right and left inverses.

LEMMA 2.1. *If* $\|x\| < 1$ *then* $e - x$ *is invertible and we have:*

$$(e-x)^{-1} = \sum_0^\infty x^n. \tag{67}$$

PROOF. Follows from the convergence of the geometric series. ☐

THEOREM 2.2. *Let* $\mathcal{A}$ *a unital Banach algebra. Then the following facts hold:*

(1) *If* $|\lambda| > \|x\|$, *then* $\lambda e - x$ *is invertible and the inverse is given by* $\sum_0^\infty \lambda^{-n-1} x^n$.
(2) *The set of invertible elements in* $\mathcal{A}$ *is open and the map* $x \mapsto x^{-1}$ *is continuous.*

PROOF. See [**10**]. ☐

DEFINITION 2.2. Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$. The *spectrum* of an element $a \in \mathcal{A}$, denoted as $\sigma(a)$, is defined by

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \text{ is not invertible }\}. \tag{68}$$

EXAMPLE 2.1. Let $\mathcal{A} = C(X)$ the algebra of continuous complex-valued functions on a compact topological space X. For every $f \in \mathcal{A}$ we have that the spectrum of f

$$\sigma(f) = \{f(x) : x \in X\} \tag{69}$$

coincides with the image of f.

EXAMPLE 2.2. Let $\mathcal{A} = \mathrm{Mat}_n(\mathbb{C})$ be the algebra of $n \times n$ matrices with entries in $\mathbb{C}$, then for every $a \in \mathcal{A}$

$$\sigma(a) = \{\lambda \in \mathbb{C} : det(a - \lambda \mathbb{I}) = 0\} \tag{70}$$

coincides with the spectrum of a in the sense of linear algebra.

In general the spectrum could be *a priori* empty. The following theorem excludes this possibility:

THEOREM 2.3. *Let* $\mathcal{A}$ *be a unital algebra over* $\mathbb{C}$. *If* $\mathcal{A}$ *is a Banach algebra, then* $\sigma(a) \neq \emptyset$ *for every* $a \in \mathcal{A}$.

PROOF. The proof uses the properties of the resolvent function. We refer to [**10**] for the details. ☐

COROLLARY 2.4 (Gelfand-Mazur). *Let* $\mathcal{A}$ *be a unital Banach algebra over* $\mathbb{C}$, *in which every nonzero element is invertible. Then* $\mathcal{A} \simeq \mathbb{C}$.

PROOF. Let $x \in \mathcal{A}$ be such that $x \notin \mathbb{C}e$. Then $\lambda e - x \neq 0$ for all $\lambda \in \mathbb{C}$, hence $\lambda e - x$ is invertible for all $\lambda \in \mathbb{C}$. This implies that $\sigma(x) = \emptyset$, which by (2.3) is absurd. □

DEFINITION 2.3. Let $x \in \mathcal{A}$. The *spectral radius* of x is defined as

$$\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}. \tag{71}$$

One can prove that $\rho(x) \leq \|x\|$.

DEFINITION 2.4. Let $\mathcal{A}$ be an Algebra.

- A *left ideal* for $\mathcal{A}$ is a subalgebra $\mathcal{I}$ of $\mathcal{A}$ such that

$$xy \in \mathcal{I} \quad \forall x \in \mathcal{A}, y \in \mathcal{I}. \tag{72}$$

- Analogously, a *right ideal* for $\mathcal{A}$ is a subalgebra $\mathcal{I}$ of $\mathcal{A}$ such that

$$yx \in \mathcal{I} \quad \forall x \in \mathcal{A}, y \in \mathcal{I}. \tag{73}$$

- A *two-sided ideal* is both a right and a left ideal $I \subseteq \mathcal{A}$.

In the next sections we will be mainly dealing with commutative algebras. Therefore, unless otherwise specified, all ideals will be two-sided. However, in the GNS construction (88) we will encounter a left ideal.

DEFINITION 2.5. Let $\mathcal{A}$ be a normed *-algebra, and I an ideal in $\mathcal{A}$.

- I is called *closed* if as a vector space it is closed in the norm topology.
- It is called *proper* if $\mathcal{I} \neq \mathcal{A}$. In the case when $\mathcal{A}$ is unital, $\mathcal{I}$ is proper if and only if $e \notin \mathcal{I}$.
- A *maximal ideal* is a proper ideal, which is not contained in any other greater proper ideal.

PROPOSITION 2.5. *Let $\mathcal{A}$ a commutative unital Banach algebra, and let $\mathcal{I} \subset \mathcal{A}$ be a proper ideal. Then the following facts hold:*

(1) *$\mathcal{I}$ does not contain any invertible elements;*
(2) *$\overline{\mathcal{I}}$, the closure of $\mathcal{I}$, is a proper ideal;*
(3) *$\mathcal{I}$ is contained in a maximal ideal;*
(4) *If $\mathcal{I}$ is maximal, then it is closed.*

PROOF. (1) Suppose there exists an invertible $x \in \mathcal{I}$. Then $e = x^{-1}x \in \mathcal{I}$ by definition (of ideal). Then by (2.5) we would have $\mathcal{I} = \mathcal{A}$, contradiction.

(2) If $\mathcal{I}$ is a proper ideal, then by (1) $\mathcal{I}$ it is contained in the set of non invertible elements of $\mathcal{A}$ which is closed by (2.2), point 2. Therefore $e \notin \overline{\mathcal{I}}$. On the other hand, it is easy to see that $\overline{\mathcal{I}}$ is an ideal.
(3) This is an application of *Zorn's lemma*; the union of an increasing family of propers ideals is again proper, since none of them contains e.
(4) Immediatly follows from (2).

□

If $\mathcal{A}$ and $\mathcal{B}$ are Banach Algebras, a Banach algebra homomorphism $\mathcal{A}$ to $\mathcal{B}$ is a bounded linear map

$$\phi : \mathcal{A} \to \mathcal{B},$$

such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathcal{A}$.
If $\mathcal{A}$ and $\mathcal{B}$ are *-algebras, a *-algebra homomorphism from $\mathcal{A}$ to $\mathcal{B}$ is a homomorphism such that $\phi(x^*) = \phi(x)^*$.

The *kernel* of ϕ is the set

(74) $$\ker(\phi) = \{a \in \mathcal{A} : \phi(a) = 0\}.$$

The kernel of a $*$-homomorphism is a two-sided ideal of $\mathcal{A}$ which is closed under the involution $*$. Moreover, if $\mathcal{A}$ is a C*-algebra $\ker(\varphi)$ is a closed C*-subalgebra of $\mathcal{A}$, with respect to the induced norm.

Definition 2.6. A representation of a C*-algebra is a couple $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space and

(75) $$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$$

is a $*$-homomorphism.
If the map π is injective, i.e. if $\ker(\pi)$ is trivial, we say that the representation is *faithful.*

3. Gelfand Theory

3.1. Multiplicative Functionals. We want to associate to every commutative C*-algebra a locally compact space. This can be done in terms of multiplicative functionals.

Definition 3.1. Let $\mathcal{A}$ be an algebra over $\mathbb{K}$. A *multiplicative functional* on $\mathcal{A}$ is a nonzero algebra homomorphism

(76) $$\chi : \mathcal{A} \to \mathbb{K}.$$

EXAMPLE 3.1. If X is a compact Hausdorff space and $\mathcal{A} = C(X)$, the evaluation map $\chi_x : f \to f(x)$ for $x \in X$ clearly defines a character.

DEFINITION 3.2. The set of all nonzero multiplicative functionals on $\mathcal{A}$ is called the *spectrum* of the algebra $\mathcal{A}$, and it is denoted by $\Sigma(\mathcal{A})$.

PROPOSITION 3.1. *Let χ be an element of $\Sigma(\mathcal{A})$. Then*

(1) $\chi(e) = 1$*;*
(2) *If $x \in \mathcal{A}$ is invertible, then $\chi(x) \neq 0$;*
(3) $|\chi(x)| \leq \|x\|$ *for all $x \in \mathcal{A}$;*
(4) χ *is continuous and of norm* 1.

PROOF. For the proof of points (1) to (3) we refer once more to [**10**][1.10].

(4) Suppose there exists some $a \in \mathcal{A}$ such that $\|a\| < 1$ e $\chi(a) = 1$. We construct a $b = \sum_{n\geq 1} a^n$. It is straightforward to show that $a + ab = b$. We get

$$\chi(b) = \chi(a) + \chi(a)\chi(b) = 1 + \chi(b) \tag{77}$$

which is absurd. Therefore $\|\chi\| \leq 1$, and since we have $\chi(e) = 1$, $\|\chi\| = 1$.

□

What is remarkable about multiplicative functionals is that, in the case of a commutative algebra, they are strictly related to maximal ideals.

THEOREM 3.2. *Let $\mathcal{A}$ be a commutative unital Banach algebra. The map $\chi \mapsto I = \ker(\chi)$ is a 1:1 correspondence between $\sigma(\mathcal{A})$ and the set of maximal ideals in $\mathcal{A}$.*

PROOF. Let $\chi \in \sigma(\mathcal{A})$, $\ker(\chi)$ is an ideal in $\mathcal{A}$ which is proper, since $\chi(e) = 1 \neq 0$, and maximal, since it has codimension 1.

Injectivity: suppose $\ker(\chi) = \ker(\psi)$. Let $x \in \mathcal{A}$. We can write $x = \chi(x)e + y$ with $y \in \ker(\psi)$. Then $\chi(x) = \psi(x)\chi(e) + \chi(y) = \psi(x)$. Then $\chi = \psi$ and therefore $\chi \mapsto \ker(\chi)$ is an injective function on the set of maximal ideals.

Let now $\mathcal{M}$ be a maximal ideal and let $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{M}$ be the projection on the quotient. Then $\mathcal{A}/\mathcal{M}$ inherits the algebra structure from $\mathcal{A}$ and becomes a Banach algebra with respect to the quotient norm $\|x + \mathcal{M}\| = inf\{\|x + m\| : m \in \mathcal{M}\}$ (see [**22**] for a proof of this fact).
$\mathcal{A}/\mathcal{M}$ has no trivial ideals. Indeed, suppose $\mathcal{I}$ is an ideal in the quotient. Then $\pi^{-1}(\mathcal{I})$ is an ideal of $\mathcal{A}$ such that $\mathcal{M} \subset \pi^{-1}(\mathcal{I}) \subset \mathcal{A}$. From the maximality of $\mathcal{M}$ it

follows that $\pi^{-1}(\mathcal{I}) = \mathcal{M}$ or $\mathcal{A}$ and this implies that $\mathcal{I} = \{0\}$ or $\mathcal{A}/\mathcal{M}$. Therefore every nonzero element in $\mathcal{A}/\mathcal{M}$ is invertible, otherwise it would generate a nontrivial ideal. By theorem 2.4 one gets that $\mathcal{A}/\mathcal{M}$ is isomorphic to $\mathbb{C}$. If we denote that isomorphism by ϕ then $\phi \circ \pi$ is a multiplicative functional on $\mathcal{A}$ with kernel $\mathcal{M}$. □

3.2. The Gelfand Transform.

DEFINITION 3.3. Let $\mathcal{A}$ be a commutative Banach algebra. The *Gelfand transform* of an element $x \in \mathcal{A}$ is the functional $\widehat{x} : \Sigma(\mathcal{A}) \to \mathbb{C}$ defined by

$$\widehat{x}(\chi) := \chi(x). \tag{78}$$

In other words $\widehat{x}$ is the evaluation in $x \in \mathcal{A}$.
The map $\mathcal{G} : x \mapsto \widehat{x}$, from $\mathcal{A}$ to $C(\Sigma(\mathcal{A}))$ is called the *Gelfand transformation.*

If we denote by $\widehat{\mathcal{A}}$ the set of all $\widehat{x}$, for $x \in \mathcal{A}$, we can topologize the spectrum $\Sigma(\mathcal{A})$. Indeed, the *Gelfand topology* of $\Sigma(\mathcal{A})$ is the weak topology induced by $\widehat{\mathcal{A}}$, i.e. the weakest topology that makes every $\widehat{x}$ continuous.
Since there is a one-to-one correspondence between maximal ideals of $\mathcal{A}$ and the elements of $\Sigma(\mathcal{A})$, the topological space $\Sigma(\mathcal{A})$, equipped with the Gelfand topology, is usually called the *maximal ideal space of* $\mathcal{A}$.

THEOREM 3.3 ([**5**], 1.2.6). *The maximal ideal space $\Sigma(\mathcal{A})$ of a unital abelian Banach algebra $\mathcal{A}$ is a compact Hausdorff space. If $\mathcal{A}$ is abelian but not unital, then $\Sigma(\mathcal{A})$ is locally compact.*

PROPOSITION 3.4. *Let $\mathcal{A}$ be a commutative unital C*-algebra. Then the Gelfand transfrorm preserves the involution. Moreover $\mathcal{G}(\mathcal{A})$ is dense in $C(\Sigma(\mathcal{A}))$.*

PROOF. See [**10**]. □

Let us now see what happens in the case of the C* algebra of continuous functions on a compact Hausdorff space.

