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Principal circle bundles, Pimsner algebras and Gysin sequences

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Dedico questa tesi ai miei genitori e a mio fratello Giuliano: senza il loro sostegno e affetto incondizionato, niente di tutto questo sarebbe stato possibile.

Preface

This thesis is the result of my Ph.D. studies at the International School for Advanced Studies (SISSA) in Trieste, Italy.

My Ph.D. project, conducted under the supervision of Prof. Giovanni Landi (Università di Trieste), with Prof. Ludwik Dąbrowski as internal co-advisor, focused on topological aspects of (noncommutative) principal circle bundles, and it culminated in the following papers (listed in chronological order):

[1] F. Arici, S. Brain, G. Landi, *The Gysin sequence for quantum lens spaces*, arXiv:1401.6788, J. Noncomm. Geom. in press.
[2] F. Arici, J. Kaad, G. Landi, *Pimsner algebras and Gysin sequences from principal circle actions*, arXiv:1409.5335, J. Noncomm. Geom. in press.
[3] F. Arici, F. D'Andrea, G. Landi, *Pimsner algebras and noncommutative circle bundles*, arXiv:1506.03109, to appear in "Noncommutative analysis, operator theory and applications NAOA2014".

This dissertation aims at presenting the results obtained in the above mentioned works in a clear, coherent and self-consistent way.

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Trieste, September 2015 *Francesca Arici*

Contents

"Would you tell me, please, which way I ought to go from here?"

"That depends a good deal on where you want to get to."

"I don't much care where."

"Then it doesn't matter which way you go."

(Lewis Carroll, Alice in Wonderland*)*

Introduction

Noncommutative topology has its roots in a deep mathematical concept, that of duality between algebra and geometry. Indeed, by Gel'fand's theorem ([37]) to any locally compact topological space X one can associate the commutative C^*-algebra $C_0(X)$ of continuous functions vanishing at infinty, and conversely every commutative C^*-algebra A is isometrically isomorphic to the algebra $C_0(\sigma(A))$ of functions on the spectrum $\sigma(A)$, which is a locally compact Hausdorff space. This correspondence is realized in terms of a contravariant functor, and yields an (anti)-equivalence of categories.

Motivated by this duality, one is led to consider noncommutative C^*-algebras as algebras of functions on some virtual dual *noncommutative space*. Examples are *quantizations* of classical commutative spaces, obtained by deforming the corresponding algebras of functions, the simplest instance of this construction being the noncommutative torus described in Example 1.5.

Noncommutative C^*-algebras also appear as the right algebraic framework for modeling dynamical systems. Group actions on a topological space are encoded in the construction of crossed product C*-algebras. Another class of algebras appearing from the study of dynamical systems are the Cuntz and Cuntz-Krieger algebras of [23] and [25], whose underlying dynamical systems are shifts and subshifts, respectively.

Gel'fand's duality can be extended to the category of complex vector bundles over a compact space X by considering projective modules of finite type over the corresponding C^*-algebra of continuous functions $C(X)$. This is the content of the Serre-Swan theorem of [78, 82], which again provides an (anti)-equivalence of categories. In order to study Hermitian vector bundles, i.e. vector bundles with a fiber-wise Hermitian product, one is then led to consider Hilbert modules over C^*-algebras. The simples example of a Hermitian vector bundle is that of an Hermitian line bundle $L \to X$; the corresponding noncommutative object is a self-Morita equivalence bimodule over a the C^*-algebra $C(X)$, which is a full Hilbert C^*-module $\mathcal{E}$ over B together with an isomorphism of B with the algebra $\mathcal{K}_B(\mathcal{E})$ of compact endomorphisms on $\mathcal{E}$.

In classical geometry complex line bundles are naturally associated to principal circle bundles. One of the aims of this thesis is studying the *noncommutative topology* of principal circle bundles–both classical and noncommutative–by means of Gysin exact sequences in K-theory and KK-theory.

Motivation

Classically, one can associate to every oriented vector bundle and to every sphere bundle a long exact sequence in (singular) cohomology, named the Gysin exact sequence. In the case of a principal circle bundle $\pi : P \to X$ the sequence reads:

$$\cdots \longrightarrow H^k(P,\mathbb{Z}) \xrightarrow{\pi_*} H^{k-1}(X,\mathbb{Z}) \xrightarrow{\alpha} H^{k+1}(X,\mathbb{Z}) \xrightarrow{\pi^*} H^{k+1}(P,\mathbb{Z}) \longrightarrow \cdots,$$

where $\pi^* : H^k(X,\mathbb{Z}) \to H^k(P,\mathbb{Z})$ and $\pi_* : H^k(P,\mathbb{Z}) \to H^{k-1}(P,\mathbb{Z})$ denote the the pull-back and the push-forward map, respectively, and $\alpha : H^{k-1}(X,\mathbb{Z}) \to H^{k+1}(X,\mathbb{Z})$ is defined on forms $\omega \in H^{k-1}(X,\mathbb{Z})$ as the cup product $\alpha(\omega) = c_1(L) \cup \omega$ with the first Chern class $c_1(L)$ of the line bundle $L \to X$ associated to the principal circle bundle $\pi : P \to X$ via the left regular representation.

The above exact sequence admits a version in topological K-theory, in the form of a six-term exact sequence, involving the K-theoretic Euler class $\chi(L)$ of the line bundle $L \to X$.

The Gysin exact sequence plays an important rôle in mathematical physics, in particular in T-duality and in Chern-Simons theory. In T-duality, the $k = 3$ segment of the Gysin sequence in singular cohomology maps the class of the H-flux, a given three-form on the total space P, to a class in $H^2(X,\mathbb{Z})$. This can be thought of as the class of the curvature of a connection on the *T-dual* circle bundle $P' \to X$. In the case of a two-dimensional base manifold X the exact sequence immediately gives an isomorphism $H^3(P,\mathbb{Z}) \simeq H^2(X,\mathbb{Z})$, hence establishing a correspondence between Dixmier-Douady classes on P and line bundles on X (*cf.* pages 385 and 391 of [10]).

In Chern-Simons theory, the importance of the Gysin sequence lies in the evaluation of the path integral on circle bundles over smooth curves, where it facilitates the counting of those circle bundles over the total space which arise as pull-backs from the base (*cf.* page 26 of the [9]).

A class of examples of circle bundles, which are of relevance for both T-duality and Chern-Simons theories, is that of lens spaces. These arise in classical geometry as quotients of odd-dimensional spheres by the action of a finite cyclic group, and they can be seen as total spaces of circle bundles over weighted projective spaces.

In the recent work [3] we focused on their noncommutative counterparts: quantum lens spaces. These have been the subject of increasing interest of late: they first appeared in [59] in the context of what we would now call *theta-deformed* topology; they later surfaced in [46] in the form of graph C^*-algebras, with certain more recent special classes described in [12, 41]. The particular case of the quantum three-dimensional real projective space was studied in [68] and [56]. In parallel with the classical construction, quantum (weighted) lens spaces are introduced as fixed point algebras for suitable actions of finite cyclic groups on function algebras over odd dimensional quantum spheres. Similarly, quantum (weighted) projective spaces are defined as fixed point algebras for a circle action on odd dimensional quantum spheres. More generally, quantum weighted projective spaces can be obtained as fixed point algebras of a circle action on the algebra of odd dimensional lens spaces–of which odd dimensional spheres are a particular example.

At the coordinate algebra level, lens spaces admit a vector space decomposition as direct sums of line bundles, which provides them with a $\mathbb{Z}$-graded algebra structure. A central rôle is played by the module of sections of the tautological line bundle over the quantum projective space. This graded decomposition and the central character played by the tautological line bundle naturally lead one to consider the notion of Pimsner algebra.

Pimsner algebras, principal circle bundles and Gysin sequences

Pimsner algebras, which were introduced in the seminal work [67], provide a unifying framework for a range of important C^*-algebras including crossed products by the integers, Cuntz-Krieger algebras [23, 25], and C^*-algebras associated to partial automorphisms [34]. Due to their flexibility and wide range of applicability, there has been an increasing interest in these algebras recently (see for instance [36, 71]). A related class of algebras, known as generalized crossed products, was independently invented in [1]. The two notions coincide in many cases, in particular in those of interest for the present work.

The connection between principal circle bundles and Pimsner algebras was spelled out, for the commutative case, in [36, Proposition 5.8]: the algebra $C(P)$ of continuous functions on the total space of a principal circle bundle $P \to X$ can be described as a Pimsner algebra generated by a classical line bundle over the compact base space. More precisely, starting from a principal circle bundle P over a compact topological space X, the module of section of any of the associated line bundles is a self-Morita equivalence bimodule for the commutative C^*-algebra $C(X)$ of continuous functions over X. Suitable tensor powers of the (sections of the) bundle are endowed with an algebra structure eventually giving back the C^*-algebra $C(P)$ of continuos functions on P.

In this thesis we extend this analogy and relate the notion of Pimsner algebra to that of a noncommutative (in general) principal circle bundle: a self-Morita equivalence bimodule $\mathpzc{E}$ over an arbitrary C^*-algebra B is thought of as a *noncommutative line bundle* and the corresponding Pimsner algebra $\mathcal{O}_{\mathpzc{E}}$ plays the rôle of the algebra of continuous functions on the total space of a noncommutative a principal circle bundle *associated* to $\mathpzc{E}$.

With a Pimsner algebra come two natural six term exact sequences in KK-theory, which relate the KK-theories of the Pimsner algebra $\mathcal{O}_{\mathpzc{E}}$ with that of the C^*-algebra B. These exact sequences are noncommutative analogues of the Gysin sequence, which, as mentioned before, in the commutative case relates the K-theories of the total space and of the base space of a circle bundle. The classical cup product with the Euler-class is replaced by a Kasparov product with the identity minus the class of the self-Morita equivalence bimodule $\mathpzc{E}$. Predecessors of these six term exact sequences are the Pimsner-Voiculescu six term exact sequences of [66] for crossed products by the integers.

Outline of the thesis

Part I deals with some prerequisites from noncommutative topology: in Chapter 1 we describe C^*-algebras and modules, stating Gel'fand duality and the Serre-Swan theorem. We describe K-theory and K-homology, which are the basic topological invariants used in the study of C^*-algebras, and conclude by describing some important classes of C^*-algebras, namely crossed product by the integers, Cuntz and Cuntz-Krieger algebras. Chapter 2 provides an overview of the theory of Hilbert modules, C^*-correspondences and KK-theory, focusing on Morita equivalence for C^*-algebras and self-Morita equivalence bimodules.

In Part II we describe the connection between Pimsner algebras and principal circle bundles. In Chapter 3 we recall the definition of the Toeplitz and Pimsner algebras of a full C^*-correspondence and the construction of generalized crossed products, focusing on the case of self-Morita equivalence bimodules. In Chapter 4 we give the definition of quantum principal circle bundles using Hopf algebras and $\mathbb{Z}$-graded algebras, and recall how principality of the action can be translated into an algebraic condition on the induced grading. This condition is particularly relevant since it resembles a similar condition appearing in the theory of generalized crossed products. We then show how all these notions are interconnected and can be seen as different descriptions of the same construction. Finally, we provide several examples: we illustrate how theta-deformed and quantum weighted projective and lens spaces fit into the framework.

Part III is devoted to exact sequences and explicit computations, and it contains the results of the recent works [3, 5]. In Chapter 5 we construct a Gysin exact sequence in K-theory for (unweighted) quantum lens spaces of any dimension. We use this exact sequence to compute the K-theory groups of the quantum lens spaces and to construct explicit generators of torsion classes as (combinations) of pulled-back line bundles. In Chapter 6 we think of weighted lens spaces as principal circle bundles over weighted projective lines. We construct Gysin exact sequences in KK-theory, which we use to compute the KK-groups of these spaces. A central character in this computation is played by an integer matrix whose entries are index pairings. The resulting computation of the KK-theory for this class of q-deformed lens spaces is, to the best of our knowledge, a novel one.

Part I

Algebras, modules and all that

Chapter 1

Some elements of noncommutative topology

This first chapter is devoted to recalling some elements of noncommutative geometry that are essential prerequisites for this work.

The material covered here can be found in standard books on operator algebras and their K-theory. Our main references are [57, 38] for the treatment of noncommutative topology, [75, 8, 87] for K-theory and [18, 44] for K-homology.

We start by recalling the duality between topological spaces and C^*-algebras that goes under the name of Gel'fand duality. This duality can be naturally extended to the category of vector bundles, leading to the Serre-Swan theorem. After that we give the definition of K-theory and K-homology for C^*-algebras, describing their properties and the index pairing. We conclude this chapter by presenting some C^*-algebras that will play a rôle later in the work as particular examples of Pimsner algebras.

1.1 Algebras as spaces

The starting idea of noncommutative geometry is that of trading spaces for algebras. In order to make this statement more precise, one needs to recall some notions.

A **-algebra* is an algebra $\mathcal{A}$ that admits an involution, i.e. a map $* : \mathcal{A} \to \mathcal{A}$ such that $*^2 = 1$, which is anti-linear and compatible with the algebra structure, i.e.

$$\begin{aligned}(\alpha a + \beta b)^* &= \overline{\alpha} a^* + \overline{\beta} b^* \\ (ab)^* &= b^* a^*.\end{aligned} \tag{1.1.1}$$

for all $a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}$.

A normed algebra $(\mathcal{A}, \|\cdot\|)$ is an algebra equipped with a norm which, in addition to the usual properties, satisfies $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in \mathcal{A}$. If the algebra is a Banach space (complete normed space) with respect to $\|\cdot\|$, it is named a *Banach algebra*.

Combining the previous definitions, a *Banach $*$-algebra* is a Banach algebra together with an involution $*$ that satisfies $\|a^*\| = \|a\|$.

A C^**-algebra* is a Banach *-algebra $(A, \|\cdot\|)$ that satisfies the stronger condition

$$\|a^*a\| = \|a\|^2 \quad \forall a \in A. \tag{1.1.2}$$

The above condition is called C^**-property*.

A C^*-algebra A is called *separable* if it contains a countable subset which is dense in the norm topology of A.

We have chosen to use the symbol A to distinguish C^*-algebras from general complex algebras, which will be denoted by $\mathcal{A}$.

Example 1.1. Let X be a topological space, that is assumed to be compact and Hausdorff. One constructs in a natural way a commutative complex algebra, denoted $C(X)$, by considering continuous complex valued functions $f : X \to \mathbb{C}$ with pointwise sum and product: for all $f, g : X \to \mathbb{C}$, one defines

$$(f+g)(x) = f(x) + g(x) \tag{1.1.3}$$
$$(f \cdot g)(x) = f(x)g(x) \tag{1.1.4}$$

Since X is compact, the *supremum norm*

$$\|f\| = \sup_{x \in X} |f(x)| \tag{1.1.5}$$

is well-defined. Moreover, since the limit of a uniformly convergent sequence of continuous functions is continuous, $C(X)$ is complete with respect to the norm (1.1.5).

One can endow $C(X)$ with an isometric involution

$$* : C(X) \to C(X) \quad f^*(x) = \overline{f(x)} \quad \forall x \in X,$$

with respect to which $C(X)$ is a commutative unital C*-algebra, with unit the constant function equal to 1.

The construction can be extended to the case of a more general locally compact Hausdorff space Y. In that case one can still construct a C^*-algebras, denoted $C_0(Y)$, consisting of complex valued functions that vanish at infinity; the algebra will however be *a priori* non-unital.

The above example exhausts all commutative C^*-algebras. Indeed one has the following result:

Theorem 1.1.1 (Gel'fand)**.** *Let A be a commutative unital C^*-algebra. There exists a compact topological space $\sigma(A)$, called the* spectrum of A, *such that $A \simeq C(\sigma(A))$ isometrically.*

There is actually more to that: if $f : X \to Y$ is a continuous mapping between two compact spaces, then $Cf := f^* : C(Y) \to C(X)$, defined as $h \mapsto h \circ f$, is a unital *-homomorphism. Moreover, if $g : Y \to Z$ is another continuous mapping, then $C(g \circ f) = Cf \circ Cg$ and $C\,\mathrm{Id}_Y$ is the identity element of $C(X)$. This can be summarized in the following:

Lemma 1.1.2. *The correspondence $X \to C(X), f \to Cf$ is a contravariant functor from the category of compact topological spaces with continuous maps to that of unital commutative C^*-algebras with *-homomorphisms.*

Moreover, one has the following theorem, due to Gel'fand:

Theorem 1.1.3 ([86, Section 1.1])**.** *The C-functor from the category of compact topological spaces with continuous maps to that of unital commutative C^*-algebras with *-homomorphism is an (anti-)equivalence of categories.*

Motivated by this duality, one can think of a *noncommutative* C^*-algebra as the algebra of functions on a virtual dual *noncommutative space*. We will now present some genuinely noncommutative examples of C^*-algebras.

Example 1.2. The algebra $\mathrm{M}_n(\mathbb{C})$ of $n \times n$ matrices with complex coefficients, with the identity matrix as unit, is a unital C*-algebra with respect to the involution $^* : T \mapsto T^*$, defined by taking the adjoint, i.e. the conjugate transpose matrix, and the norm

$$\|T\| = \text{the positive square root of the biggest eigenvalue of } T^*T. \tag{1.1.6}$$

It is possible to generalize the construction to a direct sum of complex matrix algebras, and get that

$$\mathrm{M}_{n_1}(\mathbb{C}) \oplus \mathrm{M}_{n_2}(\mathbb{C}) \oplus \cdots \oplus \mathrm{M}_{n_k}(\mathbb{C}) \tag{1.1.7}$$

is a unital C*-algebra.

Example 1.3. Generalizing the previous example, let $\mathcal{H}$ be a Hilbert space. The space $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ is a Banach space, with respect to the operator norm

$$\|T\| = \sup_{x \in \mathcal{H}, x \neq 0} \frac{\|Tx\|}{\|x\|}. \tag{1.1.8}$$

Moreover, with product the composition of operators, the identity operator as unit element and involution defined by the adjoint operator $T \mapsto T^*$, $\mathcal{L}(\mathcal{H})$ is a C*-algebra.

If the Hilbert space is finite dimensional, i.e. isomorphic to $\mathbb{C}^n$, the norm in (1.1.8) agrees with that of (1.1.6) and one gets back to Example 1.2.

Example 1.4. The subspace $\mathcal{K}(\mathcal{H})$ of compact operators on a Hilbert space $\mathcal{H}$ form a norm closed subalgebra of $\mathcal{L}(\mathcal{H})$ which is closed under adjoints, and hence a C^*-algebra. The algebra $\mathcal{K}(\mathcal{H})$ is not unital unless $\mathcal{H}$ is finite dimensional. In that case one has $\mathcal{K}(\mathcal{H}) = \mathcal{L}(\mathcal{H}) = \mathrm{M}_n(\mathbb{C})$.

Example 1.5. One of the first, and probably the most studied example of noncommutative space is the *irrational rotation algebra* A_θ, sometimes also called the *noncommutative torus*. This is the universal C^*-algebra generated by two unitaries U, V subject to the relation

$$UV = \mathrm{e}^{2\pi \mathrm{i}\theta} VU,$$

for θ any irrational number.

Generalizing Example 1.4, every norm-closed subalgebra $B \subseteq \mathcal{L}(\mathcal{H})$, which is closed under the operation of taking the adjoint, is a C*-algebra and it is called a *concrete*

C^*-algebra. We will see that every abstract C^*-algebra can be realized as a concrete C^*-algebra. To make this statement more precise, one needs to define C^*-algebra representations.

A **-morphism* between two C^*-algebras A and B is a complex linear map $\phi : A \to B$ which is a homomorphism of *-algebras, i.e. multiplicative

$$\phi(ab) = \phi(a)\phi(b), \qquad \forall a, b \in A,$$

and $*$-preserving

$$\phi(a^*) = \phi(a)^*, \qquad \forall a \in A.$$

By [38, Lemma 1.6] every *-homomorphism is norm-decreasing and continuous. Moreover, one can prove that a *-morphism is injective if and only if it is isometric, i.e. if and only if $\|\phi(a)\|_B = \|a\|_A$ for all $a \in A$.

A *representation* of a C^*-algebra A is a pair $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a Hilbert space and $\pi : A \to \mathcal{L}(\mathcal{H})$ is a *-morphism between A and the C^*-algebra $\mathcal{L}(\mathcal{H})$ of bounded operators on $\mathcal{H}$.

A representation $(\mathcal{H}, \pi)$ is called *faithful* if $\ker(\pi) = \{0\}$. Every faithful representation π of a C^*-algebra A makes it isomorphic to the concrete C^*-algebra $\pi(A)$.

Two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are said to be *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\pi_1(a) = U^* \pi_2(a) U \quad \forall a \in A.$$

A C^*-algebra is called *finite dimensional* if it admits a faithful representation on a finite dimensional Hilbert space.

The C^*-algebras in Example 1.2 are clearly finite dimensional C^*-algebras. Conversely, it is possible to show that every finite dimensional C*-algebra can be represented as a direct sum of matrix algebras. For the proof of this fact we refer to [31, Theorem III.1.1].

Theorem 1.1.4 (Gel'fand-Naĭmark)**.** *For each C^*-algebra A there exists a Hilbert space $\mathcal{H}$ and an isometric *-homomorphism $\phi : A \to \mathcal{L}(\mathcal{H})$. If A is separable, then $\mathcal{H}$ may be chosen to be separable.*

The above theorem admits a constructive proof, giving a concrete recipe on how to construct the Hilbert space explicitly. This goes under the name of GNS (Gel'fand-Naĭmark-Segal) construction (*cf.* [31, Theorem 1.9.6]).

Example 1.6. For the irrational rotation algebra A_θ, the underlying Hilbert space is the space $\mathcal{L}^2(\mathbb{S}^1)$ of square integrable functions on the circle. If one identifies functions in $\mathcal{L}^2(\mathbb{S}^1)$ with square integrable functions on the real line of period 1, the representation π of A_θ on $\mathcal{L}^2(\mathbb{S}^1)$ is given on generators by

$$\pi(U)f(z) = e^{2\pi \mathrm{i}\theta} f(z), \quad \pi(V)f(z) = f(z + \theta),$$

for all $f \in \mathcal{L}^2(\mathbb{S}^1)$.

1.2 Modules as bundles

The duality between spaces and algebras can be extended to the case of vector bundles by considering modules over algebras. We will see that the finitely generated projective ones can be thought of as *noncommutative vector bundles*. We will state all results for modules over general complex algebras $\mathcal{A}$, and will only need to use C^*-algebras in the formulation of the Serre-Swan theorem.

1.2.1 Modules over algebras

Let $\mathcal{A}$ be a complex algebra. A right $\mathcal{A}$-module is a vector space $\mathcal{E}$ together with a right action $\mathcal{E} \times \mathcal{A} \ni (\xi, a) \mapsto \xi a \in \mathcal{E}$ satisfying

$$\begin{aligned} &\xi(ab) = (\xi a)b, \\ &\xi(a+b) = \xi a + \xi b, \\ &(\xi + \eta)a = \xi a + \eta a, \end{aligned} \tag{1.2.1}$$

for all $\xi, \eta \in \mathcal{E}$ and $a \in \mathcal{A}$.

A family $\{\eta_\lambda\}_{\lambda \in \Lambda}$ of elements of $\mathcal{E}$, with Λ any directed set, is called a *generating set* for the right module $\mathcal{E}$ if any element of $\xi \in \mathcal{E}$ can be written (not necessarily in a unique way) as a combination

$$\xi = \sum_{\lambda \in \Lambda} \eta_\lambda a_\lambda, \tag{1.2.2}$$

with only a finite number of elements $a_\lambda \in \mathcal{A}$ different from zero.

A family $\{\eta_\lambda\}$ is called *free* if its elements are linearly independent over $\mathcal{A}$. It is called a *basis* if it is a free generating set, so that every element $\xi \in \mathcal{E}$ can be written uniquely as a combination of the form (1.2.2).

A module is called *free* whenever it admits a basis, and it is called *finitely generated* if it admits a finite generating set. Later on, we will refer to these objects as *algebraically finitely generated modules*, to distinguish them from the topologically finitely generated ones.

From now on the *-algebra $\mathcal{A}$ is assumed to be unital.

Example 1.7. The module $\mathcal{A}^n$, given by the direct sum of $\mathcal{A}$ with itself n-times, is a free finitely generated module for every n. The collection $\{\xi_j\}_{j=1}^n$, where ξ_j is the vector with one in the i-th entry and zeroes elsewhere, is a basis, called the *standard basis of* $\mathcal{A}^n$.

If a module $\mathcal{E}$ is finitely generated, there is always a positive integer n and a module surjection $\rho : \mathcal{A}^n \to \mathcal{E}$, such that the image of the standard basis of $\mathcal{A}^n$ is a generating set for $\mathcal{E}$ (not necessarily free).

Definition 1.2.1. A right $\mathcal{A}$-module $\mathcal{E}$ is said to be *projective* if it satisfies one of the following equivalent properties:

1. Given a surjective homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ of right A-modules, any homomorphism $\lambda : \mathcal{E} \to \mathcal{N}$ can be lifted to a homomorphism $\lambda' : \mathcal{E} \to \mathcal{M}$ such that $\lambda = \rho \circ \lambda'$.
2. Every surjective module morphism $\rho : \mathcal{M} \to \mathcal{E}$ splits, that is there exists a module morphism $s : \mathcal{E} \to \mathcal{M}$ such that $\rho \circ s = \mathrm{Id}_{\mathcal{E}}$.
3. The module $\mathcal{E}$ is a direct summand of a free module, that is there exists a free module $\mathcal{F}$ and a module $\mathcal{E}'$ such that

$$\mathcal{F} \simeq \mathcal{E} \oplus \mathcal{E}'. \tag{1.2.3}$$

For the proof of equivalence of the above conditions, we refer the reader to [88, Section 2.2] (see also the discussion after [57, Definition 7]).

Now suppose that $\mathcal{E}$ is both projective and finitely generated, with surjection $\rho : \mathcal{A}^n \to \mathcal{E}$. Then by the lifting property 1 there exists a lift $s : \mathcal{E} \to \mathcal{A}^n$ such that $s \circ \rho = \mathrm{Id}_{\mathcal{E}}$. Then $\mathbf{e} := \rho \circ s$ is an idempotent in $\mathrm{M}_n(\mathcal{A})$ satisfying $\mathbf{e}^2 = \mathbf{e}$ and $E = \mathbf{e}\mathcal{A}^n$.

The interest in finitely generated projective modules comes from the Serre-Swan theorem, which establishes a correspondence between vector bundles over a compact topological space X and finitely generated projective modules over the dual C^*-algebra of continuous functions $C(X)$. This correspondence is actually an equivalence of categories, realized in terms of the functor of sections Γ.

1.2.2 The Γ-functor

Recall that a vector bundle E over X, in symbols $E \to X$ is a *locally trivial* continuous family of *finite dimensional* vector spaces indexed by X.

The collection of continuous sections of a complex vector bundle $E \to X$, denoted by $\Gamma(E, X)$ or simply $\Gamma(E)$, is naturally a module for the commutative algebra $C(X)$ of continuous functions on the base space, the right action given by scalar multiplication in each fiber:

$$(s \cdot a)(x) = s(x)a(x), \quad \text{for all} \quad s \in \Gamma(E), a \in C(X). \tag{1.2.4}$$

If $\tau : E \to E'$ is a bundle map, then there exists a natural map $\Gamma_\tau : \Gamma(E) \to \Gamma(E')$ given by

$$\Gamma_\tau(s) = \tau \circ s,$$

which is $C(X)$-linear by linearity of each fiber map: $\tau_x : E_x \to E'_x$.

Lemma 1.2.2 ([38, Lemma 2.5])**.** *The correspondence $E \to \Gamma(E), \tau \to \Gamma_\tau$ is a functor from the category of complex vector bundles over X to the category of $C(X)$-modules.*

The Γ-functor carries the operations of duality, Whitney sum and tensor product of bundles, which are defined fiber-wise, to analogous operations on $C(X)$-modules. Indeed, one can easily check the following module isomorphisms:

$$\Gamma(E^*) \simeq \mathrm{Hom}_{C(X)}(\Gamma(E), C(X)),$$
$$\Gamma(E) \oplus \Gamma(E') \simeq \Gamma(E \oplus E').$$

Moreover, by [38, Proposition 2.6] one has:

$$\Gamma(E) \otimes_{C(X)} \Gamma(E') \simeq \Gamma(E \otimes E'),$$

where the tensor product on the left-hand side makes sense because each $\Gamma(E)$ is a $C(X)$-bimodule, by commutativity.

Let $E \to X$ be any complex vector bundle over X. By [38, Proposition 2.9] the $C(X)$-module $\Gamma(E)$ is finitely generated projective. More explicitely, for every complex vector bundle $E \to X$ there exists an idempotent $\mathbf{e}$ in the matrix algebra $\mathrm{M}_n(C(X))$ for some n such that $\Gamma(E) \simeq \mathbf{e}C(X)^n$ as modules over $C(X)$. Conversely, any $C(X)$-module of the form $\mathbf{e}C(X)^n$ is the module of sections of some vector bundle over X.

This correspondence is actually an equivalence of categories, as stated in the Serre-Swan theorem of [78, 82], that we present here in the same form of [38, Theorem 2.10].

Theorem 1.2.3 (Serre-Swan)**.** *The Γ-functor from vector bundles over a compact space X to finitely generated projective modules over $C(X)$ is an equivalence of categories.*

1.2.3 Hermitian structures over projective modules

Hermitian vector bundles, that is bundles with a fiber-wise Hermitian product, correspond to finitely generated projective $\mathcal{A}$-modules endowed with an $\mathcal{A}$-valued sesquilinear form. For C^*-algebras, the appropriate framework is that of Hilbert C^*-modules, that will be described throughly in Chapter 2, since they are at the heart of Pimsner's construction.