THEOREM 3.5. *Le X be a compact topological Hausdorff space. For every $x \in X$, we define a function $\chi_x : C(X) \to \mathbb{C}$ by $\chi_x(f) = f(x)$. Then the map $x \mapsto \chi_x$ is a homeomorphism $X :\to \Sigma(C(X))$. If we identify $x \in X$ with $\chi_x \in \Sigma(C(X))$, then the Gelfand transform becomes the identity map.*

PROOF. It is clear that every χ_x is a multiplicative functional on $C(X)$ and that $\chi_x \neq \chi_y$ for $x \neq y$, since continuous functions separate points of X. If $x_\alpha \to x$ in X, then $f(x_\alpha) \to f(x)$ for every $f \in C(X)$, and this is equivalent to saying that

$\chi_{x_\alpha} \to \chi_x$ in the weak * topology.
Therefore $x \to \chi_x$ is a continuous injection $X \to \Sigma(C(X))$.

We now prove that every continuous functional is of this form for $x \in X$. By (3.2) this is equivalent to proving that every maximal ideal in $C(X)$ is of the form $\mathcal{M}_x = \{f : f(x) = 0\}$ for some $x \in X$. It suffices to prove that every proper ideal $\mathcal{I} \subset C(X)$ is contained in some $\mathcal{M}_x$. Suppose the contrary, i.e for every $x \in X$ there exists some $f_x \in \mathcal{I}$ such that $f_x(x) \neq 0$. The open sets of the form $\{y : f_x^{-1}(y) \neq 0\}$ form a cover of X. By compactness, we can consider a finite subcover with the property that $f_1, f_2, \dots, f_n$ have no common zeros. Let $g = \sum_{j=1}^n |f_j|^2$. Then $g = \sum_{j=1}^n \overline{f}_j f_j \in \mathcal{I}$ and g is invertible in $C(X)$ since $g > 0$ almost everywhere. By proposition (2.5) this contradicts the hypothesis of $\mathcal{I}$ being proper. Therefore $\mathcal{I} \subset \mathcal{M}_x$ for some x.

Finally, since $\widehat{f}(\chi_x) = \chi_x(f) = f(x)$, if we identify h_x with x we have $\widehat{f} = f$. □

3.3. Gelfand-Naimark' Theorem. To prove the Gelfand-Naimark' Theorem, we still need a proposition about Banach algebras. We recall that in a commutative Banach algebra

$$\|\widehat{x}\|_{\sup} = \rho(x) \leq \|x\| \tag{79}$$

for all $x \in \mathcal{A}$ ([**10**][1.13]).

PROPOSITION 3.6 ([**10**],1.19). *Let $\mathcal{A}$ be a commutative unitary Banach algebra. The following facts hold:*

1. *If $x \in \mathcal{A}$, $\|\widehat{x}\|_{\sup} = \|x\|$ if and only if $\|x^{2^k}\| = \|x\|^{2^k}$ for every $k \geq 1$.*
2. *$\mathcal{G}_{\mathcal{A}}$ is an isometry if and only if $\|x^2\| = \|x\|^2$ for every $x \in \mathcal{A}$.*

THEOREM 3.7 (Gelfand-Naimark). *Let $\mathcal{A}$ be a commutative unital C*-algebra, $\mathcal{G}_{\mathcal{A}}$ is an isometric isomorphism from $\mathcal{A}$ to $C(\sigma(\mathcal{A}))$.*

PROOF. Let $x \in \mathcal{A}$. We set $y = x^*x$. Then $y = y^*$, and

$$\|y^{2^k}\| = \|(y^{2^{k-1}})^* y^{2^{k-1}}\| = \|y^{2^{k-1}}\|^2. \tag{80}$$

It follows by induction that $\|y^{2^k}\| = \|y\|^{2^k}$, and therefore $\|\widehat{y}\|_{sup} = \|y\|$ by (3.6). But this means

$$\|x\|^2 = \|y\| = \|\widehat{y}\|_{\sup} = \||\widehat{x}|^2\|_{\sup} = \|\widehat{x}\|^2_{\sup}, \tag{81}$$

hence $\mathcal{G}_{\mathcal{A}}$ is an isometry. In particular it is injective and has close image. For proposition (3.4) $\mathcal{G}_{\mathcal{A}}$ respects the involution and has dense image. □

We point out that Gelfand Naimark' theorem is much more general than how stated here: it is possible to show that the Gelfand transform $\mathcal{G}$ is a functor from the category of compact topological Hausdorff spaces to the opposite category of commutative unital C*-algebras. Moreover, it is an equivalence of categories, with inverse given by the functor Σ, which associates to every commutative unital C*-algebra its spectrum.

3.4. Towards Noncommutative Spaces. What is remarkable about this theorem is that we can completely translate a series of topological informations regarding our space in terms of properties of the corresponding algebra. We can therefore set up a dictionary:

operator algebra	**topology**
$\mathcal{A} = C(X)$ commutative C* algebra	X, locally compact, Hausdorff
* homomorphisms	continuous proper functions
* automorphisms	homeomorphisms
unitality	compactness
separability	metrizability

Moreover, using this result we have characterized *all* the commutative C* algebras.

It is interesting to see that classical mechanics completely fits into this setting. Indeed, to study the physics of a classical system we consider its *phase space*, which is a manifold endowed with all possible values of position and momentum. Observables of the system are continuous functions defined on this manifold.

In quantum mechanics we replace the algebra of real valued functions by self-adjoint operators on a Hilbert space $\mathcal{H}$, which in general do not commute.

Inspired by the noncommutativity of quantum mechanics, and keeping in mind Gelfand's duality, we weaken the hypothesis and decide to study noncommutative C* algebras as if they were the dual algebra of some strange space, which we call "noncommutative space". So one could consider general noncommutative C* algebras as the dual of the category of some *noncommutative locally compact Hausdorff* spaces. This proposal turns out to be successful, since a rich class of examples can be found, and since it is possible to extend many "classical" topological and geometrical invariants to this new class of examples. Moreover, it is possible to apply the tecniques of noncommutative geometry to situations, where the "classical" mathematical tools fail to succeed, such as in the case of singular spaces like orbit spaces(cf. Chapter 3) of group actions and foliations.

Various examples of noncommutative spaces can be found in the literature, along with a deep analysis of the motivation which lead to the birth of noncommutative geometry, for istance in [**4**], [**3**], [**11**], [**26**], [**25**]. For a brief outline on singular spaces like foliations we remind to [**9**].

4. States and the GNS construction

In this section we will describe a fundamental feature of C*-algebras, that is the possibility to construct representations of an arbitrary C*-algebra in terms of bounded operators on a Hilbert space.

Definition 4.1. Let $\mathcal{A}$ be a C*-algebra. A positive element of $\mathcal{A}$ is a self-adjoint element $a \in \mathcal{A}$ for which $\sigma(a) \subseteq [0, +\infty[$.

Clearly, for every $a \in \mathcal{A}$, the element a^*a is positive.

Definition 4.2. A linear functional $\omega : \mathcal{A} \to \mathbb{C}$ is called positive if $\omega(a^*a)$ is real and positive for all $a \in \mathcal{A}$.

If the algebra $\mathcal{A}$ is unital, then

$$0 \leq \omega(a) \leq \|a\| \, \omega(1), \qquad \forall a \in \mathcal{A}. \tag{82}$$

This implies that if $\|\omega\| = \omega(1)$, then ω is automatically continuous.

If ω and φ are positive linear functionals with $\|\omega\| = \|\varphi\|$ and if $\omega - \varphi$ is also positive, then $\varphi = \omega$.

Definition 4.3. A positive linear functional of norm 1 is called a *state* of the C*-algebra.

If $\mathcal{A}$ is unital, then any state ω satisfies $\omega(1) = 1$.

Definition 4.4. A state ω is called *faithful* if $a \geq 0$ and $\omega(a) = 0$ imply $a = 0$.

Proposition 4.1. *Let $(\pi, \mathcal{H})$ be a representation of $\mathcal{A}$. Every $\psi \in \mathcal{H}$ of norm 1 defines a state ω on $\mathcal{A}$ via:*

$$\omega(a) = \langle \psi | \pi(a) \psi \rangle \qquad \forall a \in \mathcal{A}. \tag{83}$$

PROOF. First of all

$$\begin{aligned} \omega(a^*a) = \langle\psi|\pi(a^*a)\rangle = \langle\psi|\pi(a^*)\pi(a)\psi\rangle = \\ \langle\psi|\pi(a)^*\pi(a)\psi\rangle = \langle\pi(a)\psi|\pi(a)\psi\rangle = |\pi(a)\psi|^2 \geq 0, \end{aligned} \tag{84}$$

so ω is a positive functional. Moreover, $\|\omega\| = 1$. □

We will show in the following that every state of a C*-algebra $\mathcal{A}$ is of this form for a certain representation $(\pi, \mathcal{H})$ of $\mathcal{A}$.

Every state ω of a C*-algebra $\mathcal{A}$ satisfies the *generalized Cauchy-Schwarz* inequality, i.e. we have

$$\omega(a^*b) = \overline{\omega(b^*a)} \tag{85}$$

$$|\omega(a^*b)|^2 \leq \omega(a^*a)\omega(b^*b) \tag{86}$$

for all $a, b \in \mathcal{A}$.

Since states are linear functionals, they can be linearly combined. However, the linear combination of two states is a linear functional, but in general not a state.

Nevertheless, one can prove that a convex linear combination of two states ω_1, ω_2, i.e. a functional of the form

$$\lambda\omega_1 + (1-\lambda)\omega_2, \tag{87}$$

with $0 \leq \lambda \leq 1$, is again a state.

DEFINITION 4.5. A state ω is called a *pure state* if it cannot be written (non-trivially) in the form (87).

If we denote with $\mathcal{S}(\mathcal{A})$ the set of states of the algebra $\mathcal{A}$, then $\mathcal{S}(\mathcal{A})$ is a convex subspace of the dual of $\mathcal{A}$. Moreover, it can be proven that the pure states are extremal points for $\mathcal{S}(\mathcal{A})$. We refer to [**12**], for the details.

4.1. GNS Construction. We have previously seen that, given a representation $(\pi, \mathcal{H})$ of an algebra $\mathcal{A}$, all unit vectors in $\mathcal{H}$ define a state on $\mathcal{A}$ via equation (83). Conversely, we will show in this section that, given a state ω on $\mathcal{A}$, we can always construct a representation $(\pi_\omega, \mathcal{H}_\omega)$ and a unit vector $\psi_\omega \in \mathcal{H}_\omega$ such that

$$\omega(a) = \langle\psi_\omega|\pi_\omega(a)\psi_\omega\rangle \tag{88}$$

for all $a \in \mathcal{A}$.

First of all, let

$$\mathcal{N} = \{a \in \mathcal{A} : \omega(a^*a) = 0\}. \tag{89}$$

Using property (86) of ω, we can rewrite

$$\mathcal{N} = \{a \in \mathcal{A} : \omega(b^*a) = 0 \text{ for all } b \in \mathcal{A}\}. \tag{90}$$

Therefore $\mathcal{N}$ is a closed subspace of $\mathcal{A}$. Moreover, for every $a \in \mathcal{A}$ and for every $n \in \mathcal{N}$ we have

$$\omega(b^*(an)) = \omega((a^*b)^*n) = 0, \quad \forall b \in \mathcal{A} \tag{91}$$

hence $an \in \mathcal{N}$, which implies that $\mathcal{N}$ is a left ideal in $\mathcal{A}$.

We define an inner product on $\mathcal{A}/\mathcal{N}$, the space of equivalence classes $\overline{x} := x+\mathcal{N}$, via

$$\langle \overline{x}|\overline{y}\rangle = \omega(y^*x). \tag{92}$$

This is well defined (i.e. independent of the choice of the representative). Indeed, for all $x, y \in \mathcal{A}, n_1, n_2 \in \mathcal{N}$, we have

$$\begin{aligned} \omega((y+n_2)^*(x+n_1)) = \omega(y^*x) + \omega(n_2^*x) + \omega(n_2^*n_1) + \omega(y^*n_1) = \\ \omega(y^*x) + \omega((y+n_2)^*n_1) + \overline{\omega(x^*n_2)} = \omega(y^*x). \end{aligned} \tag{93}$$

If we complete $\mathcal{A}/\mathcal{N}$ in the norm induced by the scalar product

$$\|a\|^2 = \omega(a^*a), \tag{94}$$

we obtain an Hilbert space, which we denote by $\mathcal{H}_\omega$. This is the Hilbert space on which we will define a representation of $\mathcal{A}$.