1.3 K-theory and K-homology

In this section we present the definition of K-theory and K-homology for unital C^*-algebras. Both theories can be defined for non-unital C^*-algebras, but we prefer not to dwell upon that general construction here, restricting our attention to the unital case.

1.3.1 K-theory for C^*–algebras

K theory for C^*-algebras is the noncommutative counterpart to K-theory for topological spaces. We therefore start this section by recalling the definition of topological K-theory.

If X is a compact Hausdorff space, the group $K^0(X)$ is the abelian group generated by the isomorphism classes of complex vector bundles over X, subject to the relation

$$[E] + [E'] = [E \oplus E'].$$

In view of the Serre-Swan duality described in Section 1.2.2, a complex vector bundle over X is equivalent to a finitely generated projective module over $C(X)$. In this dual picture a finitely generated projective module is more conveniently described in terms of idempotents in $\mathrm{M}_n(A)$, the C^*-algebra of $n \times n$ matrices with entries in A.

Note that by [48, Lemma 16] any idempotent in a C^*-algebra is similar to a projection, i.e. to a self-adjoint idempotent $\mathbf{p}^2 = \mathbf{p} = \mathbf{p}^*$. This leads to the possibility of describing K-theory for C^*-algebras in terms of projections only.

We define the following equivalence relation: two projections $\mathbf{p}, \mathbf{q} \in \mathrm{M}_n(A)$ are *Murray-von Neumann equivalent* if there exists $\mathbf{u} \in \mathrm{M}_n(A)$ such that $\mathbf{p} = \mathbf{u}^*\mathbf{u}$ and $\mathbf{q} = \mathbf{u}\mathbf{u}^*$. This is however not enough: in order to be able to add equivalence classes of projections one needs to consider the algebra $\mathrm{M}_\infty(A) := \bigcup_{n\in\mathbb{Z}} \mathrm{M}_n(A)$, obtained as the inductive limit over the inclusions

$$\phi : \mathrm{M}_n(A) \to \mathrm{M}_{n+1}(A), \quad a \mapsto \phi(a) := \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Two projections $\mathbf{p}, \mathbf{q} \in \mathrm{M}_\infty(A)$ are then said to be *equivalent* if there exists $\mathbf{u} \in \mathrm{M}_\infty(A)$ such that $\mathbf{p} = \mathbf{u}^*\mathbf{u}$ and $\mathbf{q} = \mathbf{u}\mathbf{u}^*$. The set $\mathcal{V}(A)$ of equivalence classes is made into an abelian semigroup by defining the group operation

$$[\mathbf{p}] + [\mathbf{q}] := \left[\begin{pmatrix} \mathbf{p} & 0 \\ 0 & \mathbf{q} \end{pmatrix} \right], \quad \forall\ [\mathbf{p}], [\mathbf{q}] \in \mathcal{V}(A).$$

Definition 1.3.1. The group $K_0(A)$ is the Grothendieck group of the semigroup $\mathcal{V}(A)$. It is realized as the collection of equivalence classes $K_0(A) := \mathcal{V}(A) \times \mathcal{V}(A)/\sim$ with respect to the equivalence relation

$$([\mathbf{p}], [\mathbf{q}]) \sim ([\mathbf{p}'], [\mathbf{q}']) \text{ if and only if } \exists\ [\mathbf{r}]\ \in \mathcal{V}(A)\ : [\mathbf{p}] + [\mathbf{q}'] + [\mathbf{r}] = [\mathbf{p}'] + [\mathbf{q}] + [\mathbf{r}]. \tag{1.3.1}$$

Note that the extra $[\mathbf{r}]$ appearing in (1.3.1) is inserted in order to get transitivity of the equivalence relation.

With this picture in mind, the group structure on $K_0(A)$ is defined via the addition

$$([\mathbf{p}], [\mathbf{q}]) + ([\mathbf{p}'], [\mathbf{q}']) = ([\mathbf{p}] + [\mathbf{p}'], [\mathbf{q}] + [\mathbf{q}']),$$

and it is independent of the representatives chosen.

The neutral element is represented by the class of

$$([\mathbf{p}],[\mathbf{p}]),\ \forall [\mathbf{p}] \in \mathcal{V}(A),$$

and it is straightforward to check that by (1.3.1) all such elements are equivalent.

Finally, the inverse in $K_0(A)$ is given by

$$-([\mathbf{p}],[\mathbf{q}]) = ([\mathbf{q}],[\mathbf{p}]).$$

Notice that there is a natural homomorphism

$$\kappa_A : \mathcal{V}(A) \to K_0(A) \qquad [\mathbf{p}] \mapsto ([\mathbf{p}],0),$$

from the semigroup $\mathcal{V}(A)$ to its Grothendieck group; the group $K_0(A)$ is universal in the following sense:

Proposition 1.3.2 ([57, Proposition 23]). *Let A be a unital C^*-algebra, G an abelian group and suppose that $\phi : \mathcal{V}(A) \to G$ is a homomorphism of semigroups such that $\phi(\mathcal{V}(A))$ is invertible in G. Then there exists a unique group homomorphism $\psi : K_0(A) \to G$ that makes the following diagram commutative:*

$$\begin{array}{ccc} \mathcal{V}(A) & & \\ {\scriptstyle \kappa_A}\downarrow & \searrow^{\phi} & \\ K_0(A) & \xrightarrow{\psi} & G \end{array} .$$

K_0 is a covariant functor from the category of C^*-algebras to that of Abelian groups: if $\alpha : A \to B$ is a morphism of C^*-algebras, then the induced map

$$\begin{aligned} \alpha_* :& \mathcal{V}(A) \to \mathcal{V}(A) \\ & \alpha_*([\mathbf{p}_{ij}]) = [\alpha(\mathbf{p}_{ij})] \end{aligned}$$

descends, by universality, to a group homomorphism

$$\alpha_* : K_0(A) \to K_0(B).$$

The group $K_1(A)$ is built in terms of unitaries, an approach that makes it less complicated to construct and to deal with than K_0.

Let $\mathcal{U}(A)$ be the group of unitary elements in a unital C^*-algebra A. One sets

$$\mathcal{U}_n(A) = \mathcal{U}(\mathrm{M}_n(A)), \qquad \mathcal{U}_\infty(A) := \bigcup_{n=1}^{\infty} \mathcal{U}_n(A).$$

There is a natural binary operation on $\mathcal{U}_\infty(A)$, given by

$$\mathbf{u} \oplus \mathbf{v} := \begin{pmatrix} \mathbf{u} & 0 \\ 0 & \mathbf{v} \end{pmatrix} \in \mathcal{U}_{n+m}(A), \quad \mathbf{u} \in \mathcal{U}_n(A),\ \mathbf{v} \in \mathcal{U}_m(A).$$

One defines an equivalence relation $\sim_1$ on $\mathcal{U}_\infty(A)$ as follows: for $\mathbf{u} \in \mathcal{U}_n(A)$, $\mathbf{v} \in \mathcal{U}_m(A)$, one writes $\mathbf{u} \sim_1 \mathbf{v}$ whenever there exists $k \geq \max\{n, m\}$ such that $\mathbf{u} \oplus \mathbf{1}_{k-n}$ and $\mathbf{v} \oplus \mathbf{1}_{k-m}$ are homotopy equivalent in $\mathcal{U}_k(A)$.

Definition 1.3.3. The group $K_1(A)$ is the quotient group $\mathcal{U}_\infty(A)/\sim_1$.

The group $K_1(A)$ can be equivalently defined as the quotient $\mathcal{U}_\infty(A)/\mathcal{U}_\infty(A)_0$, with $\mathcal{U}_\infty(A)_0$ denoting the connected component with the identity. Moreover, it admits another description in terms of invertible elements in the matrix algebra $\mathrm{M}_\infty(A)$, giving an isomorphism $K_1(A) \simeq \mathrm{GL}_\infty(A)/\mathrm{GL}_\infty(A)_0$, where again the subscript $_0$ denotes the connected component of the identity.

Like K_0, K_1 is also a covariant functor: if $\alpha : A \to B$ is a morphism of C^*-algebras, one has a well-defined map $\alpha_*[\mathbf{u}_{ij}] = [\alpha(\mathbf{u}_{ij})]$ on representatives, inducing a group homomorphism

$$\alpha_* : K_1(A) \to K_1(B).$$

In topological K-theory, the group $K^1(X)$ is defined in terms of the suspension. For a compact topological space X, the *suspension* of X, denoted by SX, is the space obtained by taking the union of two copies of the cone over X, in symbols:

$$SX := X \times [0,1]/(X \times \{0\} \cup X \times \{1\}).$$

Then one defines the group $K^1(X)$ as the K-theory of the suspension:

$$K^1(X) := K^0(SX).$$

Likewise, for a C^*-algebra A one defines the *suspension of* A as

$$SA := A \otimes C_0((0,1)) \simeq C_0((0,1), A).$$

Then one has the following crucial result:

Theorem 1.3.4 ([87, Theorem 7.2.5])**.** *There is an isomorphism*

$$\theta_A : K_1(A) \to K_0(SA)$$

such that, for every morphism of C^*-algebras $\alpha : A \to B$, the following diagram commutes:

$$\begin{array}{ccc} K_1(A) & \xrightarrow{\alpha_*} & K_1(B) \\ \downarrow{\scriptstyle \theta_A} & & \downarrow{\scriptstyle \theta_A} \\ K_0(SA) & \xrightarrow{S\alpha_*} & K_0(SB) \end{array}.$$

The next result, that goes under the name of Bott periodicity, implies that there are actually only two distinct K-theory groups.

Theorem 1.3.5 ([87, Chapter 9])**.** *There is an isomorphism*

$$\beta_A : K_0(A) \to K_1(SA)$$

that makes the following diagram commutative, whenever $\alpha : A \to B$ is a morphism of C^*-algebras:

$$\begin{array}{ccc} K_0(A) & \xrightarrow{\alpha_*} & K_0(B) \\ \big\downarrow \theta_A & & \big\downarrow \theta_A \\ K_1(SA) & \xrightarrow{S\alpha_*} & K_1(SB) \end{array} .$$

As a corollary one gets the following isomorphism:

$$K_{2n}(A) = K_0(A) \quad \text{and} \quad K_{2n+1}(A) = K_1(A),$$

which motivates the name *periodicity.*

The functors K_0 and K_1 are half-exact functors: any short exact sequence of C^*-algebras $0 \longrightarrow I \xrightarrow{i} A \xrightarrow{p} A/I \longrightarrow 0$ induces two short exact sequences of groups:

$$K_j(I) \xrightarrow{i_*} K_j(A) \xrightarrow{p_*} K_j(A/I) \quad j = 0, 1. \tag{1.3.2}$$

Furthermore, one can define a map

$$\partial : K_1(A/I) \to K_0(I) \tag{1.3.3}$$

as described in [87, Definition 8.1]. This map is known as the *connecting homomorphism* or the *index map.* Using the map in (1.3.3), together with Bott periodicity, one can combine the exact sequences of (1.3.2), obtaining the following:

Theorem 1.3.6 ([87, Theorem 9.3.2])**.** *Let $0 \longrightarrow I \xrightarrow{i} A \xrightarrow{p} A/I \longrightarrow 0$ be a short exact sequence of C^*-algebras. Then there is an exact sequence*

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{i_*} & K_0(A) & \xrightarrow{p_*} & K_0(A/I) \\ \partial \big\uparrow & & & & \big\downarrow \partial \\ K_1(A/I) & \xleftarrow[p_*]{} & K_1(A) & \xleftarrow[i_*]{} & K_1(I) \end{array} \tag{1.3.4}$$

where i_* and p_* are the maps in K-theory induced by the corresponding C^*-algebra maps, the map $\partial : K_1(A/I) \to K_0(I)$ is the index map of (1.3.3) *and $\partial : K_0(A/I) \to K_1(I)$ is the so-called exponential map, which is the composition of the suspended index map with Bott periodicity.*

We conclude this section by remarking that, for commutative C^*-algebras, operator K-theory agrees with topological K-theory. More precisely if $A = C(X)$ for a compact Hausdorff space X, then

$$K_i(A) = K^i(X), \quad i = 0, 1.$$

1.3.2 Fredholm modules and K-Homology

K-homology is the dual theory to K-theory for C^*-algebras, in the sense that there exists a well-defined pairing between the two theories, taking values in $\mathbb{Z}$.

There are several ways to define K-homology for C^*-algebras. Here we will follow Kasparov's approach, which is based on the notion of Fredholm module. Our main references are [44, Chapter 8] and [18, Section 4].

Definition 1.3.7. Let A be a C^*-algebra. An *odd* or *ungraded Fredholm module* over A is a triple $\mathcal{F} := (\rho, \mathcal{H}, F)$ consisting of a representation $\rho : A \to \mathcal{L}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$, and of an operator F on $\mathcal{H}$ such that $[F, \rho(a)], (F^2 - 1)\rho(a)$ and $(F - F^*)\rho(a)$ are in $\mathcal{K}(\mathcal{H})$ for all $a \in A$.

An *even* or *graded* Fredholm module is given by the same data, plus in addition a grading on the Hilbert space $\mathcal{H}$, i.e. a self-adjoint involution $\gamma : \mathcal{H} \to \mathcal{H}$, that commutes with the representation ρ, and such that $F\gamma - \gamma F = 0$.

Triples for which the operators $[F, \rho(a)], (F^2 - 1)\rho(a), (F - F^*)\rho(a)$ are zero for all $a \in A$ are called *degenerate* Fredholm modules.

Note that one can define Fredholm modules for a general unital *-algebra $\mathcal{A}$ without changing the definition. Moreover by [21, Proposition 4.7] if A is a C^*-algebra and $\mathcal{A} \subseteq A$ is a dense *-subalgebra which is closed under holomorphic functional calculus, any Fredholm module over $\mathcal{A}$ extends to a Fredholm module over A.

The simplest nontrivial example of an ungraded Fredholm module over A comes from $\rho : A \to \mathcal{L}(\mathcal{H})$ a representation of A and $P \in \mathcal{L}(\mathcal{H})$ a projection that commutes, modulo compacts, with every $\rho(a)$. Then if one sets $F = 2P - 1$, $(\rho, \mathcal{H}, F)$ is an odd Fredholm module over A.