First of all we take a representation of $\mathcal{A}$ on $\mathcal{A}/\mathcal{N}$: the so-called *left regular representation*:

$$\pi_0(a)\overline{x} = a\overline{x}. \tag{95}$$

Since $\mathcal{N}$ is a left ideal, this is well defined. π_0 is a *-representation, since

$$\begin{aligned} \langle \pi_0(a)\overline{x}|\overline{y}\rangle = \langle a\overline{x}|\overline{y}\rangle = \omega(y^*(ax)) = \\ = \omega((a^*y)^+x) = \langle \overline{x}|a^*\overline{y}\rangle = \langle \overline{x}|\pi_0(a^*)\overline{y}\rangle = \langle \pi_0(a^*)^*\overline{x}|\overline{y}\rangle, \end{aligned} \tag{96}$$

for all $\overline{x}, \overline{y} \in \mathcal{A}/\mathcal{N}$ and for all $a \in \mathcal{A}$. Moreover

$$\|\pi_0(a)\|^2 = \sup_{\|\overline{x}\|\leq 1} \|\pi_0(a)x\|^2 = \sup_{\|\overline{x}\|\leq 1} \omega(x^*a^*ax) \leq \sup_{\|\overline{x}\|\leq 1} \|a^*a\|\omega(x^*x) = \|a\|^2. \tag{97}$$

Therefore $\|\pi_0\| \leq 1$, which implies that, since $\mathcal{A}/\mathcal{N}$ is dense by definition, π_0 extends by continuity to a representation π of $\mathcal{A}$ on $\mathcal{H}_\omega$.

It remains to find a unit vector ψ_ω such that equation (88) holds. In the unital case, we set $\psi_\omega = \bar{e}$, where e is the multiplicative unit in $\mathcal{A}$. We have indeed that

$$\langle \pi(a)\psi_\omega | \psi_\omega \rangle = \omega(e^* a) = \omega(a). \tag{98}$$

Moreover, ψ_ω satisfies another property: it is *cyclic*, i.e. the set

$$\{\pi(a)\psi_\omega \mid a \in \mathcal{A}\} \tag{99}$$

is dense in $\mathcal{H}$.

For the nonunital case the construction of ψ_ω is more involved and uses approximate units. Since the algebra we are interested in, the noncommutative torus, is unital, we will omit this construction. It can be found in [**5**].

Therefore, we have proven the first part of the following theorem for the unital case.

THEOREM 4.2. *Let ω be a state on a C*-algebra $\mathcal{A}$. Then there exists a representation π_ω of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$ and a unit vector $\psi_\omega \in \mathcal{H}_\omega$ which is cyclic for $\mathcal{A}$ and such that*

$$\omega(a) = \langle \psi_\omega | \pi_\omega(a)\psi_\omega \rangle \tag{100}$$

for all $a \in \mathcal{A}$.

Moreover, the representation is unique up to unitary equivalence.

We conclude this section by stating a result which, together with theorem (3.3) can be considered the starting point of noncommutative geometry as the study of C*-algebras.

THEOREM 4.3 (Gelfand, Naimark, Segal). *Every C*-algebra is isometrically isomorphic to a closed *-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$*

PROOF. See ([**11**]). □

This result is highly powerful, since it gives us an explicit way of constructing C*-algebras: once we have a closed subalgebra of the algebra of bounded operators on some Hilbert space, it is guaranteed that it is indeed a C*-algebra. We are going to use this technique in the construction of the noncommutative torus, which is a noncommutative C*-algebra, and which will be realized as a closed subalgebra of the algebra of bounded operators on a certain Hilbert space.

CHAPTER 3

Noncommutative Tori

A good example is a thing of beauty. It shines and convinces. It gives insight and understanding. It provides the bedrock of belief.

M. Atiyah

1. Irrational Rotations

Our starting point is the circle S^1. Mindful of Gelfand's duality (cf. Theorem 3.3), we can look at it from an operator-algebraic point of view: since S^1 is a compact manifold, we can consider the commutative unital C*-algebra $C(S^1)$ of continuous complex valued functions $F : S^1 \to \mathbb{C}$, with the algebra operations defined pointwise, with involution given by the complex conjugate. There are several norms one can consider, the one we choose is the L^2 norm. For the sake of convenience, as pointed out in (3), we will identify functions $F : S^1 \to \mathbb{C}$ with continuous periodic functions $f : \mathbb{R} \to \mathbb{C}$ by

$$f(t) = F(e^{2\pi it}).$$

Keeping this identification in mind, from now on we will denote the algebra $C(S^1)$ with $C(\mathbb{T})$.

Let θ be a real number. We define an action[1] of the discrete Abelian group $\mathbb{Z}$ on S^1 via homeomorphism, that is, we associate to every integer $n \in \mathbb{Z}$ a homeomorphism of the circle, denoted by φ_n, which is given by

$$\varphi_n(z) = \lambda^n z, \qquad \forall n \in \mathbb{Z}, z \in S^1 \tag{101}$$

[1] Recall that a *(left) group action* of a group Γ on a group G is a map $\alpha : \Gamma \to \mathrm{Aut}(G)$ such that

$$\alpha(g_1 * g_2) = \alpha(\gamma_1) \circ \alpha(\gamma_2) \quad \forall \quad \gamma_1\gamma_2 \in \Gamma$$

$$\alpha(e) = Id,$$

where e denotes the identity element of Γ, and $*$ the group operation in Γ

where $\lambda = e^{2\pi i\theta}$. In exponential notation, for $z = e^{2\pi i\alpha}$, one gets

$$\varphi_n(z) = \varphi_n(e^{2\pi i\alpha}) = e^{2\pi i(\alpha+n\theta)}. \tag{102}$$

REMARK 1.1. For a rational θ, the orbits of this action are periodic, i.e. they will eventually close after a finite number of steps. Indeed, let $\theta = p/q$ with G.C.D.$(p, q) = 1$, then for $n = q$ we have

$$\lambda^n z = e^{2\pi ip} z = z.$$

What happens in the irrational case is completely different: the orbits are not closed. To prove this, let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and suppose there exists some $n \in \mathbb{Z}$ such that

$$\lambda^n z = e^{2\pi in\theta} z = z.$$

Then $2\pi n\theta = 2k\pi$, from which it follows that θ is rational, which is a contradiction.

Moreover, the orbit of any point $z \in S^1$ is dense in the circle. Indeed, if $z = e^{2\pi i\alpha}$ the set

$$\{(\alpha + n\theta) \mod \mathbb{Z} \mid n \in \mathbb{Z}\}$$

is dense in the interval $[0, 1[$.

We will develop this example throughout the chapter, along with more general definitions and facts. Paragraphs devoted to the study of this case will be marked with ♣.

2. C*-dynamical systems

Dynamical systems are used in classical mechanics to study the time evolution of a physical system. Basically, a dynamical system consists of 3 objects:

- A *state space*, tipically a manifold, whose elements represent all possible states of the physical system we are considering.
- A *time interval*, which can be both discrete or continuous, depending on the kind of system we are considering. Time is not an observable, but rather a parameter describing the evolution of our physical system.
- An *evolution map*, i.e. a function that contains all the information about the time evolution of the dynamical system, i.e. which assigns to every element of the time interval, the state of that system at that moment. Typically the map is required to be smooth and to satisfy the axioms for a group action.

When our system is no longer described in terms of a manifold, but rather by a C*-algebra, like in the case of quantum mechanics, we need a generalization of the notion of dynamical system, which is provided by *C*-dynamical systems.*

DEFINITION 2.1. A *continuous action* of a locally compact group G on a C*-Algebra $\mathcal{A}$ is a group homomorphism

$$\alpha : G \to \mathrm{Aut}(\mathcal{A}), \tag{103}$$

such that the map $t \mapsto \alpha_t(x)$ is continuous from G to $\mathcal{A}$ for any $x \in \mathcal{A}$.
The triple $(\mathcal{A}, G, \alpha)$ is called a *C*-dynamical System.*

EXAMPLE 2.1 (Time Evolution). The definitions outlined above may seem abstract and artificial: it is interesting, however, to see that they are inspired from quantum mechanics.
It is a known fact that in quantum mechanics the time evolution of a system with time-independent Hamiltonian operator H can be described in two equivalent pictures.
On one side we can let the state vectors $\psi \in \mathcal{H}$ evolve according to the equation

$$\psi(t) = U(t)\psi \tag{104}$$

where $U(t)$ is the unitary operator given by $U(t) = e^{iHt}$.

On the other hand one can choose to keep the state vectors fixed and let the observables evolve following

$$A(t) = U(t)AU^{\dagger}(t) \tag{105}$$

It is easy to see, that the evolution of the observables is established by a map

$$t \mapsto U(t)(\cdot)U^{\dagger}(t) \tag{106}$$

that assigns to each $t \in \mathbb{R}$ a *-automorphism of C *-algebra containing the observables of the system.

Further details about time evolution in the two pictures can be found in [**23**], Chapter 2.

♣ Let us now go back to our key example. From the C*-algebraic point of view, the construction presented in (101) leads to an action α of $\mathbb{Z}$ on functions of $C(\mathbb{T})$

by rigid rotation of angle $2\pi\theta$. More precisely, we associate to every integer $n \in \mathbb{Z}$ a map on the algebra $C(\mathbb{T})$, which we denote by α_n, given by:

$$\alpha_n(f)(t) = f(t + n\theta), \quad \forall t \in \mathbb{T} \tag{107}$$

PROPOSITION 2.1. *The triple $(C(\mathbb{T}), \mathbb{Z}, \alpha)$ is a C*-dynamical system.*

PROOF. To prove that every α_n is indeed a *-homomorphism of the algebra $C(\mathbb{T})$, let $f, g \in \mathbb{C}^(\mathbb{T})$, $\beta, \gamma \in \mathbb{C}$. We have

$$\begin{aligned} \alpha_n(\beta f + \gamma g)(t) = (\beta f + \gamma g)(t + n\theta) = \\ = \beta f(t + n\theta) + \gamma g(t + n\theta) = \beta\alpha_n(f)(t) + \gamma\alpha_n(g)(t), \end{aligned} \tag{108}$$

so the map preserves the linear structure, i.e. it is a vector space homomorphism. Moreover

$$\begin{aligned} \alpha_n\left((f \cdot g)(t)\right) = (f \cdot g)(t + n\theta) = f(t + n\theta)g(t + n\theta) = \\ = \alpha_n(f)(t)\alpha_n(g)(t) = \left(\alpha_n(f) \cdot \alpha_n(g)\right)(t), \end{aligned} \tag{109}$$

for all $f, g \in C(\mathbb{T})$, for all $t \in \mathbb{T}$. This means that the product is also preserved. Therefore α_n is an algebra homomorphism.
It remains to prove that the homomorphism preserves the involution. This follows from

$$\alpha_n(f^*(t)) = \alpha_n(\overline{f}(t)) = \overline{f}(t + n\theta) = \overline{\alpha_n(f)} = \alpha_n(f)^*. \tag{110}$$

To show that α is an action we must prove that $\alpha : \mathbb{Z} \to Aut(C(\mathbb{T}))$ is a group homomorphism, i.e. that

$$\alpha_0 = id \tag{111}$$

$$\alpha_{n+m} = \alpha_n \circ \alpha_m. \tag{112}$$

The first equation is obvious. The second follows from

$$\alpha_{n+m}f(t) = f(t + (n + m)\theta = \alpha_n\left(f(t + m\theta)\right) = \alpha_n\left(\alpha_m f(t)\right) = (\alpha_n \circ \alpha_m)f(t)$$

for all $f \in C(\mathbb{T})$, $t \in \mathbb{T}$.

Continuity is automatically satisfied, since a function from a discrete space to an arbitrary topological space is always continuous. □

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3. Crossed Products

Crossed products are the basic tool used to study C*-dynamical systems. They provide a larger C*-algebra which encodes information both on the original algebra and on the group action. In this section, we will show how to construct crossed product algebras. For our purpose we will only consider Abelian groups, since notions simplify considerably.

To every C*-dynamical system, we can associate an algebra which is constructed in the following way.
We start from the linear space $C_c(G \to \mathcal{A})$ of continuous functions from G to $\mathcal{A}$ with compact support. The linear space structure is defined point-wise. The *-algebra operation are *twisted* convolution product and involution, which for continuous groups have the form

$$(a * b)(t) = \int_G a(s)\alpha_s(b(t-s))ds \tag{113}$$

$$a^*(t) = \alpha_t(a(-t)^*). \tag{114}$$

EXAMPLE 3.1. In the case of $\mathcal{A} = \mathbb{C}$, $G = \mathbb{R}$ and $\alpha_h = id$ for every $h \in \mathbb{R}$, the previous definition coincide with the usual convolution and involution defined on the algebra of complex valued compactly supported functions on the line (cf. [**21**]).

3.1. Discrete Crossed Products. One could carry out the construction for the case of general Abelian groups. However, for the sake of convenience and motivated from our key example, in the following we will restrict our attention to discrete groups, like $\mathbb{Z}$.
In that case, the topology is the discrete one, hence compact support means finite support. Therefore, the space $C_c(G \to \mathcal{A})$ of continuous compactly supported $\mathcal{A}$ valued functions on G is just the algebra of formal sums

$$f = \sum_{t \in G} a_t t, \tag{115}$$

with the coefficients $a_t \in \mathcal{A}$ different from zero for a finite number of t's.
Equation (115) is just a formal expression meaning that the function f takes the value a_t in t.