Similarly, if $\rho : A \to \mathcal{L}(\mathcal{H})$ is a representation of A and $U \in \mathcal{L}(\mathcal{H})$ is a unitary that commutes, modulo compacts, with every $\rho(a)$, one can define a graded Fredholm module $(\mathcal{H}', \rho', F)$ over A by setting

$$\mathcal{H}' := \mathcal{H} \oplus \mathcal{H}, \quad \rho' = \rho \oplus \rho, \quad F = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}.$$

Definition 1.3.8 ([21, Definition 4.3])**.** A Fredholm module $(\rho, \mathcal{H}, F)$ for A is $(p+1)$-*summable*, for $p \in \mathbb{N}$, if there exists a dense subalgebra $\mathcal{A} \subset A$ such that for all $a \in \mathcal{A}$ one has

$$[F, \rho(a)] \in \mathcal{L}^{p+1}(\mathcal{H}),$$

where $\mathcal{L}^{p+1}(\mathcal{H})$ denotes the Schatten ideal

$$\mathcal{L}^{p+1}(\mathcal{H}) := \{T \in \mathcal{L}(\mathcal{H}) \mid \mathrm{Tr}(|T|^p) < \infty\}.$$

There is a natural binary operation of *direct sum* on Fredholm modules, given by taking the direct sum of the Hilbert spaces, of the representations and of the operators.

In order to define the Kasparov K-homology groups, we need to define some notions of equivalence for Fredholm modules, that will work both in the graded and in the ungraded case.

Definition 1.3.9. Let $\mathcal{F} = (\rho, \mathcal{H}, F)$ and $\mathcal{F}' = (\rho', \mathcal{H}', F')$ be two Fredholm modules over the same C^*-algebra A. We say that they are *unitarily equivalent*, if there exists a (grading preserving) homomorphism $U : \mathcal{H} \to \mathcal{H}'$ that intertwines the representations ρ and ρ' and the operators F and F'.

Definition 1.3.10. A Fredholm module $(\rho, \mathcal{H}, F')$ is a *compact perturbation* of the Fredholm module $(\rho, \mathcal{H}, F)$ if and only if $(F - F')\rho(a) \in \mathcal{K}(\mathcal{H})$ for all $a \in A$.

Definition 1.3.11. Suppose that $\mathcal{F}_t = (\rho, \mathcal{H}_t, F_t)$ is a family of Fredholm modules parametrized by $t \in [0, 1]$, in which the representation remains constant but F_t varies with t. If the function $t \mapsto F_t$ is norm continuous, one says that the family defines an *operator homotopy* between the Fredholm modules $\mathcal{F}_0 = (\rho, \mathcal{H}_0, F_0)$ and $\mathcal{F}_1 = (\rho, \mathcal{H}_1, F_1)$, and that the two Fredholm modules are *operator homotopic*.

It is easy to see that compact perturbation implies operator homotopy, given by the linear path from F to F'.

We are now ready to define the Kasparov K-homology groups.

Definition 1.3.12. Let $i = 0, 1$. The K-homology group $K^i(A)$ is the abelian group with one generator for every unitary equivalence class of Fredholm modules, (even or graded if $i = 0$, odd or ungraded if $i = 1$), with the following relations:

- If $\mathcal{F}_0$ and $\mathcal{F}_1$ are operator homotopic Fredholm modules of the same degree, then $[\mathcal{F}_0] = [\mathcal{F}_1] \in K^i(A)$.
- If $\mathcal{F}_0$ and $\mathcal{F}_1$ are Fredholm modules of the same degree, $[\mathcal{F}_0] + [\mathcal{F}_1] = [\mathcal{F}_0 \oplus \mathcal{F}_1]$.

For every degenerate Fredholm module $\mathcal{F}_0$, and for every Fredholm module $\mathcal{F}$ one has that $[\mathcal{F} \oplus \mathcal{F}_0] = [\mathcal{F}]$, hence the class of a degenerate Fredholm module is zero in K-homology.

K-homology is a contravariant functor from the category of separable (unital) C^*-algebras to that of abelian groups. Indeed, if $\alpha : A \to B$ is a morphism of C^*-algebras, and $(\rho, \mathcal{H}, F)$ a Fredholm module over B, then $(\rho \circ \alpha, \mathcal{H}, F)$ is a Fredholm module over A. This gives a map

$$\begin{aligned} \alpha_* : &K^i(B) \to K^i(A) \\ &[(\rho, \mathcal{H}, F)] \mapsto [(\rho \circ \alpha, \mathcal{H}, F)], \end{aligned}$$

for $i = 0, 1$.

Like for K-theory, one can obtain exact sequences in K-homology from an extension of C^*-algebras, but only if an extra condition is fulfilled.

Theorem 1.3.13. *Let* $0 \longrightarrow I \xrightarrow{\ i\ } A \xrightarrow{\ p\ } A/I \longrightarrow 0$ *an exact sequence of* C^**-algebras admitting a completely positive cross section* $A/I \to A$*. Then there is an exact sequence*

$$\begin{array}{ccccc} K^0(I) & \xleftarrow[i^*]{} & K^0(A) & \xleftarrow[p^*]{} & K^0(B) \\ \Big\downarrow \partial & & & & \partial \Big\uparrow \\ K^1(B) & \xrightarrow{p^*} & K^1(A) & \xrightarrow{i^*} & K^1(I) \end{array} \quad , \tag{1.3.5}$$

where i^* and p^* are the maps in K-homology induced by the corresponding C^*-algebra maps, and ∂ are again connecting maps defined in terms of an index.

1.3.3 Pairings

Recall that a bounded linear operator F on a Hilbert space $\mathcal{H}$ is called *Fredholm* if it has closed range and finite dimensional kernel and cokernel. Then one defines the index of the operator as

$$\mathrm{Ind}(F) := \dim \ker(F) - \dim \mathrm{coker}(F).$$

The Fredholm index is constant under compact perturbations and it is a homotopy invariant (*cf.* [44, Proposition 2.1.6]).

The pairing between K-theory and K-homology is given by the index of a suitable Fredholm operator. For this reason it is sometimes named the *index pairing*.

We start from the ungraded case. Let $\mathcal{F} = (\rho, \mathcal{H}, F)$ be an odd Fredholm module over A and $\mathbf{u} \in \mathrm{M}_k(A)$ a unitary. One denotes by $\mathcal{H}^k$ the Hilbert space $\mathbb{C}^k \otimes \mathcal{H}$, by P_k the operator $1 \otimes \frac{1}{2}(1+F)$ on $\mathcal{H}^k$ and by U the unitary operator $(1 \otimes \rho)(u)$ on $\mathcal{H}^k$. By [44, Proposition 8.7.1] the operator

$$P_k(U)P_k - (1 - P_k) : \mathcal{H}^k \to \mathcal{H}^k$$

is Fredholm and, by invariance under homotopy and compact perturbations, its Fredholm index only depends on the K-theory class $[\mathbf{u}] \in K_1(A)$ and on the K-homology class $[\mathcal{F}] \in K^1(A)$, not on the chosen representatives. This gives a well-defined pairing

$$\langle \cdot, \cdot \rangle : K_1(A) \times K^1(A) \to \mathbb{Z}.$$

Let now $\mathcal{F} = (\rho, \mathcal{H}, F)$ be an even Fredholm module over A and $\mathbf{p} \in \mathrm{M}_k(A)$ a projection. One denotes by $\mathcal{H}^k$ the Hilbert space $\mathcal{H} \otimes \mathbb{C}^k$–that now is seen as a graded Hilbert space–and by P the projection $(1 \otimes \rho)(p)$ in $\mathcal{L}(\mathcal{H})$. If one writes

$$F = \begin{pmatrix} 0 & F_- \\ F_+ & 0 \end{pmatrix}$$

with respect to the graded decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, then by [44, Proposition 8.7.2] the operator

$$P(1 \otimes F_+)P : P(\mathbb{C}^k \otimes \mathcal{H}_+) \to P(\mathbb{C}^k \otimes \mathcal{H}_-)$$

is Fredholm and its Fredholm index only depends on the K-theory class $[\mathbf{p}] \in K_0(A)$ and on the K-homology class $[\mathcal{F}] \in K^0(A)$. This gives a well-defined pairing

$$\langle \cdot, \cdot \rangle : K_0(A) \times K^0(A) \to \mathbb{Z}.$$

The existence of a pairing between K-theory and K-homology can be seen as an instance of the properties of the Kasparov product, that we will describe in Section 2.4.

1.4 Further examples of C^*-algebras

We finish this chapter with a description of some notable C^*-algebras, that we will encounter later as particular cases of Pimsner's construction. These are crossed products by the integers, Cuntz algebras and Cuntz-Krieger algebras.

At the end of this section we will describe how these algebras can be naturally endowed with a strongly continuous circle action. Our interest in algebras with circle action is two-fold: on one hand, these seem to yield the correct framework for modeling principal circle actions on spaces, provided they satisfy an extra assumption; on the other hand, we will see later in Subsection 3.2.1 that Pimsner algebras are naturally endowed with a strongly continuous action, called the *gauge action*.

1.4.1 Crossed Products by the integers

Crossed products are the basic tool used to study groups acting on a C^*-algebra. They provide a larger C^*-algebra which encodes information both on the original algebra and on the group action. Although crossed products can be defined for any locally compact group, we will focus here on the particular case of crossed products by the integers. Before that, we need to recall a more general definition:

Definition 1.4.1. Let B a C^*-algebra. A *continuous action* of a locally compact group G on B is a group homomorphism

$$\alpha : G \to \mathrm{Aut}(B), \tag{1.4.1}$$

such that the map $t \mapsto \alpha_t(x)$ is continuous from G to B for any $x \in B$, with respect to the norm topology on B.
The triple (B, G, α) is called a *C^*-dynamical System*.

Definition 1.4.2. Given a C^*-dynamical system (B, G, α), a *covariant representation* is a pair (π, u), where π is a representation of B on a Hilbert space $\mathcal{H}$ and u is a unitary representation of G on the same Hilbert space satisfying

$$u_t \pi(a) u_t^* = \pi(\alpha_t(a)) \qquad \text{for all } a \in B, t \in G. \tag{1.4.2}$$

Equation (1.4.2) is an identity between operators: it must hold for every element $\varphi \in \mathcal{H}$.

As already mentioned, we shall now restrict our attention to the group of integers $\mathbb{Z}$. The Haar measure on $\mathbb{Z}$ is the counting measure and the topology is the discrete one; if one considers the space $C_c(\mathbb{Z}, B)$ of continuous compactly supported B valued functions on $\mathbb{Z}$, this is just the algebra of formal sums

$$f = \sum_{t\in\mathbb{Z}} a_t t, \tag{1.4.3}$$

with the coefficients $a_t \in B$ different from zero for a finite number of t's. The formula in (1.4.3) should be interpreted as the statement that the function f takes the value a_t in t.

One has a twisted convolution product given by

$$f * g = \sum_{s\in\mathbb{Z}} \left(\sum_{t\in\mathbb{Z}} a_t \alpha_t(b_{s-t}) \right) s, \tag{1.4.4}$$

and an involution given by

$$f^* = \sum_{t\in\mathbb{Z}} \alpha_t({a_{-t}}^*) t, \tag{1.4.5}$$

that turn $C_c(\mathbb{Z}, B)$ into a *-algebra.

The algebra B can be embedded in $C_c(\mathbb{Z}, B)$ using functions supported at the identity element of $\mathbb{Z}$. Likewise, the group $\mathbb{Z}$ is represented in $C_c(\mathbb{Z}, B)$ by associating to every $s \in \mathbb{Z}$ the delta-function

$$\delta_s(t) = \begin{cases} 1 & t = s \\ 0 & t \neq s \end{cases} .$$

The algebra $C_c(\mathbb{Z}, B)$ is a good candidate for an algebra encoding information about both the algebra B and the group $\mathbb{Z}$. For each f in $C_c(\mathbb{Z}, B)$ one can define the L^1-norm

$$\|f\|_1 := \sum_{t\in\mathbb{Z}} \|a_t\|_B, \tag{1.4.6}$$

where $\|\cdot\|_B$ denotes the C*-algebra norm of B. The norm (1.4.6) turns $C_c(\mathbb{Z}, B)$ into a normed algebra with isometric involution. One denotes by $\mathcal{L}^1(\mathbb{Z}, B)$ its completion in the norm (1.4.6), which is a Banach algebra but in general not a C*-algebra, as one can see in the following example:

Example 1.8. For $B = \mathbb{C}$, the resulting algebra is $\ell^1(\mathbb{Z})$ equipped with twisted involution and convolution. It is easy to see that the C^*-property (1.1.2) does not hold. A counterexample is given by the sequence $a = (a_n)$ with

$$a_n = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \\ 0 & \text{otherwise} \end{cases} . \tag{1.4.7}$$

One should therefore look for another way of making $C_c(\mathbb{Z}, B)$ into a C^*-algebra. Note that given a covariant representation $(\pi, u, \mathcal{H})$ of the discrete C*-dynamical system $(B, \mathbb{Z}, \alpha)$, one can always construct a representation $\pi \times u$ of the algebra $C_c(\mathbb{Z}, B)$ on $\mathcal{L}(\mathcal{H})$ given by

$$(\pi \times u)(f) = \sum_{t \in \mathbb{Z}} \pi(a_t) u_t. \tag{1.4.8}$$

Moreover, given any nondegenerate faithful representation $\pi : B \to \mathcal{L}(\mathcal{H})$, there always exists a covariant representation (π_α, λ) of (B, G, α), with λ the left-regular representation, satisfying:

$$\pi_\alpha(x)\xi(t) = \pi(\alpha_{t^{-1}}(x))\xi(t) \qquad (\lambda(t)\xi)(s) = \xi(t^{-1}s), \tag{1.4.9}$$

If one considers the image of the representation $\pi_\alpha \times \lambda$ in $\mathcal{L}(\mathcal{H})$ this is by definition a concrete C^*-algebra.

Definition 1.4.3. The *crossed product* $B \rtimes_\alpha \mathbb{Z}$ of the C^*-algebra B by $\mathbb{Z}$ is the concrete C*-algebra

$$B \rtimes_\alpha \mathbb{Z} = (\pi_\alpha \times \lambda)(C_c(\mathbb{Z}, B)).$$

Example 1.9. The irrational rotation algebra A_θ of Example 1.5 can be realized as a crossed product by the integers, by considering the C^*-dynamical system $(C(\mathbb{S}^1), \alpha, \mathbb{Z})$ defined as follows.

We identify functions in $C(\mathbb{S}^1)$ with continuous functions on the real line of period 1. The action α of $\mathbb{Z}$ induced by rotation of the angle θ, and it is given, for every $n \in \mathbb{Z}$, by

$$\alpha_n f(t) = f(t + n\theta).$$

For the corresponding crossed product algebra one has the isomorphism

$$C(\mathbb{S}^1) \rtimes_\alpha \mathbb{Z} \simeq A_\theta.$$

Remark 1.4.4. For a general group G, the definition of the crossed product algebra is slightly more involved. One constructs the algebra $B \rtimes_\alpha G$ starting from the algebra $C_c(G, B)$ of compactly supported B valued functions on G, with twisted convolution and involution, and completing in the norm

$$\|f\| = \sup_\sigma \|\sigma(f)\|, \tag{1.4.10}$$

with σ running over *all possible* *-representations of $C_c(G, B)$. The supremum is always bounded by the $\mathcal{L}^1$ norm of f, and it is taken over a non-empty family of representations.

If instead of taking all possible *-representations of $C_c(G,B)$, one completes in the norm induced by the left regular representation, then the resulting algebra is, in general, smaller than $\mathcal{B} \rtimes_\alpha G$. It is named the *reduced crossed product algebra* and it is denoted by $B \rtimes_{\mathrm{red}} G$.

It is a known fact that for abelian groups, and hence for the particular case of the integers, the reduced crossed product and the full one agree. This fact is true more generally for a bigger class of groups, that of amenable groups, as stated for instance in [90, Theorem 7.13].

1.4.2 Cuntz and Cuntz-Krieger algebras

Cuntz algebras and Cuntz-Krieger algebras are universal C^*-algebras generated by (partial) isometries subject to certain additional conditions. They naturally appear in the study of dynamical systems.

Before describing these algebras in detail, one needs to recall some facts about isometries and partial isometries.

Definition 1.4.5. Let S be a bounded linear operator on a Hilbert space. S is an isometry if and only if $S^*S = \mathrm{Id}$.

Theorem 1.4.6. *Let S be a bounded linear operator on a Hilbert space. The following are equivalent:*

1. *S is a partial isometry;*
2. *S^*S is a projection;*
3. *SS^* is a projection;*
4. *$SS^*S = S$;*
5. *$S^*SS^* = S^*$.*

In this case, S^*S is the projection on $(\ker S)^\perp$ and SS^* is the projection on the range of S.

Definition 1.4.7. Let $n \geq 2$. The Cuntz algebra $\mathcal{O}_n$ is the universal C^*-algebra generated by n-isometries $S_1, \dots, S_n$ subject to the additional condition

$$\sum_{i=1}^{n} S_i {S_i}^* = \mathrm{Id}\,. \tag{1.4.11}$$

Note that condition (1.4.11) implies in particular that the corresponding range projections are orthogonal, i.e. $S_i^* S_j = \delta_{ij}\,\mathrm{Id}_n$ for all i, j.

Theorem 1.4.8 ([23, Theorem 3.1])**.** *Suppose that S_i and T_i, $i = 1, \dots, n$ are two families of non-zero partial isometries satisfying* (1.4.11)*. Then the map $S_i \mapsto T_i$ extendes to an isomorphism $C^*(S_1, \dots, S_n) \simeq C^*(T_1, \dots, T_n)$.*

Cuntz algebras admit an extension by compact operators:

Proposition 1.4.9 ([23, Proposition 3.1])**.** *Let $V_1, \ldots, V_n$ be isometries on a Hilbert space $\mathcal{H}$ such that $\sum_{i=1}^n V_i V_i^* \leq 1$. Then the projection $P := \mathrm{Id} - \sum_{i=1}^n V_i V_i^*$ generates a closed two sided ideal $\mathcal{I}$ in $C^*(V_1, \ldots, V_n)$ which is isomorphic to $\mathcal{K}(\mathcal{H})$ and contains P as a minimal projection. The quotient $C^*(V_1, \ldots, V_n)/\mathcal{I}$ is isomorphic to $\mathcal{O}_n$.*

The Toeplitz extension

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow C^*(V_1, \ldots, V_n) \longrightarrow \mathcal{O}_n \longrightarrow 0. \tag{1.4.12}$$

is used to prove the following result:

Theorem 1.4.10 ([24, Theorem 3.7])**.** *For all $n \in \mathbb{N}$, the K-theory group of the Cuntz algebra equal*

$$K_0(\mathcal{O}_n) \simeq \mathbb{Z}_{n-1}, \quad K_1(\mathcal{O}_n) \simeq \mathbb{Z}.$$

This in particular implies that the Cuntz algberas are pairwise non isomorphic, i.e. $\mathcal{O}_n \simeq \mathcal{O}_m$ if and only if $n = m$.

Cuntz-Krieger algebras are a generalization of the above construction.

Definition 1.4.11. Let $\mathbf{A}$ be a matrix with entries in $\{0, 1\}$ with no rows or columns equal to zero.

The Cuntz-Krieger algebra $\mathcal{O}_{\mathbf{A}}$ is the universal C*-algebra generated by partial isometries S_i with pairwise orthogonal range projections, subject to the relations

$$\begin{aligned} \sum_{i=1}^n S_i {S_i}^* &= \mathrm{Id} \\ {S_i}^* S_i &= \sum_{j=1}^n \mathbf{A}_{ij} S_j {S_j}^*. \end{aligned} \tag{1.4.13}$$

Clearly for $\mathbf{A}$ the matrix with all entries equal to one, one gets back the Cuntz algebras $\mathcal{O}_n$.

Any family $\{S_i\}_{i=1}^n$ of partial isometries satisfying the conditions in (1.4.13) is called a *Cuntz-Krieger* $\mathbf{A}$*-family*, and there is a uniqueness results similar to Theorem 1.4.8. We state it here under mildly stronger assumptions than those of the original paper.

Theorem 1.4.12 (*cf.*[25, Theorem 2.13])**.** *Let $\mathbf{A}$ be an $n \times n$ matrix with entries in $\{0, 1\}$ which is irreducible and not a permutation matrix, and assume that S_i and T_i, $i = 1, \ldots, n$ are Cuntz-Krieger $\mathbf{A}$-families. Then the map $S_i \mapsto T_i$ extends to an isomorphism $C^*(S_1, \ldots, S_n) \simeq C^*(T_1, \ldots, T_n)$.*

The K-theory groups of the Cuntz-Krieger algebras can be computed in terms of the algebraic properties of the matrix $\mathbf{A}$.

Theorem 1.4.13 ([24, Theorem 4.2])**.** *One has group isomorphisms*

$$K_0(\mathcal{O}_{\mathbf{A}}) \simeq \mathrm{coker}(1 - \mathbf{A}^t) \qquad K_1(\mathcal{O}_{\mathbf{A}}) \simeq \ker(1 - \mathbf{A}^t). \tag{1.4.14}$$

Remark 1.4.14. Given any $n \times n$ matrix $\mathbf{A}$ with entries in $\{0, 1\}$, one can construct a graph Γ with n vertices and an edge joining the i-th and the j-th vertex whenever $\mathbf{A}_{ij} = 1$. The resulting Cuntz-Krieger algebra $\mathcal{O}_{\mathbf{A}}$ agrees with the C^*-algebra of the graph Γ, as defined in [54, 53].

1.4.3 Circle actions

We conclude this chapter by explaining how the algebras described in this section are naturally equipped with a strongly continuous circle action, a feature they share with all Pimsner algebras, as we will see in Subsection 3.2.1.

Definition 1.4.15. An action $\sigma := \{\sigma_w\}_{w\in\mathbb{S}^1}$ of the circle $\mathbb{S}^1$ on a C^*-algebra A is *strongly continuous* if for every convergent sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathbb{S}^1$, with limit w, and every $a \in A$, the sequence $\sigma_{w_n}(a)$ converges to $\sigma_w(a)$.

If σ is a strongly continuous action, then for every $a \in A$, the map $w \to \sigma_w(a)$ is a continuous function from $\mathbb{S}^1 \to A$, with respect to the norm topology.

Let $A = B \rtimes \mathbb{Z}$ be the crossed product C^*-algebras of B with the integers. There is a natural action $\hat{\alpha}$ of the Pontryagin dual group $\hat{\mathbb{Z}} := \mathbb{S}^1$ on A. This action is defined, at the level of functions, as

$$\hat{\alpha}_\gamma\left(f\right)\left(t\right) = \gamma(t)f(t),$$

for all $\gamma \in \hat{\mathbb{Z}}$ and $t \in \mathbb{Z}$; it is called the *dual action*.

Similarly, given the Cuntz algebras $\mathcal{O}_n$, for every $z \in \mathbb{S}^1$ the family $\{zS_i\}_{i=1}^n$ is a family which generates $\mathcal{O}_n$ as well, and by Theorem 1.4.8 the assignment $\sigma_z(S_i) = zS_i$ gives a homomorphism $\sigma_z : \mathcal{O}_n \to \mathcal{O}_n$. The homomorphism $\sigma_{\overline{z}}$ is an inverse to σ_z, so that σ_z is actually in $\mathrm{Aut}(\mathcal{O}_n)$. Moreover, it is easy to see that $\sigma := \{\sigma_z\}_{z\in\mathbb{S}^1}$ is a strongly continuous circle action on $\mathcal{O}_n$, that is named the *gauge action*.

The construction works readily for Cuntz-Krieger algebras: one uses the fact that for any $z \in \mathbb{S}^1$ the family $\{zS_i\}_{i=1}^n$ is also a Cuntz-Krieger $\mathbf{A}$-family, and by Theorem 1.4.12 one has a well-defined homomorphism $\sigma_z : \mathcal{O}_{\mathbf{A}} \to \mathcal{O}_{\mathbf{A}}$ such that $\sigma_z(S_i) = zS_i$. An inverse is given by $\sigma_{\overline{z}}$, so that $\sigma_z \in \mathrm{Aut}(\mathcal{O}_{\mathbf{A}})$ and $\sigma := \{\sigma_z\}_{z\in\mathbb{S}^1}$ gives a strongly continuous circle action on $\mathcal{O}_{\mathbf{A}}$, named the *gauge action* as well.

We will investigate algebras endowed with a strongly continuous circle action in Section 3.2, and relate them to Pimsner algebras.

Chapter 2

Hilbert modules, Morita equivalence and KK-theory

In this chapter, we will describe the building blocks of this thesis: Hilbert modules, C^*-correspondences and self-Morita equivalence bimodules. We will start by defining Hilbert C^*-modules, showing how to construct new modules from existing ones, and we will describe the properties of operators between Hilbert modules, leading to the notion of module frames. We will then define C^*-correspondences and show how they appear in the definition of Morita equivalence for C^*-algebras. A central character in this thesis will be played by a particular class of correspondences that go under the name of self-Morita equivalence bimodules; we will think of these as sections of a noncommutative Hermitian line bundle. We conclude this chapter by recalling some notions of Kasparov's bivariant K-theory.

Our main references for this chapter are [8, 55, 70, 87].

2.1 Hilbert modules, operators and module frames

Hilbert C^*-modules play a central rôle in the modern developments of noncommutative geometry and index theory. Roughly speaking, they are a generalization of Hilbert spaces in which the complex scalars are replaced by a C^*-algebra. They were first introduced by Kaplansky in [48] for the commutative unital case. Later Paschke proved in [64] that most facts hold true in the case of an arbitrary C^*-algebra. Around the same time, the theory of Hilbert C^*-modules was developed and used by Rieffel in his work [72] on induced representations, leading to the notion of strong Morita equivalence. Hilbert C^*-modules are a central character in Kasparov's bivariant K-theory (see for instance [50]) and from a geometrical point of view, they can be thought of as modules of sections of a *noncommutative* Hermitian vector bundle.

In the following we will denote an arbitrary C^*-algebra with the letter B.

Definition 2.1.1. A *pre-Hilbert module* over B is a right B-module $\mathcal{E}$ with a B-valued Hermitian product, i.e. a map $\langle \cdot, \cdot \rangle_B : \mathcal{E} \times \mathcal{E} \to B$ satisfying

$$\begin{aligned}\langle \xi, \eta \rangle_B &= \langle \eta, \xi \rangle_B{}^*;\\ \langle \xi, \eta b \rangle_B &= \langle \xi, \eta \rangle_B b;\\ \langle \xi, \xi \rangle_B &\geq 0;\\ \langle \xi, \xi \rangle_B = 0 &\Rightarrow \xi = 0\end{aligned}$$

for all $\xi, \eta \in \mathcal{E}$ and for all $b \in B$.

To lighten notation we shall write $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_B$ whenever possible.

Note that we use here the so-called *physicists' notation* with linearity in the second entry. This may seem to be inconsistent with the usual notion of Hilbert space found in mathematics textbooks, like for instance Definition 12.1 in [76], but it has some practical advantages: we will use right modules and linear operators acting on the left. For this reason, if not otherwise stated, we will always consider our modules to be right-modules. Clearly one may just as well define left-modules by adapting the above definition.

For a pre-Hilbert module $\mathcal{E}$, one can define a scalar valued norm $\|\cdot\|$ using the C^*-norm on B:

$$\|\xi\|^2 = \|\langle \xi, \xi \rangle_B\|_B. \tag{2.1.1}$$

Definition 2.1.2. A *Hilbert C^*-module* $\mathcal{E}$ is a pre-Hilbert module that is complete in the norm (2.1.1).

If one defines $\langle \mathcal{E}, \mathcal{E} \rangle$ to be the linear span of elements of the form $\langle \xi, \eta \rangle$ for $\xi, \eta \in \mathcal{E}$, then its closure its a two-sided ideal in B. We say that the Hilbert module $\mathcal{E}$ is *full* whenever $\langle \mathcal{E}, \mathcal{E} \rangle$ is dense in B.

Example 2.1. Clearly, a Hilbert C^*-module over the field of complex numbers $\mathbb{C}$ is nothing but a Hilbert space.

Example 2.2. If B is a commutative unital C^*-algebra, then by Gel'fand's Theorem 1.1.1, $B \simeq C(X)$. Moreover, by the Serre-Swan Theorem 1.2.3, a vector bundle $E \to X$ is equivalently described by the right $C(X)$-module of sections $\mathcal{E} := \Gamma(X)$, with right action given, for every $\xi \in \mathcal{E}$ and $f \in C(X)$, by the section $t \mapsto \xi(t)f(t)$. If in addition E is an Hermitian vector bundle with fiber E_t, i.e. if one has an Hermitian product $\langle \cdot, \cdot \rangle_{E_t}$ in every fiber, which varies continuously with t, then for any two sections $\xi, \eta \in \Gamma(X)$ one can define their Hermitian product as the section

$$t \mapsto \langle \xi(t), \eta(t) \rangle_{E_t}.$$

Then $\mathcal{E}$ is a Hilbert C^*-module over the commutative C^*-algebra $B \simeq C(X)$.

Example 2.3. The simplest (in general) noncommutative example of Hilbert C^*-module is given by the algebra B itself, with Hermitian product

$$\langle a, b \rangle = a^* b, \tag{2.1.2}$$

and right action given by the algebra product. This module will be denoted by B_B. By the existence of approximate units for C^*-algebras (*cf.* [31, Theorem 1.4.8]), the module B_B is automatically full.

Example 2.4. Since every C^*-algebra is naturally a Hilbert module over itself, one can use this fact to define, for any natural number n, the B module B^n, in analogy with Example 1.7. It consists of n-tuples of elements $\xi_i \in B$, with component-wise right multiplication and well-defined Hermitian product

$$\langle \xi, \eta \rangle = \sum_{i=1}^{n} \langle \xi_i, \eta_i \rangle_B, \tag{2.1.3}$$

for $\xi = (\xi_i)$ and $\eta = (\eta_i)$, which turn it into a Hilbert C^*-module.