The twisted convolution product is given by

$$f * g = \sum_{s \in G} \left(\sum_{t \in G} a_t \alpha_t(b_{s-t}) \right) s, \tag{116}$$

while the involution is given by

$$f^* = \sum_{t \in G} \alpha_t(a^*_{-t})t. \tag{117}$$

The algebra $\mathcal{A}$ can be embedded in $C_c(G \to \mathcal{A})$ in terms of functions supported by the identity element of G. The group G is also represented in $C_c(G \to \mathcal{A})$ by associating to every $s \in G$ the delta-function

$$\delta_s(t) = \begin{cases} 1 & t = s \\ 0 & t \neq s. \end{cases}$$

We see that the algebra $C_c(G \to \mathcal{A})$ is a good candidate for an algebra encoding informations about both the algebra $\mathcal{A}$ and the group G.

Moreover, for each f in $C_c(G \to \mathcal{A})$ we can define the L^1-norm

$$\|f\|_1 := \sum_{t \in G} \|a_t\|_{\mathcal{A}}, \tag{118}$$

where $\| \cdot \|_{\mathcal{A}}$ denotes the C*-algebra norm of $\mathcal{A}$. The norm (118) turns $C_c(G \to \mathcal{A})$ into a normed algebra with isometric involution. We denote by $L^1(G \to \mathcal{A})$ its completion in the norm (118), which is a Banach algebra but in general not a C*-algebra, as the following example shows.

Example 3.2. If $\mathcal{A} = \mathbb{C}$ and $G = \mathbb{Z}$, $C_c(\mathbb{Z} \to \mathbb{C})$ is the algebra $\ell^1(\mathbb{Z})$ equipped with twisted involution and convolution:

$$a^*_n = \alpha_n(\overline{a}_{-n}) \tag{119}$$

$$a * b = c \qquad c_n = \sum_m a_n \alpha_n(b_{n-m}). \tag{120}$$

which is a Banach algebra but not a C*-algebra.

Indeed, for $\alpha = id$, $C_c(\mathbb{Z} \to \mathbb{C})$ coincides with the space of complex valued sequences $(a_n)_{n \in \mathbb{Z}}$, with involution and convolution given by

$$a^*_n = \overline{a}_{-n} \tag{121}$$

$$a * b = c \qquad c_n = \sum_m a_n (b_{n-m}). \tag{122}$$

If we complete it in the norm

$$\|a\|_1 := \sum_{n \in \mathbb{Z}} \|a_n\|, \tag{123}$$

and we get the Banach space $\ell^1(\mathbb{Z})$ with twisted convolution and involution. However, the C* property (60) does not hold. To see this, one can take for instance the element a given by

$$a_n = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \\ 0 & \text{otherwise} \end{cases}, \tag{124}$$

and do explicit computations.

♣ In our case $\mathcal{A} = C(\mathbb{T})$ and $G = \mathbb{Z}$. The elements of $C_c(\mathbb{Z} \to C(\mathbb{T})$ are of the form

$$a = \sum_{n \in \mathbb{Z}} a_n n, \tag{125}$$

with the coefficient a_n being continuous functions living in $C(\mathbb{T})$. The algebra operations are twisted convolution and involution. For every $a, b \in C_c(\mathbb{Z} \to C(\mathbb{T}))$ the convolution is given by

$$(a * b)_n = \sum_m a_n \alpha_n(b_{n-m}), \tag{126}$$

that is, the n-th coeffiecient is the function given by

$$(a * b)_n(t) = \sum_m a_n(t) b_{n-m}(t + n\alpha). \tag{127}$$

We have seen in example (3.2) that the L^1 norm is not the correct one to consider if we want to turn the algebra $C_c(G \to \mathcal{A})$ into a C*-algebra. In the following, we will show that it is possible to construct other possible norms in terms of representation of the algebra itself. This effort will turn out to be successful.

Recall that if we want to study an algebra, we can represent it on a Hilbert space, as in definition 3.1. Similarly, if we want to study a group, we can consider unitary representations on Hilbert spaces.

Definition 3.1. Let G be a topological group and $\mathcal{H}$ a Hilbert space. A *unitary representation* u of G on $\mathcal{H}$ is a group homomorphism from G to the group $\mathcal{U}(\mathcal{H})$ of unitary operators on $\mathcal{H}$:

$$u : G \to \mathcal{U}(\mathcal{H}), \tag{128}$$

i.e. $u(g) =: u_g$ is a unitary operator on $\mathcal{H}$ for every $g \in G$.

When one is dealing with a C*-dynamical system, the two objects, the algebra $\mathcal{A}$ and the group G are connected, since the latter acts on $\mathcal{A}$ by automorphisms. This motivates the following definition, which sets a compatibility between the representations of the algebra and the action of the group G on the algebra itself.

DEFINITION 3.2. A *covariant representation* of a C*-dynamical system $(\mathcal{A}, G, \alpha)$ is a pair (π, u), where π is a representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and u is a unitary representation of G on the same Hilbert space satisfying

$$u_g \pi(a) u_g^* = \pi(\alpha_g(a)) \qquad \text{for all } a \in \mathcal{A}, g \in G. \tag{129}$$

Equation (129) is an identity between operators: it must hold for every element $\varphi \in \mathcal{H}$.

It can be proven that, given a C*-dynamical system, covariant representations can always be constructed. For the construction we refer to [**12**]. In the following, we will see an explicit example of a covariant representation for our setting.

♣ Starting from our C*-dynamical system $(C(\mathbb{T}), \mathbb{Z}, \alpha)$, a covariant representation for can be constructed by taking the Hilbert space $\mathcal{H} = L^2(\mathbb{T})$.

The representation $\pi : C(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ is given in terms of multiplication operator: to every function $f \in C(\mathbb{T})$ we associate the operator $\pi(f) = M_f$ on Ł$^2(\mathbb{T})$ given by

$$M_f(\varphi)(t) = f(t)\varphi(t), \tag{130}$$

for every $\varphi \in L^2(\mathbb{T})$.
This is indeed an operator on $L^2(\mathbb{T})$, since

$$\|f\varphi\|_2^2 = \int |f(x)\varphi(x)|^2 dx \leq \sup_{x \in \mathbb{T}} |f(x)|^2 \; \|\varphi\|_2^2. \tag{131}$$

A unitary representation of the group $\mathbb{Z}$ on the same Hilbert space is constructed in the following way: we associate to every integer n an operator u_n acting on square summable functions on $\mathbb{T}$ via

$$u_n(f(t)) = f(t + n\theta). \tag{132}$$

PROPOSITION 3.1. *The couple (π, u) is a covariant representation.*

PROOF. We omit the proof that that π, u are a C*-algebra and a unitary group representation, which is straightforward.

We want to show, for every $n \in \mathbb{Z}$, $f \in C(\mathbb{T})$, we have

$$(u_n M_f u_n^*)\varphi(t) = M_{\alpha_n(f)}\varphi(t). \tag{133}$$

Indeed, using equation (107) we have:

$$\begin{aligned}(u_n M_f u_n^*)\varphi(t) &= (u_n M_f)\varphi(t - n\theta) = \\ = u_n(f(t)\varphi_{t+n\theta}) &= f(t+n\theta)\varphi(t) = M_{\alpha_n(f)}\varphi(t),\end{aligned} \tag{134}$$

which finishes the proof. □

The remarkable property of covariant representations $(\pi, u, \mathcal{H})$ of a C*-dynamical system $(\mathcal{A}, G, \alpha)$ is that one can always construct a representation λ of the algebra $C_c(G \to \mathcal{A})$ on $\mathcal{B}(\mathcal{H})$. For every

$$a = \sum_{t \in G} a_t t \tag{135}$$

the corresponding operator in $\mathcal{B}(\mathcal{H})$ is given by

$$\lambda(a) = \sum_{t \in G} \pi(a(t))u_t. \tag{136}$$

$\lambda : C_c(G \to \mathcal{A}) \to \mathcal{B}(\mathcal{H})$ is called the *left regular representation* associated to the covariant representation (π, u).

♣ The covariant representation $(\pi, u, \mathcal{H})$ of $(C(\mathbb{T}), \mathbb{Z}, \alpha)$ yields a faithful *-representation ρ of $C_c(G \to \mathcal{A})$ on $\mathcal{B}(\mathcal{H})$. For every

$$a = \sum_{n \in \mathbb{Z}} a_n n,$$

the corresponding operator in $\mathcal{B}(\mathcal{H})$ is given by

$$\rho(a) = \sum_{n \in \mathbb{Z}} \pi(a_n)u_n. \tag{137}$$

This operator associates to every $\varphi \in L^2(\mathbb{T})$ the function defined by

$$\rho(a)(\varphi)(t) = \sum_{n \in \mathbb{Z}} (\pi(a_n)u_n)(\varphi(t)) = \sum_{n \in \mathbb{Z}} \pi(a_n)\varphi(t+n\theta) = \sum_{n \in \mathbb{Z}} a_n(t+n\theta)\varphi(t+n\theta). \tag{138}$$

To turn $C_c(G \to \mathcal{A})$ into a C*-algebra, we define the norm

$$\|f\| = \sup_{\sigma} \|\sigma(f)\|. \tag{139}$$

with σ running over *all possible* *-representations of $C_c(G \to \mathcal{A})$.
The supremum is always bounded by the L^1 norm of f, and it is taken over a non-empty family of representations, because we know that certain representations can be explicitly constructed from the covariant representation, as shown left regular representation defined in (136).

Definition 3.3. The *crossed product* (also called *covariance algebra*) of the C* dynamical system $(\mathcal{A}, G, \alpha)$ is the closure of $C_c(G, \mathcal{A})$ in the norm 139, and it is denoted by

$$C^*(\mathcal{A}, G) = \mathcal{A} \rtimes_\alpha G. \tag{140}$$

This definitions is quite complicated, since it requires to consider all *-representations of the algebra. However, since we already have the representation defined in (136), there's another (*a priori*) smaller C*-algebra we can consider: we take the closure of $\lambda(C_c(G \to \mathcal{A}))$ in $\mathcal{B}(\mathcal{H})$, which is a C*-subalgebra. This motivates the following definition:

Definition 3.4. The *reduced crossed product* of $\mathcal{A}$ by G is the C*-algebra

$$\mathcal{A} \rtimes_{\alpha,\mathrm{red}} G = \lambda(\mathcal{A}G).$$

It is a known fact, of which we omit the proof, since it would be too complicated, that in the case of *amenable* groups, a class of groups which contains both Abelian and compact groups, the reduced crossed product equals the full one ([**14**], Theorem 1.10.13). This is precisely what happens in our case: $\mathbb{Z}$ is a commutative and therefore amenable group.

4. The Irrational Rotation Algebra

Definition 4.1. We define the *irrational rotation algebra* $\mathcal{A}^2_\theta$ as the crossed product

$$C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}. \tag{141}$$

Since for $\mathbb{Z}$ the reduced crossed equals the full crossed product, we will construct the irrational rotation algebra using the reduced one.

In the literature this algebra is sometimes referred to as the *noncommutative torus*. However, we will call *noncommutative torus* the smooth subalgebra that we will define in equation (147).

In the spirit of the previous chapter, we have constructed the irrational rotation algebra $\mathcal{A}_\theta^2$ as a closed *-subalgebras of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space.

Let us consider again the multiplication operator M_f and the unitary shift operator V on $L^2(\mathbb{T})$:

$$M_f(\varphi)(t) := f(t)\varphi(t)$$
$$V(\varphi)(t) := \varphi(t+\theta).$$

We have another operator, called the *unitary multiplication operator*, which acts on functions in $L^2(\mathbb{T})$ via

$$U(\varphi)(t) = e^{2\pi i t}\varphi(t). \tag{142}$$

PROPOSITION 4.1. *The operator U generates the representation of $C(\mathbb{T})$ on $L^2(\mathbb{T})$.*

PROOF. Recalling that, by equation (15), we can expand every $f \in C(\mathbb{T})$ in Fourier series:

$$f(t) = \sum_n a_n e^{2\pi n i t},$$

one gets

$$M_f(\varphi)(t) = f(t)\varphi(t) = \sum_n a_n e^{2\pi i n t}\varphi(t) = \sum_n a_n U^n \varphi(t).$$

□

Moreover, since

$$(VU\xi)(t) = (U\xi)(t+\theta) = e^{2\pi i(t+\theta)}\xi(t+\theta) = e^{2\pi i\theta}(UV\xi)(t),$$

these unitaries obey the *twisted commutation relation*

$$\boxed{VU = e^{2\pi i\theta}UV.} \tag{143}$$

Therefore the algebra $\mathcal{A}_\theta^2$ is generated by the two unitary operators U and V on $\mathcal{B}(L^2(\mathbb{T}))$, subject to the twisted commutation relation (143).
By *generated* we mean that the algebra $\mathcal{A}_\theta^2$ is the closure of the linear span of all possible products of U and V's. Therefore, every element $a \in \mathcal{A}_\theta^2$ has the form

$$a = \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n, \tag{144}$$

with the $a_{m,n}$ living in $\mathbb{C}$.

The following proposition ensures us that every C*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by two unitaries satisfying (143) is isomorphic to the irrational rotation algebra.

PROPOSITION 4.2. *Let $\mathcal{A}$ be the universal C*-algebra generated by two unitary operators u, v on some Hilbert space $\mathcal{H}$ obeying the twisted commutation relation:*

$$vu = e^{2\pi i\theta}uv. \tag{145}$$

The map $u \mapsto U$, $v \mapsto V$ determines an isomorphism of $\mathcal{A}$ onto $\mathcal{A}^2_\theta$.

PROOF. The linear map that takes finite sums $\sum_{m,n} a_{m,n}u^m v^n$ to the corresponding sum $\sum_{m,n} a_{m,n}U^m V^n$ is multiplicative because of (145) and extends to a unital isomorphism $\phi : \mathcal{A} \to \mathcal{A}^2_\theta$.

To define an *inverse morphism* it is sufficient to obtain a covariant representation of $(C(\mathbb{T}), \mathbb{Z}, \alpha)$ whose image is isomorphic to $\mathcal{A}^2_\theta$. We can suppose that $\mathcal{A}^2_\theta$ is faithfully represented on a Hilbert space $\mathcal{H}$ and define (π, λ) on $\mathcal{H}$ as follows: set $\lambda_m := v^m$, and if $f(t) = \sum_n c_n e^{2\pi int}$ is a finite Fourier series, let $\pi(f) := \sum_n c_n u^n$. It is easy to see that (107) implies that $\lambda_m \pi(f)\lambda^*_m = \sum_n c_n e^{2\pi imn\theta}u^n = \pi(\alpha^m(f))$. Therefore π extends to $C(\mathbb{T})$ to give the desired covariant representation.
The associated representation of $\mathcal{A}^2_\theta$ is an isomorphism $\rho : \mathcal{A}^2_\theta \to \mathcal{A}$ satisfying $\rho(U) = u$ and $\rho(V) = v$, which is inverse to ϕ. □

Therefore any C*-algebra generated by two unitaries satisfying (145) is a quotient of $\mathcal{A}^2_\theta$. Actually, since $\mathcal{A}^2_\theta$ is simple for $\theta \in [0, 1] \setminus \mathbb{Q}$ ([**11**] Corollary 12.12), any such algebra is isomorphic to $\mathcal{A}^2_\theta$.

In virtue of this isomorphism, in the following we will always think of the irrational rotation algebra $\mathcal{A}^2_\theta$ as the universal C*-algebra generated by two unitaries subject to the relation (143) and of its elements in terms of series of the form (144).

4.1. The noncommutative torus. From (145) it follows that $\mathcal{A}^2_\theta$ is Abelian if and only if $e^{2\pi i\theta} = 1$, or equivalently if and only if θ is an integer. We may identify $\mathcal{A}_0$ with the C*-algebra $C(\mathbb{T})$ of continuous functions on the 2-torus with coordinates (x, y) by taking $u := e^{2\pi ix}$, $v := e^{2\pi iy}$ and writing the elements of the algebra in terms of Fourier series.

Under certain condition on the coefficients of the series, one gets convergence. It is in general convenient to restrict to the pre C*-algebra $C^\infty(\mathbb{T})$. In particular, we know from Theorem 2.6 that the Fourier series for $a \in C^\infty(\mathbb{T})$ converges back (both uniformly and in the L^2 norm) to a and its coefficients belong to $\mathcal{S}(\mathbb{Z}^2)$ (rapidly decreasing double sequencese), i.e:

$$\sup_{r,s}(1+r^2+s^2)^k|a_{rs}|^2 < \infty \qquad \text{for all } k. \tag{146}$$

By analogy with the commutative case we give the following definition.

DEFINITION 4.2. We call *noncommutative torus* the dense subalgebra $\mathcal{A}_\theta^\infty$ of $\mathcal{A}_\theta^2$ given by

$$\mathcal{A}_\theta^\infty = \{a = \sum_{m,n} a_{m,n}U^mV^n | \{a_{m,n}\} \in \mathcal{S}(\mathbb{Z}^2)\}. \tag{147}$$

In particular, the unit $1 \in \mathcal{A}_\theta^2$ belongs to $\mathcal{A}_\theta^\infty$.

4.2. A trace on $\mathcal{A}_\theta^2$. Before defining a trace on the rotation algebra, we recall some facts and definitions about traces on a C*-algebra.

DEFINITION 4.3. A state τ on an algebra $\mathcal{A}$ is called *tracial* if $\tau(ab) = \tau(ba)$.

DEFINITION 4.4. A *trace* on $\mathcal{A}$ is a nontrivial tracial state.

We can endow the algebra $\mathcal{A}_\theta^2$ equipped with a faithful tracial state τ. We define it first on the dense subalgebra $\mathcal{A}_\theta^\infty$:

$$\tau : \mathcal{A}_\theta^\infty \longrightarrow \mathbb{C} \tag{148}$$

$$a = \sum_{m,n\in\mathbb{Z}} a_{m,n}U^mV^n \longmapsto a_{00}. \tag{149}$$

The proof that τ is continuous can be found in [**11**], proposition 12.10. Moreover $\tau(1) = 1$, which implies that τ is a normalized linear functional on $\mathcal{A}_\theta^\infty$. By a continuity argument, since $\mathcal{A}_\theta^\infty$ is dense in $\mathcal{A}_\theta^2$, τ extends to a state on $\mathcal{A}_\theta^2$. Since

$$\tau(a^*a) = \sum_{m,n\in\mathbb{Z}} |a_{m,n}|^2 \geq 0, \tag{150}$$

and $\tau(a^*a) > 0$ for every non-zero $a \in \mathcal{A}_\theta^n$, τ is faithful. The state is tracial, since from the formula for the explicit product (cf. appendix A) in $\mathcal{A}_\theta^2$, we get that

$$\tau(ab) = (ab)_{00} = \sum_{m,n} b_{m,n}a_{-m,-n}e^{2\pi imn\theta} = \tag{151}$$

$$= \sum_{m,n} b_{-m,-n}a_{m,n}e^{2\pi imn\theta} = \tag{152}$$

$$\sum_{m,n} a_{m,n}b_{-m,-n}e^{2\pi imn\theta} = (ba)_{00} = \tau(ba). \tag{153}$$

THEOREM 4.3. *If $\theta \in [0,1] \setminus \mathbb{Q}$, then the tracial state τ on $\mathcal{A}_\theta^2$ is unique.*

PROOF. For the proof we refer to [**11**], proposition 12.11, where the statement is proven in the n-dimensional case. ☐

We have seen in section 4 that states can be used to construct a scalar product: we can therefore define a scalar product on the rotation algebra in terms of the tracial state τ: for every $a, b \in \mathcal{A}_\theta^2$, we defined a $\mathbb{C}$-valued scalar product

$$\langle a, b\rangle = \tau(b^* a). \tag{154}$$

The fact that the bilinear map $\langle\cdot,\cdot\rangle$ is indeed a scalar product can be easily checked by explicit calculations.

The definition of the scalar product (154) allows us to establish an analogy with what happens in the noncommutative case: the decomposition of an arbitrary element of the algebra $\mathcal{A}_\theta^2$ in terms of multiples of $U^m V^n$ should be viewed as a *noncommutative analogue of the Fourier series decomposition*, with the $a_{m,n}$ as a sort of *Fourier coefficients*. Indeed, for $a \in \mathcal{A}_\theta^2$ but not necessarily in $\mathcal{A}_\theta^\infty$, the Fourier coefficients are well defined and satisfy the analogue of the equation that defines the Fourier coefficients in the commutative case:

$$a_{m,n} = \langle a, U^m V^n\rangle, \tag{155}$$

which justifies the term *Fourier series expansion*.

Moreover, they also satisfy $|a_{m,n}| \leq \|a\|$.

We remark that the Fourier series expansion is only a formal expression and need not converge in the topology of $\mathcal{A}_\theta^2$, just as in $C(\mathbb{T})$ one has functions whose Fourier series do not converge absolutely or even pointwise.

REMARK 4.1. Via the Fourier expansion described above, provided we only look at the linear structure and forget noncommutativity, we get an isomorphism by

$$\begin{aligned} C^\infty(\mathbb{T}^2) &\longrightarrow \mathcal{A}_\theta^\infty \\ (x \mapsto e^{2\pi i x}) &\mapsto U \\ (y \mapsto e^{2\pi i y}) &\mapsto V \end{aligned}$$

and therefore we can identify

$$f(x,y) = \sum_{m,n} f_{m,n} e^{2\pi i(mx+ny)} \mapsto \sum_{m,n} f_{m,n} U^m V^n.$$

5. Isomorphisms between the various $\mathcal{A}_\theta^2$'s

Having defined our C*-algebra in terms of generators and relations between them, we immediately get the isomorphism $\mathcal{A}_\theta^2 \simeq \mathcal{A}_{\theta+n}^2$ for all $n \in \mathbb{Z}$, since equation (145) remains unchanged by $\theta \mapsto \theta + n$. Therefore we can, whenever convenient, restrict the range of the parameter θ to the interval $0 \leq \theta < 1$. On the other hand, it follows from (145) that $uv = e^{2\pi i(1-\theta)}vu$, and this implies that the correspondence $u \mapsto v, v \mapsto u$ estends to an isomorphism $\mathcal{A}_\theta^2 \simeq \mathcal{A}_{1-\theta}^2$.

REMARK 5.1. As we shall see later, these are the only isomorphisms between the torus algebras.

For nonintegral values of θ, the rational and irrational case are very different. We are mainly interested in the irrational case.

THEOREM 5.1 ([**11**]12.7). *If α and β are irrational numbers in the interval $[0, 1/2]$, and if $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ are isomorphic, then $\alpha = \beta$.*

PROOF. The proof is due to Rieffel [**18**] and uses existence and properties of non-trivial projectors. □

6. Morita Equivalence of Noncommutative Tori

We have seen (5.1) that an irrational number $\theta \in [0, \frac{1}{2}]$ parametrizes a family of mutually non-isomorphic noncommutative tori. However, there's a weaker equivalence relation we can define for algebras.

6.1. Morita Equivalence of C*-algebras.

DEFINITION 6.1. A left pre-C* module over a C*-algebra $\mathcal{B}$ is a complex vector space $\mathcal{F}$ that is also a left $B-$module equipped with a sesquilinear pairing $\mathcal{F}\times\mathcal{F} \to \mathcal{B}$ satisfying

$$\langle r|s+t\rangle = \langle r|s\rangle + \langle r|t\rangle \tag{156}$$

$$\langle br|s\rangle = b\langle r|s\rangle \tag{157}$$

$$\langle r|s\rangle = \langle s|r\rangle^* \tag{158}$$

$$\langle s|s\rangle > 0 \qquad \forall s \neq 0 \tag{159}$$

For all $r, s \in \mathcal{F}$, $b \in \mathcal{B}$.

A left pre C*-B-module $\mathcal{F}$ is *full* if $\langle \mathcal{F}|\mathcal{F}\rangle$ is dense in B. A left C*-module over $\mathcal{B}$ is obtained by completing $\mathcal{F}$ in the norm

$$s \mapsto \|\langle s|s\rangle\|^2.$$

If $\mathcal{E}$ is any right $\mathcal{A}$-module, its conjugate space $\overline{\mathcal{E}}$ is a left $\mathcal{A}$-module: using the obvious notation $\overline{\mathcal{E}} = \{\overline{s} : s \in \mathcal{F}\}$ we can define

$$a\overline{s} = \overline{sa^*}.$$

If $\mathcal{E}$ is a right pre C*-module over $\mathcal{A}$, there is an obvious pairing that makes $\overline{\mathcal{E}}$ a left pre C*-$\mathcal{A}$-module, namely:

$$(r|s) = \langle \overline{r}|\overline{s}\rangle \tag{160}$$

Definition 6.2. A pre C*-$\mathcal{B}$-$\mathcal{A}$-bimodule is a complex vector space $\mathcal{E}$ that is both a left pre-C*-$\mathcal{B}$-module and a right pre C*-$\mathcal{A}$-module, and moreover satisfies

$$r(s|t) = \langle r|s\rangle t \qquad \text{for every } r, s, t \in \mathcal{E}. \tag{161}$$

We say that $\mathcal{E}$ is *right full* if $(\mathcal{E}|\mathcal{E})$ is dense in $\mathcal{A}$ or *left full* if $\langle \mathcal{E}|\mathcal{E}\rangle$ is dense in $\mathcal{B}$. We call it simply *full* if both conditions hold.

Remark 6.1. The two norms naturally defined on a pre-C*-$\mathcal{B}$-$\mathcal{A}$-bimodule coincide.