Example 2.5. Generalizing the previous example, one can start from $\{\mathcal{E}_i\}_{i=1}^n$ a finite set of Hilbert C^*-modules over B. The direct sum $\bigoplus_{i=1}^n \mathcal{E}_i$ is a B-module in the obvious way (point-wise multiplication) with inner product defined, as in (2.1.3). If all $\mathcal{E}_i = \mathcal{E}$, then $\bigoplus_i^n \mathcal{E}_i$ will be denoted by $\mathcal{E}^n$.

Things become subtler if $\{\mathcal{E}_i\}_{i\in I}$ is an infinite set of Hilbert B-modules. One defines $\bigoplus_{i\in I} \mathcal{E}_i$ as the set of sequences (ξ_i), with $\xi_i \in \mathcal{E}_i$ and such that $\sum_{i\in I} \langle \xi_i, \xi_i \rangle$ converges in B. Then for $\xi = (\xi_i)$ and $\eta = (\eta_i)$ the inner product

$$\langle \xi, \eta \rangle = \sum_{i\in I} \langle \xi_i, \eta_i \rangle$$

is well-defined and the module $\bigoplus_{i\in I} \mathcal{E}_i$ is complete, hence a Hilbert C^*-module.

Example 2.6. If $\mathcal{H}$ is a Hilbert space, the algebraic tensor product $\mathcal{H} \otimes_{\text{alg}} B$ has a natural structure of right B-module and it can be endowed with a B-valued Hermitian product

$$\langle \xi \otimes a, \eta \otimes b \rangle := \langle \xi, \eta \rangle a^* b,$$

which turns it into a pre-Hilbert module. We denote its completion with $\mathcal{H} \otimes B$. If the Hilbert space $\mathcal{H}$ has an orthonormal basis, then $\mathcal{H} \otimes B$ can be naturally identified with the direct sum module $\bigoplus_i B$ defined previously. If $\mathcal{H}$ is a separable, finite dimensional Hilbert space, then $\mathcal{H} \otimes B$ is denoted with $\mathcal{H}_B$ and it is referred to as the *standard Hilbert module.*

A Hilbert C^*-module $\mathcal{E}$ is *topologically finitely generated* if there exists a finite set $\{\eta_1, \ldots, \eta_n\}$ of elements of $\mathcal{E}$ such that the B-linear span of the η_i's is dense in $\mathcal{E}$. It is said to be *algebraically* finitely generated if every element $\xi \in \mathcal{E}$ is of the form $\sum_{i=1}^n b_i \eta_i$ for some b_i.

Definition 2.1.3 (*cf.* Definition 1.2.1)**.** Let B a unital C^*-algebra. A Hilbert C^*-module $\mathcal{E}$ is *projective* if it is a direct summand in the free module B^n for some n.

Proposition 2.1.4 ([87, Corollary 15.4.8])**.** *Every algebraically finitely generated Hilbert C^*-module over a unital algebra is projective.*

Let now $\mathcal{E}, \mathcal{F}$ be two Hilbert C^*-modules over the same C^*-algebra B.

Definition 2.1.5. A map $T : \mathcal{E} \to \mathcal{F}$ is said to be an *adjointable operator* if and only if there exists another map $T^* : \mathcal{F} \to \mathcal{E}$ with the property that

$$\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle \qquad \text{for all} \quad \xi \in \mathcal{E}, \eta \in \mathcal{F} \ .$$

Every adjointable operator is automatically linear, and by the Banach-Steinhaus theorem, it is bounded. However, the converse is in general not true: a bounded linear map between Hilbert modules need not be adjointable. To see this, let A be a unital C^*-algebra, B a proper ideal and $i : B \to A$ the inclusion. If i were adjointable, we would have $i^*(1) = 1$, which is not the case, since $1 \notin B$.

The collection of adjointable operators from $\mathcal{E}$ to $\mathcal{F}$ is denoted $\mathcal{L}_B(\mathcal{E}, \mathcal{F})$. When $\mathcal{E} = \mathcal{F}$, the adjointable operators form a C^*-algebra, that is denoted by $\mathcal{L}_B(\mathcal{E})$.

Inside the space of adjointable operators one can single out a particular class, which is analogous to that of finite rank operators on a Hilbert space. More precisely, for every $\xi \in \mathcal{F}, \eta \in \mathcal{E}$ one defines the operator $\theta_{\xi,\eta} : \mathcal{E} \to \mathcal{F}$ as

$$\theta_{\xi,\eta}(\zeta) = \xi \langle \eta, \zeta \rangle, \qquad \forall \zeta \in \mathcal{E} \tag{2.1.4}$$

This is an adjointable operator, with adjoint $\theta^*_{\xi,\eta} : \mathcal{F} \to \mathcal{E}$ given by $\theta_{\eta,\xi}$.

We denote by $\mathcal{K}_B(\mathcal{E}, \mathcal{F})$ the closed linear subspace of $\mathcal{L}_B(\mathcal{E}, \mathcal{F})$ spanned by

$$\{\theta_{\xi,\eta} \,|\, \xi, \eta \in \mathcal{E}\}, \tag{2.1.5}$$

which we refer to as the space of *compact adjointable operators*.

In particular $\mathcal{K}_B(\mathcal{E}) := \mathcal{K}_B(\mathcal{E}, \mathcal{E}) \subseteq \mathcal{L}_B(\mathcal{E})$ is an ideal, hence a C^*-subalgebra, whose elements are referred to as *compact endomorphisms*. For any C^*-algebra B, seen as a Hilbert module over itself, we have $\mathcal{K}_B(B) \simeq B$.

Finally, the C^*-algebraic dual of $\mathcal{E}$, denoted by $\mathcal{E}^*$ is defined as the space

$$\mathcal{E}^* := \{\phi \in \mathcal{L}_B(\mathcal{E}, B) \mid \exists \xi \in \mathcal{E} \text{ with } \phi(\eta) = \langle \xi, \eta \rangle \ \forall \eta \in \mathcal{E}\}\,. \tag{2.1.6}$$

Thus, with $\xi \in \mathcal{E}$, if $\lambda_\xi : \mathcal{E} \to B$ is the operator defined by $\lambda_\xi(\eta) = \langle \xi, \eta \rangle$, for all $\eta \in \mathcal{E}$, every element of $\mathcal{E}^*$ is of the form λ_ξ for some $\xi \in \mathcal{E}$.

One says that a module $\mathcal{E}$ is *self-dual* if the C^*-algebraic dual $\mathcal{E}^*$ coincides with $\mathcal{L}_B(\mathcal{E}, B)$. If B is unital, then B^n is self dual. As a consequence, every finitely generated projective Hilbert C^*-module over a unital C^*-algebra is also self-dual.

We finish by describing how finite projective modules can be characterized in terms of their algebra of compact operators.

Theorem 2.1.6 ([38, Proposition 3.9])**.** *Let B be a unital C^*-algebra and $\mathcal{E}$ a right B-module. Then $\mathcal{E}$ is finitely generated projective if and only if*

$$1 = \mathrm{Id}_{\mathcal{E}} \in \mathcal{K}_B(\mathcal{E}).$$

Proof. In one direction the proof is straightforward: let $\mathcal{E} \simeq \mathbf{p}B^n$, it is then enough to consider the image of the standard basis of B^n.

Conversely, since $\mathrm{Id}_{\mathcal{E}} \in \mathcal{K}_B(\mathcal{E})$, and $\mathcal{K}_B(\mathcal{E})$ is an ideal, we have an isomorphism $\mathcal{K}_B(\mathcal{E}) \simeq \mathcal{L}_B(\mathcal{E})$. Since $\mathcal{K}_B(\mathcal{E})$ is a unital C^*-algebra, and the invertible elements in a unital C^*-algebra form an open set, the dense ideal of finite rank operators cannot be proper, so $\mathrm{Id}_{\mathcal{E}}$ is actually finite rank. This means that the identity operator is of the form

$$\mathrm{Id}_{\mathcal{E}} = \sum\nolimits_{j=1}^{n} \theta_{\zeta_j,\zeta'_j}, \tag{2.1.7}$$

for a finite number of $\zeta_j, \zeta'_j \in \mathcal{E}$. Using the trivial identity $\mathrm{Id}_{\mathcal{E}} = \mathrm{Id}^*_{\mathcal{E}}\,\mathrm{Id}_{\mathcal{E}}$ we can actually select a finite family $\{\eta_j\}_{j=1}^n$ in $\mathcal{E}$ such that

$$\mathrm{Id}_{\mathcal{E}} = \sum\nolimits_{j=1}^{n} \theta_{\eta_j,\eta_j}. \tag{2.1.8}$$

As a consequence, one can reconstruct any element of $\xi \in \mathcal{E}$ as

$$\xi = \sum\nolimits_{j=1}^{n} \eta_j \langle \eta_j, \xi \rangle_B. \tag{2.1.9}$$

The matrix $\mathbf{p} = (\mathbf{p}_{jk})$ with elements $\mathbf{p}_{jk} = \langle \eta_j, \eta_k \rangle_B$ is a projection in the matrix algebra $\mathrm{M}_n(B)$. By construction $(\mathbf{p}_{jk})^* = \mathbf{p}_{kj}$ and, using (2.1.9) one shows

$$\begin{aligned}(\mathbf{p}^2)_{jl} &= \sum_{k=1}^{n} \langle \eta_j, \eta_k \rangle_B \langle \eta_k, \eta_l \rangle_B \\ &= \sum_{k=1}^{n} \langle \eta_j, \eta_k \langle \eta_k, \eta_l \rangle_B \rangle_B = \langle \eta_j, \eta_l \rangle_B = \mathbf{p}_{jl}\,.\end{aligned}$$

This establishes the finite right B-module projectivity of $\mathcal{E}$ with the isometric identification $\mathcal{E} \simeq \mathbf{p}B^n$. Furthermore, $\mathcal{E}$ is self-dual for its Hermitian product. ☐

The proof of the theorem motivates the following definition:

Definition 2.1.7 ([74]). A *finite standard module frame* for the right Hilbert B-module $\mathcal{E}$ is a finite family of elements $\{\eta_i\}_{j=1}^n$ of $\mathcal{E}$ such that, for all $\xi \in \mathcal{E}$, the reconstruction formula (2.1.9), holds true.

Remark 2.1.8. More generally, one could consider frames with countable elements, with (2.1.9) replaced by a series convergent in $\mathcal{E}$, or equivalently (2.1.8) replaced by the condition that the series $\sum_j \theta_{\eta_j,\eta_j}$ is strictly convergent to the unit of $\mathcal{L}_B(\mathcal{E})$ ($\mathcal{K}_B(\mathcal{E})$ need not be unital). We refer to [35] for details.

As we have seen in the proof of Theorem 2.1.6, the existence of a finite frame is a geometrical condition: whenever one has a right Hilbert B-module $\mathcal{E}$ with a finite standard module frame, the module itself is finitely generated and projective.

Actually, for $\mathcal{E}$ to be finitely generated projective it is enough to have two finite sets $\{\zeta_i\}_{i=1}^n$ and $\{\zeta'_i\}_{i=1}^n$ of elements of $\mathcal{E}$ satisfying (2.1.7). Then, any element $\xi \in \mathcal{E}$ can be reconstructed as

$$\xi = \sum_{j=1}^{n} \zeta_j \langle \zeta'_j, \xi \rangle_B,$$

and the matrix with elements $\mathbf{e}_{jk} = \langle \zeta_j, \zeta'_k \rangle_B$ is an idempotent in $\mathrm{M}_n(B)$, $(\mathbf{e}^2)_{jk} = \mathbf{e}_{jk}$, and $\mathcal{E} \simeq \mathbf{e}B^n$ as a right B-module.

2.2 C^*-correspondences, bimodules and Morita equivalence

C^*-correspondences are the building blocks of Kasparov's bivariant K-theory, and they play a crucial rôle in Morita equivalence.

2.2.1 C^*-correspondences

We have described in the previous section how any C^*-algebra B is naturally a Hilbert B-module. Given a second C^*-algebra A and a C^*-homomorphism $\phi : A \to B$, the algebra B itself can be endowed with a left A-module structure, with action

$$a \cdot b = \phi(a)b,$$

satisfying $\langle \phi(a)b, c\rangle = \langle b, \phi(a)^* c\rangle$.

Note that whenever we have a Hilbert B-module $\mathcal{E}$, by its very definition, the algebra $\mathcal{L}_B(\mathcal{E})$ and its subalgebra $\mathcal{K}_B(\mathcal{E})$ act adjointably on $\mathcal{E}$ from the left. More generally, whenever one has a map $\phi : A \to \mathcal{L}_B(\mathcal{E})$, it is possible to endow the Hilbert module $\mathcal{E}$ with a left A-module structure:

$$a \cdot \xi = \phi(a)(\xi) \tag{2.2.1}$$

for all $\xi \in \mathcal{E}$ and $a \in A$. This motivates the following:

Definition 2.2.1. A *C^*-correspondence* $(\mathcal{E}, \phi)$ *from A to B*, also named an (A, B)-*correspondence*, is a right Hilbert B-module $\mathcal{E}$ endowed with a *-homomorphism $\phi : A \to \mathcal{L}_B(\mathcal{E})$. If $A = B$ we refer to $(\mathcal{E}, \phi)$ as a *C^*-correspondence over B*.

When no confusion arises, we will refer to the pair $(\mathcal{E}, \phi)$ by using the compact notation $\mathcal{E}_\phi$.

Two C^*-correspondences $\mathcal{E}_\phi$ and $\mathcal{F}_\psi$ over the same algebra B are called *isomorphic* if and only if there exists a unitary $U \in \mathcal{L}_B(\mathcal{E}, \mathcal{F})$ intertwining ϕ and ψ.

C^*-correspondences can be *composed*: given an (A, B)-correspondence $\mathcal{E}_\phi$ and a (B, C)-correspondence $\mathcal{F}_\psi$, one can construct an (A, C)-correspondence, named the *interior tensor product* of $\mathcal{E}_\phi$ and $\mathcal{F}_\psi$.

As a first step, one constructs the *balanced tensor product* $\mathcal{E} \otimes_B \mathcal{F}$ which is a quotient of the algebraic tensor product $\mathcal{E} \otimes_{\mathrm{alg}} \mathcal{F}$ by the subspace generated by elements of the form

$$\xi b \otimes \eta - \xi \otimes \psi(b)\eta, \tag{2.2.2}$$

for all $\xi \in \mathcal{E},\ \eta \in \mathcal{F},\ b \in B$.

This has a natural structure of right module over C given by

$$(\xi \otimes \eta)c = \xi \otimes (\eta c)$$

and a C-valued inner product defined on simple tensors as

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_C := \langle \eta_1, \psi(\langle \xi_1, \xi_2 \rangle_B)\eta \rangle_C, \tag{2.2.3}$$

and extended by linearity.

The inner product is well-defined and has all required properties; in particular, the null space $N = \{\zeta \in \mathcal{E} \otimes_{\mathrm{alg}} \mathcal{F} \ ; \ \langle \zeta, \eta \rangle = 0\}$ is shown to coincide with the subspace generated by elements of the form in (2.2.2).