$\mathcal{E}$ is called a C*-$\mathcal{B}$-$\mathcal{A}$-bimodule if it is complete with respect to the norm

$$\|s\|^2 = \|\,(s|s)\,\| = \|\langle s|s\rangle\|^2. \tag{162}$$

Definition 6.3. If $\mathcal{F}$ is a C* C* $\mathcal{A}-\mathcal{B}-$bimodule and $\mathcal{G}$ is a C* $\mathcal{B}-\mathcal{C}-$bimodule, the tensor product $\mathcal{F} \otimes_{\mathcal{B}} \mathcal{G}$ becomes a C* $\mathcal{A} - \mathcal{C}-$bimodule, with the pairings on tensors given by:

$$(r_1 \times s_1|r_2 \times s_2) := ((r_2|r_1)_{\mathcal{B}}\, s_1|s_2)_{\mathcal{C}} \tag{163}$$

$$\langle r_1 \times s_1|r_2 \times s_2\rangle := \langle r_1|r_2\langle s_2|s_1\rangle_{\mathcal{B}}\rangle_{\mathcal{A}}. \tag{164}$$

With these C*-bimodule techniques we can introduce an important equivalence relations between C* algebras, which is weaker than isomorphism.

Definition 6.4. We say that two C*-algebras, $\mathcal{A}$ and $\mathcal{B}$ are *strongly Morita equivalent* when there is a C*-$\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{E}$ and a C*-$\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{F}$ such that

$$\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} \simeq \mathcal{B} \qquad \mathcal{F} \otimes_{\mathcal{B}} \mathcal{E} \simeq \mathcal{A} \tag{165}$$

as $\mathcal{B}$ and $\mathcal{A}$ bimodules respectively. We refer to $\mathcal{E}$ and $\mathcal{F}$ as *equivalence bimodules.*

THEOREM 6.1. *The algebras $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ are strongly Morita equivalent if and only if α and β are in the same orbit of the action of $GL(2,\mathbb{Z})$ on irrational numbers by linear fractional transformations:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \theta = \frac{a\theta + b}{c\theta + d}. \tag{166}$$

Proof Sketch: The proof is due to Rieffel ([**18**]). The crucial observation is the fact that the C*-algebras $\mathcal{A}^2_\theta$ and $\mathcal{A}^2_{(\theta^{-1})}$ are strongly Morita equivalent ([**11**], proposition 12.19). Moreover, we already know that $\mathcal{A}^2_{\theta+n}$ is isomorphic to $\mathcal{A}^2_\theta$.

Let now $GL(2,\mathbb{Z})$ be the group of 2×2 invertible matrices with entries in $\mathbb{Z}$. If we let it act on the set of irrational numbers via fractional linear transformations as in (166), we immediatly see that the two matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{167}$$

generate $GL(2,\mathbb{Z})$ and carry θ to θ^{-1} and to $\theta + 1$ respectively.
It follows that if θ and ϕ are two irrational numbers in the same orbit of the action (166), then $\mathcal{A}^2_\theta$ and $\mathcal{A}^2_\phi$ are strongly Morita equivalent. □

COROLLARY 6.2. *For θ rational, the algebra $\mathcal{A}^2_\theta$ is Morita equivalent to ths algebra of functions on $\mathbb{T}^2$.*

CHAPTER 4

Differential Operators

The study of differential operators is certainly mainly motivated by physics. The Laplace operator appears in many equations of mathematical physics, like the wave and heat equation, and also in the Schrödinger equation. The Dirac operator, for instance, appears in the Dirac equation which was developed to study the relativistic dynamics of the electron (for an historical outline see the introduction of [**24**]).

However, the two operators have wide application also in mathematics, for instance in complex analysis and algebraic geometry. Solutions to the Cauchy-Riemann equation, with the Cauchy Riemann operator being a two dimensional Dirac operator, are holomorphic functions, while solution to the Laplace equation are harmonic functions. Examples of harmonic functions are the real and imaginary part of a holomorphic function. From this fact, one immediately guesses that the two operator must be connected. Indeed, the Dirac operator is, roughly speaking, a *square root* of the Laplacian.

In this section we will introduce the noncommutative analogues of the Dirac and Laplace operator for the case of noncommutative tori.

1. Derivations and Differential Operators

Derivations are a generalization of the well known concept of derivative of a function. Since noncommutative spaces, like the noncommutative torus, are C*-algebras, we need to define what derivations on an algebra are.

Definition 1.1. Let $\mathcal{A}$ a $\mathbb{C}$-algebra. A linear map $d : \mathcal{A} \to \mathcal{A}$ is called a *derivation* on $\mathcal{A}$ if, for all $a, b \in \mathcal{A}$, $\lambda \in \mathbb{C}$ the Leibniz rule:

$$d(ab) = d(a)b + ad(b) \tag{168}$$

is satisfied.

Example 1.1. Let $\mathcal{M}$ be a smooth compact manifold, $C^\infty(\mathcal{M})$ the algebra of smooth complex valued functions on $\mathcal{M}$. Every smooth vector field X on $\mathcal{M}$ defines a derivation on $C^\infty(\mathcal{M})$. (see [**7**] for the details).

Let us come back to the algebra $\mathcal{A}_\theta^2$, which we have seen to be generated by two unitary operators U and V (4.2 and fol.). Thus, a derivation will be completely determined by how it acts on the two generators. In particular, we can define two derivations, called the *basic derivations*, which act separately on the unitaries U and V.

We define the derivations on the noncommutative torus, i.e. on the dense smooth subalgebra $\mathcal{A}_\theta^\infty$, since in $\mathcal{A}_\theta^\infty$ we have convergence of the Fourier series (cf. Section 4.1).

DEFINITION 1.2. The *basic derivations* δ_1, δ_2 on the noncommutative torus $\mathcal{A}_\theta^\infty$, are defined by the following rules:

$$\delta_1(U) = 2\pi i U \quad \delta_1(V) = 0 \tag{169}$$

$$\delta_2(V) = 2\pi i V \quad \delta_2(U) = 0 \tag{170}$$

By routine calculation, we get that for every $a \in \mathcal{A}_\theta^\infty$,

$$a = \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n$$

the basic derivations act in the following way.

$$\delta_1(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n) = 2\pi i \sum_{m,n\in\mathbb{Z}} m\ a_{m,n} U^m V^n \tag{171}$$

$$\delta_2(\sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n) = 2\pi i \sum_{m,n\in\mathbb{Z}} n\ a_{m,n} U^m V^n \tag{172}$$

The Leibniz rule can be checked by induction and using the explicitly formula for the product (A).

We point out that in the commutative case, i.e. for the algebra $\mathcal{A}_0^\infty = C^\infty(\mathbb{T})$, this definition coincides with the partial derivatives $\delta_1 = \frac{\partial}{\partial x}$ and $\delta_2 = \frac{\partial}{\partial y}$. Indeed, for

$$f(x,y) = \sum_{m,n} a_{m,n} e^{2\pi i(mx+ny)}$$

we have

(173)

$$\delta_1(\sum_{m,n\in\mathbb{Z}} a_{m,n} u^m v^n) = 2\pi i \sum_{m,n\in\mathbb{Z}} m a_{m,n} u^m v^n = 2\pi i \sum_{m,n\in\mathbb{Z}} m a_{m,n} e^{2\pi i(mx+ny)} =$$

$$= \frac{\partial}{\partial x} \sum_{m,n\in\mathbb{Z}} a_{m,n} e^{2\pi i(mx+ny)} = \frac{\partial}{\partial x} f(x,y)$$

and an analogue equation holds for δ_2.

PROPOSITION 1.1. *The basic derivations on $\mathcal{A}_\theta^\infty$ commute, i.e:*

$$[\delta_1, \delta_2] = 0. \tag{174}$$

PROOF. Let $a \in \mathcal{A}_\theta^\infty 2$, $a = \sum_{m,n} a_{m,n} U^m V^n$. From definition 1.1 we get

(175)

$$\delta_2(\delta_1(\sum_{m,n\in\mathbb{Z}} a_{m,n}U^mV^n)) = 2\pi i\delta_2(\sum_{m,n\in\mathbb{Z}} ma_{m,n}U^mV^n) = -4\pi^2(\sum_{m,n\in\mathbb{Z}} nma_{m,n}U^mV^n)$$

(176)

$$\delta_1(\delta_2(\sum_{m,n\in\mathbb{Z}} a_{m,n}U^mV^n)) = 2\pi i\delta_1(\sum_{m,n\in\mathbb{Z}} na_{m,n}U^mV^n) = -4\pi^2(\sum_{m,n\in\mathbb{Z}} mna_{m,n}U^mV^n)$$

And since n, m are just numbers, we get commutativity. □

PROPOSITION 1.2. *For every $a \in \mathcal{A}_\theta^\infty$, $\tau(\delta_i(a)) = 0$, for $i = 1, 2$.*

PROOF. This follows from the definition. Indeed, for $a = \sum_{m,n} a_{m,n} U^m V^n$ we have

$$\tau(\delta_1(a)) = \tau\left(2\pi i \sum_{m,n\in\mathbb{Z}} ma_{m,n}U^mV^n\right) = 0$$

and similarly for δ_2. □

COROLLARY 1.3 (Integration by parts). *If $a, b \in \mathcal{A}_\theta^\infty$, then*

$$\tau(\delta_i(a)b) = -\tau(a\delta_i(b)), \qquad i = 1, 2$$

PROOF. By 1.2 have

$$0 = \tau(\delta_i(ab)) = \tau(\delta_i(a)b) + \tau(a\delta_i(b)).$$

□

2. Cauchy Riemann Operator

Given the two derivations δ_1 and δ_2, we can form complex linear combinations to construct new derivations. The first one we consider is the noncommutative analogue of the so-called *Cauchy-Riemann Operator*. In the commutative case, the importance of this operator is given by the fact that holomorphic functions belong to its kernel. We recall that a function $f : \mathbb{R}^2 \to \mathbb{C}$ is holomorphic, if and only if it satisfies the Cauchy Riemann equation.

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} =: \overline{\partial} f(x, y) = 0. \tag{177}$$

With abuse of notation, we define the noncommutative Cauchy-Riemann operator on the noncommutative torus as

$$\overline{\partial} = \delta_1 + i\delta_2. \tag{178}$$

PROPOSITION 2.1. (1) *The operator* $\overline{\partial} : \mathcal{A}_\theta^\infty \to \mathcal{A}_\theta^\infty$ *has a kernel given by scalar multiples of the identity and restricts on a bijection on* $\ker(\tau)$.

(2) *The spectrum of* $\overline{\partial}$ *consists of complex numbers of the form* $2\pi(im - n)$, *with* $(m, n) \in \mathbb{Z}^2$, *with corresponding eigenfunctions* $U^m V^n$.

PROOF. The statements follow from the fact that for $a = \sum_{m,n} a_{m,n} U^m V^n$, we have

$$\overline{\partial} a = 2\pi i \sum_{m,n} (m + in) a_{m,n} U^m V^n. \tag{179}$$

Indeed, if $a \in \ker(\overline{\partial})$ the only coefficient which can differ from zero is $a_{0,0}$. Therefore a must be a scalar multiple of the identity. From equation 179 we also conclude that $\tau(\overline{\partial} a) = 0$.

Now let $a \in \mathcal{A}_\theta^\infty \cap \ker(\tau)$. Then $a = \sum_{(m,n) \neq (0,0)} a_{m,n} U^m V^n$, with the coefficients satisfying condition 25. Then also the coefficients of $\overline{\partial}(a)$, which are given by $\{(m + in) a_{m,n}\}_{(m,n) \in \mathbb{Z}^2}$, belong to $\mathcal{S}(\mathbb{Z}^2)$. Therefore $\overline{\partial}(a)$ automatically belongs to $\mathcal{A}_\theta^\infty \cap \ker(\tau)$.

To prove injectivity, consider $a = \sum_{m,n} a_{m,n} U^m V^n$ with $\overline{\partial} a = 0$.
Since $m + in \neq 0$ for every $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, it follows that $a_{m,n} = 0$ for every $(m, n) \neq 0$ and since a has zero trace, a must be the trivial element.

Let $a \in \mathcal{A}_\theta^\infty \cap \ker(\tau)$. Since $a = \sum_{(m,n) \neq (0,0)} a_{m,n} U^m V^n$ we can build a preimage for a, namely $\sum_{m,n} b_{m,n}$ and chose $b_{m,n} = (m\tau + n)^{-1} a_{m,n}$. It is immediate to check that the coefficients $b_{m,n}$ satisfy (25).

From equation (179) it follows that the eigenvalues of $\overline{\partial}$ have the form

$$2\pi i(m + in) \;=\; 2\pi(im - n) \qquad (m, n) \in \mathbb{Z}^2,$$

with corresponding eigenvectors $U^m V^n$.
Conversely, since $\overline{\partial} a = 2\pi i \sum_{m,n} (m + in) a_{m,n} U^m V^n$, the only possible eigenfunctions for the operator are those of the form $U^m V^n$, which concludes the proof, since by the spectral theorem we have that the operator has pure point spectrum.[1] □

[1] To prove this, we need to extend $\overline{\partial}$ to an operator on the Hilbert space $\mathcal{H}_0$ carrying the GNS representation (cf. Chapter 2, Section 4) of $\mathcal{A}_\theta^\infty$, i.e. the completion of $\mathcal{A}_\theta^2$ in the norm induced by the tracial state τ. We will encounter this Hilbert space again in Section 4.

The fact that the kernel is given by scalar multiples of the identity has a commutative analogue: every holomorphic function on the torus, which is a compact manifold, is constant by Liouville's theorem.

We can also consider the complex conjugate of the operator 178:

$$\partial = \delta_1 - i\delta_2 \tag{180}$$

which satisfies similar properties.

PROPOSITION 2.2. (1) *The operator* $\partial : \mathcal{A}_\theta^\infty \to \mathcal{A}_\theta^\infty$ *has a kernel given by scalar multiples of the identity and restricts on a bijection on* $\ker(\tau)$*.*

(2) *The spectrum of* ∂ *consists of complex numbers of the form* $2\pi(im+n)$*, with* $(m,n) \in \mathbb{Z}^2$*, with corresponding eigenfunctions* $U^m V^n$*.*

PROOF. The proof is the same as above (cf. 2.1). □

Moreover, we have the following:

PROPOSITION 2.3. *For every* $a \in \mathcal{A}_\theta^\infty$ *we have*

$$\overline{\partial}(a)^* = \partial(a^*). \tag{181}$$

PROOF. It follows from explicit calculations:

$$\partial(a^*) = \partial((\sum_{n,n} a_{m,n} U^m V^n)^*) = \partial(\sum_{m,n} \overline{a_{m,n}} U^{-m} V^{-n}) =$$

$$= 2\pi i \sum_{m,n} (-m+in)\overline{a_{m,n}} U^{-m} V^{-n} = -2\pi i \sum_{m,n} (m-in)\overline{a_{m,n}} U^{-m} V^{-n} =$$

$$= (2\pi i \sum_{m,n} (m+in) a_{m,n} U^m V^n)^* = \overline{\partial}(a^*).$$

□

Summarizing, in this section we have defined a noncommutative Cauchy-Riemann operator, and we have seen that it has a lot in common with its commutative analogue. Furthermore, as we have already pointed out, we can consider general complex linear combinations of our basic derivations, which will lead us to the definition of more general differential operators. We will develop this idea in Section 4.

2.1. A family of derivations on the noncommutative Torus. In analogy with [**26**], we define a family of derivations ∂_ω, with ω being a complex number, $Im(\omega) > 0$. We will explain in the following why the parameter ω is supposed to have imaginary part strictly greater than zero.

Definition 2.1. Given a number $\omega \in \mathbb{C} \setminus \mathbb{R}$ we define a derivation $\partial_\omega : \mathcal{A}_\theta^\infty \to \mathcal{A}_\theta^\infty$, which, up to a scaling factor, is a linear combination of the two basic derivations δ_1, δ_2:

$$\partial_\omega := \delta_1 + \omega\delta_2 \qquad \omega \in \mathbb{C}. \tag{182}$$

Using the Fourier-like expansion for $a \in \mathcal{A}_\theta^\infty$, behaves in the following way:

$$\sum a_{m,n} U^m V^n \mapsto 2\pi i \cdot \sum_{m,n} (m + n\omega) a_{m,n} U^m V^n. \tag{183}$$

We observe that, since we could replace ∂_ω by $-\omega^{-1}\partial_\omega = \delta_2 + \omega^{-1}\delta_1$, we may assume that $Im(\omega) > 0$.

We remark that the Cauchy Riemann operator defined in (178) is a particular case of this operator, with $\omega = i$.

Proposition 2.4. *The derivation ∂_ω defined in (2.1) satisfies following properties:*

(1) *The operator $\partial_\omega : \mathcal{A}_\theta^\infty \to \mathcal{A}_\theta^\infty$ has a kernel given by scalar multiples of the identity and restricts on a bijection on* $\ker(\tau)$.
(2) *The spectrum of ∂_ω consists of complex numbers of the form $m + \omega n$, with $(m, n) \in \mathbb{Z}^2$, with corresponding eigenfunctions $U^m V^n$.*
(3) $\partial_\omega(a)^* = \partial_{\overline{\omega}}(a^*)$,

Proof. The proof is a generalization of the ones for Proposition 2.1 and 2.3. □

The importance of this family of operators, is that it allows us to construct a family of Dirac-like operators on the noncommutative torus. The construction of such operators will be pursued in Section 4.

3. The Laplacian on The Noncommuative Torus

This section is mainly based on a recent paper by Rosenberg ([**20**]), where an analogue of the Laplace operator for the noncommutative torus is constructed.
This operator is an important example of noncommutative elliptic differential operator, whose definition was given by Connes in his breakthrough paper [**2**].

Definition 3.1. A *noncommutative partial differential operator* of order k is a sum

$$D = \sum_{|\alpha| \leq k} a_\alpha \delta^\alpha, \qquad a_\alpha \in \mathcal{A}_\theta^\infty \tag{184}$$

In (184) we have used the multi-index notation, i.e. $\alpha = (\alpha_1, \alpha_n)$ is a couple of integers and $\delta^\alpha = \delta_1^{\alpha_1}\delta_2^{\alpha_2}$.

Moreover, we say that D is elliptic if for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$, the principal symbol $\sigma(\xi) = \sum_{|\alpha|=k} a_\alpha \xi^\alpha$ is invertible.

Typically, one is interested in first and second order differential operator. We remark that the operators $\overline{\partial}$, ∂ and the family of operators ∂_ω are examples of first order elliptic differential operators. As example of second order differential operator we define the following.

Definition 3.2. Using the derivations δ_i as in Definition 1.2, we define the *noncommutative Laplace operator* as

$$\Delta = \delta_1^2 + \delta_2^2. \tag{185}$$

For the upcoming propositions and for their proofs we refer once again to [**20**].

Proposition 3.1. *The spectrum of the Laplace operator (185) is made of real numbers of the form*

$$\{-4\pi^2(m^2 + n^2) \mid m, n \in \mathbb{Z}^2\}. \tag{186}$$

Proof. Definition (1.1) implies $\delta_1(U^mV^n) = \delta_1(U^m)V^n$ and $\delta_2(U^mV^n) = U^m\delta_2(V^n)$. Therefore

$$\begin{aligned}\Delta(U^mV^n) = \delta_1^2(U^mV^n) + \delta_2^2(U^mV^n) &= \\ = \delta_1(\delta_1(U^mV^n) + \delta_2(\delta_2(U^mV^n) &= \\ = \delta_1(\delta_1(U^m)V^n) + \delta_2(U^m\delta_2(V^n)) &= \\ = 2\pi i(m\delta_1(U^mV^n) + n\delta_2(U^mV^n)) &= \\ = 2\pi i(m(2\pi imU^mV^n) + n(2\pi inU^mV^n)) &= \\ = -4\pi^2(m^2+n^2)U^mV^n&.\end{aligned}$$

Hence $\{-4\pi^2(m^2 + n^2) \mid (m, n) \in \mathbb{Z}^2\} \subseteq \sigma(\Delta)$.
Conversely, since $\Delta(\sum_{m,n} a_{m,n}U^mV^n) = \sum_{m,n} a_{m,n}(-4\pi^2(m^2+n^2))U^mV^n$, the only possible eigenfunctions for the operator are those of the form U^mV^n. As in the proof of 2.1, the operator has pure point spectrum, which concludes the proof. □

So the spectrum of the Laplacian on the noncommutative torus coincides with the spectrum of the Laplace operator in the commmutative case (cf. 1 3.3).

PROPOSITION 3.2. *For any $\lambda > 0$ (or not of the form $-4\pi^2 n$, $n \in \mathbb{Z}$)*

$$-\Delta + \lambda : \mathcal{A}_\theta^\infty \to \mathcal{A}_\theta^\infty \tag{187}$$

is bijective.

PROOF. We first observe that

$$(-\Delta + \lambda)\sum_{m,n} a_{m,n}U^mV^n = \sum_{m,n}(4\pi^2(m^2+n^2)+\lambda)U^mV^n$$

with $(4\pi^2(m^2+n^2)+\lambda) > 0$. Moreover

$$a = \sum_{m,n} a_{m,n}U^mV^n \in \ker(-\Delta+\lambda) \Longleftrightarrow$$
$$\Longleftrightarrow (-\Delta+\lambda)\sum_{m,n} a_{m,n}U^mV^n = 0 \Longleftrightarrow$$
$$\Longleftrightarrow \sum_{mn}(4\pi^2(m^2+n^2)+\lambda)U^mV^n = 0 \Longleftrightarrow$$
$$\Longleftrightarrow (4\pi^2(m^2+n^2)+\lambda)a_{m,n} = 0 \text{ for all } m,n \in \mathbb{Z} \Longleftrightarrow$$
$$\Longleftrightarrow a_{m,n} = 0 \text{ for all } m,n \in \mathbb{Z}.$$

Therefore $-\Delta + \lambda$ is injective.
The inverse of an element $\sum_{m,n\in\mathbb{Z}} c_{m,n}U^mV^n$ is given by

$$\sum_{m,n} \frac{1}{4\pi(m^2+n^2)+\lambda} c_{m,n}. \tag{188}$$

This yields a map : $\mathcal{A}_\theta^\infty \to \mathcal{A}_\theta^\infty$, since, for $c_{m,n}$ rapidly decreasing, $(4\pi^2(m^2+n^2)+\lambda)^{-1}c_{m,n}$ is rapidly decreasing as well. □

PROPOSITION 3.3. *The image of $\Delta : \mathcal{A}_\theta^\infty \to \mathcal{A}_\theta^\infty$ is precisely $\mathcal{A}_\theta^\infty \cap \ker(\tau)$, i.e the smooth elements with zero trace.*

PROOF. We have

$$\Delta(\sum_{m,n} a_{m,n}U^mV^n) = -4\pi^2\sum_{m,n}(m^2+n^2)a_{m,n}U^mV^n =$$
$$= -4\pi^2 \sum_{(m,n)\neq(0,0)} (m^2+n^2)a_{m,n}U^mV^n,$$

since the factor m^2+n^2 kills the term with $m = n = 0$. Therefore $Im(\Delta) \subseteq \ker(\tau)$.
Conversely, let $a \in \ker(\tau) \cap \mathcal{A}_\theta^\infty$. This means $a_{00} = 0$ and $a_{m,n}$ rapidly decreasing.

Then $a_{m,n}/m^2 + n^2$ is rapidly decreasing and

$$\sum_{(m,n)\neq(0,0)} \frac{-a_{m,n}}{4\pi^2(m^2+n^2)} U^m V^n \tag{189}$$

converges to an element $b \in \mathcal{A}_\theta^\infty$ with $\Delta b = a$. □

The operators defined in this chapter have a lot of similarities with their commutative analogues.

THEOREM 3.4. *The operators* $\Delta, \partial, \overline{\partial}$ *satisfy*

$$\Delta = \overline{\partial}\partial. \tag{190}$$

PROOF. This can be easily proven by explicit calculations. □

DEFINITION 3.3. Let $a \in \mathcal{A}_\theta^\infty$. An element a is said to be *harmonic*, if $\Delta a = 0$, *subarmonic* if $\Delta a \geq 0$.

THEOREM 3.5. *Let* $a \in \mathcal{A}_\theta^\infty$ *be subarmonic. Then* a *is constant.*

PROOF. By proposition (3.3), we have $\tau(\Delta a) = 0$. We know that the tracial state defined on $\mathcal{A}_\theta^\infty$ is faithful (section 3.4.2), which means that if $b \geq 0$ satisfies $\tau(b) = 0$. We get that $\Delta a = 0$, which implies $a \in \ker(\Delta)$. Therefore it is a scalar multiple of the identity. □

COROLLARY 3.6. *Every harmonic on the noncommutative torus is constant.*

This is a noncommutative generalization of the fact that harmonic functions on the torus, which is a compact manifold, are constant. This does not happen, for instance, in the complex plane, where real and imaginary part of every holomorphic function are harmonic and, in general, not constant.

4. The Dirac Operator

In this section, we show how to construct analogues of the Dirac Operator for the noncommutative torus. The definition differs from the one we gave in (184), since, in complete analogy with the commutative case, the operator is constructed as a linear combinations of derivatives, with the coefficients living in some matrix algebra.

In the commutative case, one needs to consider Clifford algebras and Spinors. In the noncommutative case, the construction is pursued in terms of a *spectral triple*, an object which was conceived by Connes ([**4**]) to generalize *spin geometry* and *index theory* to noncommutative spaces.

Roughly speaking, a spectral triple is a set of 3 objects $(\mathcal{A}, \mathcal{H}, \not{D})$, where $\mathcal{H}$ is a Hilbert space, $\mathcal{A}$ is an algebra represented on $\mathcal{H}$ and $\not{D}$ a densely defined self-adjoint operator on $\mathcal{A}$.