One then defines $\mathcal{E}\widehat{\otimes}_\psi \mathcal{F}$ to be the right Hilbert module obtained by completing $\mathcal{E} \otimes_B \mathcal{F}$ in the norm induced by (2.2.3).

Moreover for every $T \in \mathcal{L}_B(\mathcal{E})$, the operator defined on simple tensors by

$$\xi \otimes \eta \mapsto T(\xi) \otimes \eta$$

extends to a well-defined operator $\phi_*(T) := T \otimes \mathrm{Id}$. It is adjointable with adjoint given by $T^* \otimes \mathrm{Id} = \phi_*(T^*)$. In particular, this means that there is a left action of A defined on simple tensors by

$$(\phi \otimes_\psi \mathrm{Id})(a)(\xi \otimes \eta) = \phi(a)\xi \otimes \eta,$$

and extended by linearity to a map

$$\phi \otimes_\psi \mathrm{Id} : A \to \mathcal{L}_C(\mathcal{E}\widehat{\otimes}_\psi \mathcal{F}),$$

thus turning $\mathcal{E}\widehat{\otimes}_\psi \mathcal{F}$ into an (A, C)-correspondence.

We have already encountered an example of interior tensor product: if $\mathcal{E} = \mathcal{H}$ a Hilbert space seen as a correspondence over $\mathbb{C}$, and $\mathcal{F} = B_B$ with ι the complex action by multiples of the identity, then $\mathcal{H} \otimes_\iota B$ is unitarily equivalent, as a correspondence over $\mathbb{C}$, to the module $\mathcal{H} \otimes B$ presented in Example 2.6.

Proposition 2.2.2 ([7, Proposition 5.2])**.** *Given two *-algebra homomorphisms $\phi : A \to B$ and $\psi : B \to C$, one has an isomorphism of (A, C)-correspondences.*

$$\mathcal{E}_\phi \widehat{\otimes}_\psi \mathcal{E}_\psi = \mathcal{E}_{\psi \circ \phi}.$$

Taking the interior tensor power is an associative operation on isomorphism classes of C^*-correspondences.

Remark 2.2.3. In the paper [67], the name *Hilbert bimodule* was used to denote a right Hilbert module with a left action by adjointable operators. We however prefer describing a pair $(\mathcal{E}, \phi)$ with the name *C^*–correspondence*, which has recently emerged as a more common one, reserving the terminology Hilbert bimodule to the more restrictive case where one has both a left and a right inner product satisfying an extra compatibility relation.

2.2.2 Hilbert bimodules

Given a right Hilbert B-module $\mathcal{E}$, by construction, compact endomorphisms act on the left on $\mathcal{E}$. Then by defining,

$$ {}_{\mathcal{K}_B(\mathcal{E})}\langle \xi, \eta \rangle := \theta_{\xi,\eta}, \tag{2.2.4} $$

one obtains a natural $\mathcal{K}_B(\mathcal{E})$-valued Hermitian product on $\mathcal{E}$. Note this is *left* linear over $\mathcal{K}_B(\mathcal{E})$, that is ${}_{\mathcal{K}_B(\mathcal{E})}\langle T\xi, \eta \rangle = T({}_{\mathcal{K}}\langle \xi, \eta \rangle)$ for $T \in \mathcal{K}_B(\mathcal{E})$. Thus $\mathcal{E}$ is a *left* Hilbert $\mathcal{K}_B(\mathcal{E})$-module and by the very definition of $\mathcal{K}_B(\mathcal{E})$, $\mathcal{E}$ is full over $\mathcal{K}_B(\mathcal{E})$. One easily checks the compatibility condition

$$ \xi \langle \eta, \zeta \rangle_B = {}_{\mathcal{K}_B(\mathcal{E})}\langle \xi, \eta \rangle \zeta, \qquad \text{for all} \quad \xi, \eta, \zeta \in \mathcal{E}. \tag{2.2.5} $$

In particular, the B-valued and $\mathcal{K}_B(\mathcal{E})$-valued norms coincide [70, Lemma 2.30].

This motivates the following definition:

Definition 2.2.4. Given two C^*-algebras A and B, a *Hilbert* (A, B)*-bimodule* $\mathcal{E}$ is a right Hilbert B-module with B-valued Hermitian product $\langle \ , \ \rangle_B$, which is at the same time a left Hilbert A-module with A-valued Hermitian product ${}_A\langle \ , \ \rangle$ and such that the Hermitian products are compatible, that is,

$$ \xi \langle \eta, \zeta \rangle_B = {}_A\langle \xi, \eta \rangle \zeta, \qquad \text{for all} \quad \xi, \eta, \zeta \in \mathcal{E}. \tag{2.2.6} $$

Note that $\langle \ , \ \rangle_B$ is right B-linear, while ${}_A\langle \ , \ \rangle$ is left A-linear.

A Hilbert (A, B)-bimodule is a very special object: an (A, B)-correspondence $\mathcal{E}_\phi$ in the sense of Definition 2.2.1 need not have, in general, a left A-valued inner product. However, whenever the map ϕ is injective with $\mathcal{K}_B(\mathcal{E}) \subseteq \phi(A)$, the map ϕ^{-1} is well-defined on $\mathcal{K}_B(\mathcal{E})$. One can use this fact together with the left $\mathcal{K}_B(\mathcal{E})$-valued inner product on $\mathcal{E}$ to define an A-valued inner product

$$ {}_A\langle \xi, \eta \rangle = \phi^{-1}({}_{\mathcal{K}_B(\mathcal{E})}\langle \xi, \eta \rangle), \qquad \text{for all} \quad \xi, \eta \in \mathcal{E}. \tag{2.2.7} $$

Note that the module $\mathcal{E}$ need not be full with respect to this left inner product. If in addition ϕ is an isomorphism onto $\mathcal{K}_B(\mathcal{E})$, then by the very definition of $\mathcal{K}_B(\mathcal{E})$, $\mathcal{E}$ is also full as left Hilbert A-module.

2.2.3 Morita equivalence

Definition 2.2.5. An (A, B)-*equivalence bimodule* is a full (A, B)-correspondence $\mathcal{E}_\phi$ where the left action $\phi : A \to \mathcal{L}_B(\mathcal{E})$ is an isomorphism onto $\mathcal{K}_B(\mathcal{E})$. One says that two C^*-algebras A and B are Morita equivalent if such an (A, B)-equivalence bimodule exists.

Every full Hilbert B-module $\mathcal{E}$ is a $(\mathcal{K}_B(\mathcal{E}), B)$-equivalence bimodule.

Morita equivalence is a weaker equivalence relation that isomorphism. Indeed, given an isomorphism $\phi : A \to B$, the C^*-correspondence (B_B, ϕ) is an (A, B)-equivalence bimodule.

Remark 2.2.6. Let (A, B) two unital C^*-algebras. Any C^*-correspondence $\mathcal{E}_\phi$ implementing the Morita equivalence between the two algebras is finitely generated projective as an A-module. To see this, one can use the fact that $A \simeq \mathcal{K}_B(\mathcal{E})$ is unital and Theorem 2.1.6 to obtain the claim. Moreover, this in particular implies that $\mathcal{E}$ admits a finite module frame in the sense of Definition 2.1.7.

Proposition 2.2.7 ([70, Proposition 3.8])**.** *If $\mathcal{E}_\phi$ is an (A, B)-equivalence bimodule, then $\mathcal{E}_\phi$ is a Hilbert (A, B)-bimodule in the sense of Definition 2.2.4.*

Theorem 2.2.8. *Morita equivalence is an equivalence relation.*

Proof. Morita equivalence is clearly symmetric since for every C^*-algebra one has $B \simeq \mathcal{K}_B(B)$. It is reflexive since by [55, Proposition 7.1] $B \simeq \mathcal{K}_A(\mathcal{E})$ implies that $A \simeq \mathcal{K}_B(\mathcal{F})$ for $\mathcal{F}$ the Hilbert module $\mathcal{F} = \mathcal{K}_A(E, A)$. Moreover, $\mathcal{F}$ is full whenever $\mathcal{E}$ is.

Transitivity is obtained by taking the interior tensor product of correspondences. To see this, one needs to check that $\phi \otimes_\psi \mathrm{Id}$ gives an isomorphism $A \simeq \mathcal{K}_C(\mathcal{E} \otimes_\psi \mathcal{F})$, whenever ϕ and ψ are isomorphisms.

First of all, every $\xi \in \mathcal{E}$ and $\eta \in \mathcal{F}$ the equation $S_\xi(\eta) = \xi \otimes \eta$, defines an element $S_\xi \in \mathcal{L}_C(\mathcal{F}, \mathcal{E}\widehat{\otimes}_\psi\mathcal{F})$ whose adjoint is just $S_\xi^*(\zeta \otimes \eta) = \psi(\langle \xi, \zeta \rangle)\eta$, for $\xi, \zeta \in \mathcal{E}$, $\eta \in \mathcal{F}$. Finally, for $b \in B$ and $\xi_1, \xi_2, \zeta \in \mathcal{E}$ and $\eta \in \mathcal{F}$, one computes:

$$\begin{aligned} S_{\xi_1}\psi(b)S_{\xi_2}^*(\zeta \otimes \eta) &= S_{\xi_1}\psi(b)\psi(\langle \xi_2, \zeta \rangle)(\eta) = \xi_1 \otimes \psi(b\langle \xi_2, \zeta \rangle)(\eta) \\ &= \xi_1 b\langle \xi_2, \zeta \rangle \otimes \eta = (\theta_{\xi_1 b, \xi_2}(\zeta)) \otimes \eta \\ &= \phi_*(\theta_{\xi_1 b, \xi_2})(\zeta \otimes \eta) \end{aligned}$$

Thus

$$\phi_*(\theta_{\xi_1 b, \xi_2}) = S_{\xi_1}\psi(b)S_{\xi_2}^*,$$

which is in $\mathcal{K}_C(\mathcal{E}\widehat{\otimes}_\psi\mathcal{F})$ since $\psi(b) \in \mathcal{K}_B(\mathcal{E})$.
Since ψ is non degenerate, it follows, by using an approximate unit for B, that $\psi_*(\theta_{\xi_1,\xi_2}) = S_{\xi_1}S_{\xi_2}^*$, and by the definition of $\mathcal{K}_B(\mathcal{E})$ it follows that $\phi_*(\mathcal{K}_B(\mathcal{E}))$ is contained in $\mathcal{K}_C(\mathcal{E}\widehat{\otimes}_\psi\mathcal{F})$. By [55, Proposition 4.7], whenever ϕ is an isomorphism, ϕ_* is an isomorphism as well. In particular, for $A \simeq \mathcal{K}_B(\mathcal{E})$ one has that $A \simeq \mathcal{K}_C(\mathcal{E}\widehat{\otimes}_\psi\mathcal{F})$ via the isomorphism ϕ_*. □

Morita equivalence is a purely noncommutative notion. Indeed, Morita equivalent algebras have isomorphic centers (*cf.* [52, Section 2.3]), and therefore two commutative C^*-algebras are Morita equivalent if and only if they are isomorphic.

In noncommutative topology Morita equivalence is the most natural equivalence relation to consider: Morita equivalent C^*-algebras have, among other things, the same representation theory and the same K-theory and K-homology (and also bivariant K-theory) groups.

2.3 Self-Morita equivalence bimodules and the Picard group

Let B be a C^*-algebra. A self-Morita equivalence bimodule over B is any C^*-correspondence $\mathcal{E}_\phi$ over B which implements the reflexivity of Morita equivalence for a C^*-algebra B. The simplest example of self-Morita equivalence bimodule is the algebra B itself together with the identity map. Equivalently, self-Morita equivalence bimodules can be defined in terms of (B, B)-bimodules, but we prefer adopting the former approach.

Definition 2.3.1. A *self-Morita equivalence bimodule* over B is a C^*-correspondence $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is full and $\phi : B \to \mathcal{K}_B(\mathcal{E})$ is an isomorphism.

Two self-Morita equivalence bimodules are isomorphic if and only if they are isomorphic as C^*-correspondences; by [2, Corollary 1.2] the left inner product is automatically preserved.

The prototypical commutative example of a self-Morita equivalence bimodule is provided by $B = C(X)$, the C^*-algebra of continuous functions on a compact topological space X, $\mathcal{E} = \Gamma(X)$ the $C(X)$-module of sections of a Hermitian line bundle $L \to X$ and ϕ the trivial action. For this reason, one is led to think of self-Morita equivalence bimodules as *noncommutative* line bundles.

Similarly to line bundles in classical geometry, self-Morita equivalence bimodules are *invertible* in some sense. More precisely, if $(\mathcal{E}, \phi)$ is a self-Morita equivalence bimodule over B, the dual Hilbert module $\mathcal{E}^*$ as defined in (2.1.6), can be made into a self-Morita equivalence bimodule over B as well.

First of all, $\mathcal{E}^*$ is given the structure of a (right) Hilbert C^*-module over B by means of the map ϕ. Recall that the elements of $\mathcal{E}^*$ are of the form λ_ξ for some $\xi \in \mathcal{E}$, with $\lambda_\xi(\eta) = \langle \xi, \eta \rangle$, for all $\eta \in \mathcal{E}$. The right action of B on $\mathcal{E}^*$ is given by

$$\lambda_\xi \, b := \lambda_\xi \circ \phi(b) = \lambda_{\phi(b)\xi} \, ,$$

the second equality being easily established. The B-valued Hermitian product on $\mathcal{E}^*$ uses the left $\mathcal{K}_B(\mathcal{E})$-valued Hermitian product on $\mathcal{E}$:

$$\langle \lambda_\xi, \lambda_\eta \rangle := \phi^{-1}(\theta_{\xi,\eta}) \, ,$$

and $\mathcal{E}^*$ is full as well. Next, define a $*$-homomorphism $\phi^* : B \to \mathcal{L}_B(\mathcal{E}^*)$ by

$$\phi^*(b)(\lambda_\xi) := \lambda_{\xi \cdot b^*},$$

which is in fact an isomorphism $\phi^* : B \to \mathcal{K}_B(\mathcal{E}^*)$. Thus, the pair $(\mathcal{E}^*, \phi^*)$ is a self-Morita equivalence bimodule over B, according to Definition 2.3.1. It is the *inverse* to the self-Morita equivalence bimodule $(\mathcal{E}, \phi)$ with respect to the operation given by the interior tensor product.

For a C^*-algebra B, the collection of unitary equivalence classes of self-Morita equivalence bimodules over B has a natural group structure with respect to the interior tensor product, with identity element the class of the self-Morita equivalence bimodule (B, Id). Thinking of self-Morita equivalence bimodules as *noncommutative line bundles*, this group is named the *Picard group* of B, denoted by $\mathrm{Pic}(B)$, in analogy with the classical situation for which the Picard group of a space X is the group of isomorphism classes of line bundles over X.