There are several axioms that the elements of a spectral triple must satisfy and that are modelled on the properties of the Dirac opreator in the commutative case. For all the details, we refer to [**26**], in particular to Chapter 3 for the axiomatic foundation and to Chapter 4 for the construction and the proof that the triple, that we will present in a while, satisfies all axioms.

The ingredients of our spectral triple are the following:

(1) Our algebra is the dense subalgebra $\mathcal{A}_\theta^\infty$.
(2) The Hilbert space is $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0$, where $\mathcal{H}_0$ is the Hilbert space carrying the GNS representation (cf. Section 4) of $\mathcal{A}_\theta^2$.
More precisely, $\mathcal{H}_0$ is the completion of $\mathcal{A}_\theta^2$ in the norm induced by the tracial state τ: for all $a \in \mathcal{A}_\theta^2$

$$\|a\|_2 = \sqrt{\tau(a^* a)}.$$

Since the tracial state τ is faithful, we have the inclusion $\mathcal{A}_\theta^\infty \hookrightarrow \mathcal{H}_0$. We will denote the image of $a \in \mathcal{A}_\theta^\infty$ in $\mathcal{H}_0$ with $\underline{a}$.
The GNS representation is the left regular representation, i.e. for all $a \in \mathcal{A}_\theta^\infty$, the operator $\pi(a)$ is given by

$$\pi(a) : \underline{b} \mapsto \underline{a}\ \underline{b}.$$

(3) We recall that in the commutative case, the 2-dimensional Dirac operator has the form

$$\not{D} = -i\sigma_i \delta_i = -i(\sigma_1 \partial_x + \sigma_2 \partial_y), \tag{191}$$

with σ_1, σ_2 being the first 2-Pauli matrices[2], and it is defined on the space of Dirac spinors.

Analogously, we define the noncommutative Dirac operator as:

(192)
$$\not{D} = -i\sigma_i\delta_i = -i(\sigma_1\delta_1 + \sigma_2\delta_2),$$

with δ_1, δ_2 being the basic derivations (1.2). In matrix form, the operator becomes

$$\not{D} = \begin{pmatrix} 0 & \delta_1 \\ \delta_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i\delta_2 \\ i\delta_2 & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & \partial \\ \overline{\partial} & 0 \end{pmatrix},$$

i.e. it involves both derivations $\partial, \overline{\partial}$ defined in 2.

Clearly, it is also possible to define a family of spectral triples, as anticipated in 2.1 using the derivations ∂_ω: we can consider the spectral triple $(\mathcal{A}_\theta^\infty, \mathcal{H}, \not{D}_\omega)$ with the Dirac Operator given by

$$\not{D} = -i \begin{pmatrix} 0 & \partial_\omega \\ \overline{\partial}_\omega & 0 \end{pmatrix}.$$

We have seen that the differential operators presented in this chapter have a lot of similarities with their commutative analogues. This comes from the commutativity of the derivations (cf. Proposition 1.1) and from the fact that the algebras $\mathcal{A}_\theta^\infty$ and $C^\infty(\mathbb{T})$, as already remarked in 4.1, are isomorphic as vector spaces. Indeed, the noncommutativity of $\mathcal{A}_\theta^\infty$ is present only at the level of the algebra product, and when one writes a general element in terms of its Fourier series

$$a = \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n,$$

the noncommutativity of the generators is already included in the coefficients $a_{m,n}$.

[2]Recall that the Pauli matrices are a set of 2×2 complex Hermitian matrices which are defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

and which satisty following commutation relations:

$$\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij}\mathbb{I}.$$

CHAPTER 5

Conclusions

After a brief introduction on Hilbert space theory and Fourier analysis, in this work it has been our intention to give a brief insight into the ideas of Noncommutative Geometry, in particular into some techniques coming from functional analysis, like the theorem of Gelfand and Naimark on commutative C*-algebras (3.3) and the GNS construction (cf. Section 4), which allow us to define a notion of *noncommutative space*.

We have chosen to investigate one space in particular, the *irrational rotation algebra* which is maybe the easiest and certainly the most known and best studied example of noncommutative space. In Chapter 3 we have shown how this space can be constructed as a *crossed product* algebra, and we have studied some of its features, like uniqueness of the trace, isomorphisms and Morita equivalence between different rotation algebras. Moreover, we have constructed its smooth dense subalgebra, the *noncommutative torus*, which we have used mainly to define derivations and differential operators, like the Dirac operator and the Laplacian.

As we have already pointed out, the space we have taken into account has been considered with such attention, that it has a broaden literature on it; therefore our treatment is clearly not exhaustive. There are many other possible topics one could focus on, and the space itself can be considered from other points of view. In this section we want to give a brief insight into some possible further developments and alternative approaches.

1. Higher Dimensional Nonccomutative Tori

As a first topic, one could choose to consider higher dimensional rotation algebras, which are defined in terms of skew symmetric real matrices.

Definition 1.1. Let θ be a real skew symmetric matrix with entries θ_{jk}. We define the *n-dimensional rotation algebra* $\mathcal{A}_\theta^n$ as the universal C*-algebra generated by n unitaries $U_1, \dots, U_n$ subject to the commutation relations

$$U_k U_j = e^{2\pi i \theta_{jk}} U_j U_k, \qquad i, j = 1, \dots, n. \tag{193}$$

It is easy to see that in the two-dimensional case the definition is consistent with the one we gave for the rotation algebra $\mathcal{A}^2_\theta$: every skew symmetric 2×2 real matrix is determined by a real parameter θ and has the form

$$\begin{pmatrix} 0 & \theta \\ -\theta & 0. \end{pmatrix} \tag{194}$$

Using (193) we get the same commutation relation as (143).

In the two dimensional case, one is mainly interested in the case of an irrational parameter θ. In higher dimensions, we are not dealing with a single real number anymore, but rather with a matrix.

DEFINITION 1.2 ([**11**]). A skew-symmetric $\theta \in \mathrm{Mat}_n(\mathbb{R})$ is said to be *quite irrational* if the lattice Λ_θ generated by its columns is such that $\Lambda_\theta + \mathbb{Z}^n$ is dense in $\mathbb{R}^n$.

It is easy to see, using the exponential map, that a 2×2 skew-symmetric real matrix, parametrized by a real number θ, is quite irrational if and only if $\theta \notin \mathbb{Q}$.

The importance of this definition is that, as anticipated in theorem 4.3, there is a uniqueness result for the trace on $\mathcal{A}^n_\theta$: quite irrationality of the matrix provides a sufficient condition.

PROPOSITION 1.1 ([**11**], proposition 12.11). *If θ is quite irrational, the tracial state τ on $\mathcal{A}^n_\theta$ is unique.*

We recall from linear algebra that any skew-symmetric real $n \times n$ matrix θ can be brought into the canonical skew diagonal form with skew eigenvalues θ_i, by an orthogonal transformation, i.e.

$$\theta \sim \theta' = R^T \Theta R \qquad \text{with} \quad R^T R = R R^T = \mathbb{I} \tag{195}$$

$$\theta' = \begin{pmatrix} 0 & \theta_1 & & & & \\ -\theta_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \theta_r & \\ & & & -\theta_r & 0 & \\ & & & & & \mathbf{0}_{n-2r} \end{pmatrix} \tag{196}$$

Clearly a skew-symmetric matrix θ in the block diagonal form is quite irrational if and only if $\theta_i \in \mathbb{R} \setminus \mathbb{Q}$ for all i.

Given an arbitrary matrix θ and denoted with θ' its block-diagonal form, a natural question is whether there is some connection between the rotation algebras generated by θ and θ'. Those algebras are in general non-isomorphic ([**1**]). On the other hand, we already know that we have a weaker equivalence relation for noncommutative spaces. If we ask for Morita equivalence, there are several results in this direction, due to Rieffel et al. ([**17**], [**8**], [**13**]).
The Morita equivalence is obtained in terms of an action of the group $SO(n,n|\mathbb{Z})$ on the space of $n \times n$ skew-symmetric real matrices. More precisely, the group $SO(n,n|\mathbb{Z})$ is the group of $2n \times 2n$ matrices with integer entries and determinant 1 preserving the quadratic form

$$x_1 x_{n+1} + \cdots + x_n x_{2n}.$$

The elements of $SO(n,n|\mathbb{Z})$ can be written in 2×2 block form:

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{197}$$

with A, B, C, D being $n \times n$ matrices satisfying

$$A^t C + C^t A = 0 = B^t D + D^t B \qquad A^t D + C^t B = \mathbb{I}. \tag{198}$$

The action of $SO(n,n|\mathbb{Z})$ on an arbitrary skew symmetric matrix θ is defined as

$$G\theta = (A\theta + B)(C\theta + D)^{-1}, \tag{199}$$

whenever $C\theta + D$ is invertible. This action is clearly a generalization to higher dimension of the action defined in (166).

In [**13**] it is shown that whenever $G\theta$ is defined, the rotation algebras generated by θ and by $G\theta$ are strongly Morita equivalent.

An interesting development would be to show whether the process of bringing a matrix θ in skew-symmetric diagonal form can always be realized in terms of the action (199) for a suitable $G \in SO(n,n|\mathbb{Z})$. This would give a complete answer to the problem addressed above.

2. Differential Operators and Spectral Triples

The construction of the Dirac and Laplace operators we have pursued in Chapter 4 admit generalizations to higher dimensions as well.
First of all, one needs to consider the noncommutative n-dimensional torus, i.e. the smooth subalgebra $\mathcal{A}^{\infty}_{\theta,n}$ of formal series with coefficients living in $\mathcal{S}(\mathbb{Z}^n)$.

In complete analogy with Definition 1.2, one can define n basic derivations $\delta_1, \ldots, \delta_n$ on $\mathcal{A}^\infty_{\theta,n}$, which act separately on the generators.
The Laplacian on $\mathcal{A}^\infty_{\theta,n}$ is then defined as

$$\Delta = \delta_1^2 + \cdots + \delta_n^2. \tag{200}$$

For the Dirac operator, it is natural to consider operators of the form

$$\not{D} = \sum_{i=1}^{n} \gamma^i \delta_i \tag{201}$$

where the γ_i's are matrices satisfying

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} \tag{202}$$

Again, the correct setting is the one of *spectral triples* (cf. Chapter 4, Section 4).

An interesting problem would be to study how the behavior of the Dirac and Laplace operator changes if we move from a general rotation algebra, given in terms of a matrix θ, to the one determined by the block diagonal matrix θ' which is unitary equivalent to θ.

Another possible direction is trying to solve general differential equations in the noncommutative case. For some non-linear equations involving the Laplacian, this is done in [**20**].

Moreover, if one tries to solve the equation

$$\partial_\omega x = xa \tag{203}$$

with $a \in \mathcal{A}^\infty_\theta$ and ∂_ω defined as in Section (2.1), one gets to the definition of a noncommutative analogue of the exponential function. This was recently done by Polishchuck ([**15**]) using techniques coming from algebraic geometry and combinatorics and should be related to the theory of so-called *noncommutative elliptic curves.*

In conclusion, the noncommutative torus has, despite its apparent simplicity, many interesting features, which are connected with different areas of Mathematics. This is perhaps the reason why it has been so intensively studied over the last decades.

APPENDIX A

Explicit Formulas for the Product in $\mathcal{A}_\theta^\infty$

In this appendix, we show how to compute the product of elements in the non-commutative torus $\mathcal{A}_\theta^\infty$.

If $a = \sum_{m,n} a_{mn} U^m V^n$ and $b = \sum_{m,n} b_{mn} U^m V^n$, then ab has Fourier coefficients given by the twisted convolution of the coefficients:

$$
\begin{aligned}
ab = \left(\sum_{m,n} a_{mn} U^m V^n\right)\left(\sum_{k,l} b_{kl} U^k V^l\right) &= \\
= \sum_{m,n,k,l} a_{mn} b_{kl} U^m V^n U^k V^l &= \\
= \sum_{m,n,k,l} a_{mn} b_{kl} e^{-2\pi i \theta k n} U^m U^k V^n V^l &= \\
= \sum_{m,n,k,l} a_{mn} b_{kl} e^{-2\pi i \theta k n} U^{m+k} V^{n+l} &= \\
= \sum_{p,q} f_{pq} U^p V^q &
\end{aligned} \tag{204}
$$

where

$$f_{pq} = \sum_{m,n} c_{mn} d_{p-m,q-n} e^{-2\pi i (p-m) n \theta}. \tag{205}$$

Sometimes it is also useful to introduce this compact notation: for every $v = (m, n) \in \mathbb{Z}^2$, we define the so-called *Weyl element*

$$U_v = exp(-\pi i \theta m n) U^m V^n. \tag{206}$$

Then we have the following rule in $\mathcal{A}_\theta^\infty$:

$$U_v \cdot U_v' = exp(2\pi i < v, v' >) U_{v+v'} \tag{207}$$

where we have set

$$< v, v' >=< v, v' >_\theta = \frac{1}{2}(mn' - m'n). \tag{208}$$

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