Recall that every automorphism $\alpha \in \mathrm{Aut}(B)$ yields a self-Morita equivalence bimodule (B, α) for B. If β is another automorphism, then by Proposition 2.2.2 their product is equivalent to $(B, \beta\alpha)$. Thus one has an anti-homomorphism from the group of automorphisms of B to the Picard group, that is denoted by

$$\Phi_B : \mathrm{Aut}(B) \to \mathrm{Pic}(B).$$

Let us now assume, for the sake of simplicity, that B is unital. Then if u is a unitary element in B, one denotes by Ad_u the automorphism of B defined by $\mathrm{Ad}_u(b) = ubu^*$. $\mathrm{Ad}(u)$ is called an *inner automorphism* of B and one denotes the group of inner automorphisms of B with $\mathrm{Inn}(B)$. Since isomorphism of correspondences is implemented by unitaries, one gets that $\mathrm{Inn}(B)$ is contained in the kernel of the map Φ_B. By [11, Theorem 3.1] one actually has an isomrphism $\mathrm{Inn}(B) \simeq \ker(\Phi_B)$.

Recall now that for any unital algebra B one also has a short exact sequence of groups

$$1 \longrightarrow \mathrm{Inn}(B) \longrightarrow \mathrm{Aut}(B) \longrightarrow \mathrm{Out}(B) \longrightarrow 1\ .$$

For every automorphism α, we will denote by $[\alpha]$ its corresponding class in $\mathrm{Out}(B)$; then every element $[\alpha] \in \mathrm{Out}(B)$ yields an element in $\mathrm{Pic}(B)$.

Remark 2.3.2. If B is not unital, an analogous result holds true, with the group $\mathrm{Inn}(B)$ replaced by the group of generalized inner automorphisms $\mathrm{Gin}(B)$, the group of inner automorphisms of the multiplier algebra $\mathcal{M}(B)$ of B.

The Picard group of a commutative C^*-algebra

Since a class of examples of self-Morita equivalence bimodules that are of interest for the present thesis comes from modules over commutative algebras, it is crucial to understand the structure of the group $\mathrm{Pic}(B)$ in the commutative case.

Recall that for a commutative C^*-algebra B, the classical Picard group $\mathrm{CPic}(B)$ is the group of Hilbert line bundles over the spectrum $\sigma(B)$ of B (*cf.* [32]), and one can prove that it agrees with the group $\mathrm{Pic}(\sigma(X))$. The following result relates the noncommutative Picard group with its classical counterpart:

Theorem 2.3.3 ([2, Theorem 1.12])**.** *Let B a commutative C^*-algebra. Then* $\mathrm{CPic}(B)$ *is a normal subgroup in* $\mathrm{Pic}(B)$ *and*

$$\mathrm{Pic}(B) \simeq \mathrm{CPic}(B) \rtimes \mathrm{Aut}(B),$$

where the action of $\mathrm{Aut}(B)$ *is given by conjugation.*

We will see an example of this phenomenon in Section 4.3.

2.4 Kasparov modules and KK-theory

We conclude this section with a quick overview of Kasparov's bivariant K-theory.

Definition 2.4.1. Let A and B two C^*-algebras. We define $\mathbb{E}(A,B)$ to be the set of all triples $(\mathcal{E},\phi,F)$, where $(\mathcal{E},\phi)$ is a countably generated (A,B)-correspondence, with grading $\gamma : E \to E$, and $F \in \mathcal{L}_B(\mathcal{E})$ is an operator of degree one, such that $[F,\phi(a)], (F^2-1)\phi(a), (F-F^*)\phi(a)$ are in $\mathcal{K}_B(\mathcal{E})$ for all $a \in A$.

The elements of $\mathbb{E}(A,B)$ are called *even Kasparov modules*. Triples $(\mathcal{E},\phi,F)$ for which the operators $[F,\phi(a)], (F^2-1)\phi(a), (F-F^*)\phi(a)$ are all zero are called *degenerate* Kasparov modules. We denote their collection with $\mathfrak{D}(A,B)$.

Example 2.7. If $\phi : A \to B$ is a graded homomorphism, then the triple $(B,\phi,0)$ is a Kasparov (A,B)-module. One special case is when $A = B$ and $\phi = \mathrm{Id}_A$.

Example 2.8. Whenever one has a C^*-correspondence $(\mathcal{E},\phi)$ satisyfying the additional assumption that the image of ϕ is contained in $\mathcal{K}_B(\mathcal{E})$, then the triple $(\mathcal{E},\phi,0)$ is an even Kasparov (A,B)-module.

There is a naturally binary operation on Kasparov modules given by direct sum, and $\mathbb{E}(A,B)$ is closed under direct sum. If $(\mathcal{E}_i,\phi_i,F_i)$ are Kasparov (A_i,B)-modules for $i=1,2$, then $(\mathcal{E}_1\oplus\mathcal{E}_2,\phi_1\oplus\phi_2,F_1\oplus F_2)$ is a Kasparov $(A_1\oplus A_2,B)$-module. Similarly, if $(\mathcal{E}_i,\phi_i,F_i)$ are Kasparov (A,B_i)-modules for $i=1,2$, then $(\mathcal{E}_1\oplus\mathcal{E}_2,\phi_1\oplus\phi_2,F_1\oplus F_2)$ is a Kasparov $(A,B_1\oplus B_2)$-module.

In order to define KK-theory, one needs to introduce some equivalence relation on Kasparov modules, which are reminiscent of the equivalence relations for K-theory and K-homology we encountered in Section 1.3.

Definition 2.4.2. Two Kasparov modules $(\mathcal{E}_1,\phi_1,F_1),(\mathcal{E}_2,\phi_2,F) \in \mathbb{E}(A,B)$ are *unitarily equivlent* if there is a unitary in $\mathcal{L}_B^*(\mathcal{E}_1,\mathcal{E}_2)$, of degree 0, intertwining the ϕ_i's and the F_i's.

Definition 2.4.3. A Kasparov module $(\mathcal{E},\phi,F')$ is a *compact perturbation* of $(\mathcal{E},\phi,F)$ if and only if $(F-F')\phi(a) \in \mathcal{K}_B(\mathcal{E})$ for all $a \in A$.

One defines $\sim_{\mathrm{cp}}$ to be the equivalence relation on $\mathbb{E}(A,B)$ generated by unitary equivalence, compact perturbation and addition of degenerate elements. Moreover, one can define a *stabilized* version of this equivalence relation, that is denoted by $\sim_{\mathrm{c}}$: $(\mathcal{E}_1,\phi_1,F_1) \sim_{\mathrm{c}} (\mathcal{E}_2,\phi_2,F)$ if and only if there exist $(\mathcal{E}_1',\psi_1,G_1)$ and $(\mathcal{E}_2',\psi_2,G_2)$ such that $\sim_{\mathrm{c}}$: $(\mathcal{E}_1,\phi_1,F_1)\oplus(\mathcal{E}_1',\psi_1,G_1) \sim_{\mathrm{cp}} (\mathcal{E}_2,\phi_2,F)\oplus(\mathcal{E}_2',\psi_2,G_2)$. This equivalence relation has different names in the literature: in [26] it is named *cobordism*. The set of equivalence classes of $\mathbb{E}(A,B)$ under the equivalence relation $\sim_c$ will be denoted by $KK_{\mathrm{c}}(A,B)$.

Definition 2.4.4. Two Kasparov modules $(\mathcal{E}_1,\phi_1,F_1),(\mathcal{E}_2,\phi_2,F) \in \mathbb{E}(A,B)$ are *homotopy equivalent* if there is an element $(\mathcal{E},\phi,F) \in \mathbb{E}(A,C([0,1],B))$ such that the element $(\mathcal{E}\otimes_{f_i} B, f_i\cdot\phi_i, f_i(F))$ is unitarily equivalent to $(\mathcal{E}_i,\phi_i,F_i)$.

The set of equivalence classes of $\mathbb{E}(A,B)$ under the equivalence relation $\sim_h$ will be denoted by $KK(A,B)$.

From now on we will assume that A is separable and B is σ-unital. In that case the equivalence relations $\sim_{\mathrm{h}}$ and $\sim_{\mathrm{c}}$ coincide and by [26, Theorem 3.7] we have a group isomorphism $KK_{\mathrm{c}}(A,B) \simeq KK(A,B)$.

Definition 2.4.5. An *odd Kasparov module* is a triple $(\mathcal{E}, \phi, F)$ as in Definition 2.4.1, were $\mathcal{E}$ is trivially graded and $F \in \mathcal{L}_B(\mathcal{E})$ is an operator such that $[F, \phi(a)]$, $(F^2-1)\phi(a)$ and $(F - F^*)\phi(a)$ are in $\mathcal{K}_B(\mathcal{E})$ for all $a \in A$[1].

The group $KK_1(A,B)$ is defined as the set of equivalence classes of odd Kasparov modules under unitary equivalence and homotopy. Alternatively, it can be defined using the theory of extensions. This was acutally Kasparov's approach [50].

Remark 2.4.6 (Higher KK-groups). The group $KK_1(A,B)$ can be equivalently defined as $KK(A, B\hat{\otimes}\mathbb{C})$. More generally, one sets

$$KK^d(A,B) := KK(A, B\widehat{\otimes}\mathbb{C}^n),$$

for all $d \in \mathbb{N}$, where $\widehat{\otimes}$ denotes the graded tensor product of C^*-algebras (see for instance [8, Section 14.5]).

The KK groups are functorial: if $f : A' \to A$ and $g : B \to B'$ are homomorphisms of C^*-algebras, one has a map $f^* : \mathbb{E}(A,B) \to \mathbb{E}(A',B)$ given by

$$f^*(\mathcal{E}, \phi, F) = (f^*\mathcal{E}, \phi \circ f, F),$$

and a map $g_* : \mathbb{E}(A,B) \to \mathbb{E}(A,B')$ given by

$$g_*(\mathcal{E}, \phi, F) = (\mathcal{E} \otimes_B B', \phi \otimes_g \mathrm{Id}, F\hat{\otimes}1).$$

Both maps pass to the quotient KK.

2.4.1 The unbounded picture

Definition 2.4.7. An *unbounded* Kasparov module is a pair $(\mathcal{E}, \mathfrak{D})$, where $\mathcal{E}$ is an (A,B)-bimodule, $\mathfrak{D} : \mathfrak{Dom}(\mathfrak{D}) \to \mathcal{E}$ is an odd self adjoint regular operator and for all $a \in A$, the operator $a(1+\mathfrak{D}^2)^{-1}$ is a compact endomorphism. Moreover, the subalgebra

$$\mathcal{A} := \{a \in A \mid [\mathfrak{D}, a] \in \mathcal{L}_B(\mathcal{E})\}$$

is dense in A.

The set of unbounded Kasparov (A,B)-modules is denoted by $\Psi_1(A,B)$.

Given any unbounded Kasparov module $(\mathcal{E}, \mathfrak{D}) \in \Psi_1(A,B)$, one can construct a bounded Kasparov module in $\mathbb{E}(A,B)$ by considering the bounded transform of $\mathfrak{D}$, which is defined as the operator:

[1] Since $\mathcal{E}$ is trivially graded, there is no requirement on the degree of F.

$$F := \mathfrak{D}(1+\mathfrak{D}^2)^{-\frac{1}{2}}.$$

The relation between the bounded and unbounded picture of K-theory was established in [6] and it is contained in the following:

Theorem 2.4.8. *Let $(\mathcal{E}, \mathfrak{D})$ be an unbounded Kasparov module and let F be the bounded transform of $\mathfrak{D}$. Then the following facts hold:*

- *$(\mathcal{E}, \phi, F)$ is a bounded Kasparov module.*
- *Two unbounded Kasparov modules are equivalent if their bounded transforms are homotopic. Any Kasparov module is homotopic to the bounded transform of an unbounded one.*

Unbounded representatives can be defined for higher KK-groups by looking at modules in $\Psi_1(A, B\widehat{\otimes}\mathbb{C}^n)$.

2.4.2 The Kasparov product

There is a well-defined bilinear pairing

$$\otimes_B : KK(A,B) \times KK(B,C) \to KK(A,C) \tag{2.4.1}$$

denoted with $(x, y) \mapsto x \otimes_B y$, covariant in C and contravariant in A. It is compatible with composition of morphisms: given $\phi : A \to B$ and $\psi : B \to C$ morphisms of C^*-algebras, then

$$[\phi] \otimes_B [\psi] = [\psi \circ \phi] \in KK(A,C).$$

Proposition 2.4.9. *The product is associative, meaning that if $\alpha \in KK(A,B)$, $\beta \in KK(B,C)$, and $\gamma \in KK(C,D)$, then*

$$\alpha \otimes_B (\beta \otimes_C \gamma) = (\alpha \otimes_B \beta) \otimes_C \gamma \in KK(A,D).$$

The Kasparov product is functorial, i.e. it is compatible with pull-back and push-forward. In particular, one has the following three possibilities:

1. If A, A' are separable, $f : A' \to A$ is a homomorphism, and $x_1 \in KK(A,B)$, $x_2 \in KK(B,C)$, then $f^*(x_1) \otimes_B x_2 = f^*(x_1 \otimes_B x_2) \in KK(A', C)$.
2. If $g : C \to C'$ is a homomorphism and $x_1 \in KK(A,B)$, $x_2 \in KK(B,C)$, then $x_1 \otimes_B g_*(x_2) = g_*(x_1 \otimes_B x_2) \in KK(A, C')$.
3. If $h : B \to B'$ is a homomorphism, $x_1 \in KK(A,B)$, $x_2 \in KK(B', C)$, then $x_1 \otimes_B h^*(x_2) = h_*(x_1) \otimes_{B'} x_2 \in KK(A,B)$.

Proposition 2.4.10. *Let $[1_A] \in KK(A,A)$ denote the class of the Kasparov module (A, Id_A). Then for any $\alpha \in KK(A,B)$ and any $\beta \in KK(B,A)$, $1_A \otimes_A \alpha = \alpha$ and $\beta \otimes_A 1_A = \beta$.*

If A is separable, it follows from associativity that $KK(A, A)$ is a ring under the intersection product. Moreover, $KK(\mathbb{C}, \mathbb{C}) \simeq \mathbb{Z}$ as a ring.

KK-theory incorporates both K-homology and K-theory of C^*-algebras, by setting the first or the second variable equal to $\mathbb{C}$, respectively:

$$KK^i(\mathbb{C}, B) = K^i(B), \quad KK^i(A, \mathbb{C}) = K_i(A).$$

As a consequence the intersection product

$$KK^i(\mathbb{C}, A) \times KK^i(A, \mathbb{C}) \to KK(\mathbb{C}, \mathbb{C})$$

agrees with the index pairing between K-theory and K-homology described in Subsection 1.3.3.

The Kasparov product 2.4.1 admits a simpler description in terms of unbounded Kasparov modules. This is still a central topic of active research, which however goes beyond the scope of the present work. We limit ourself to mentioning the following partial result, which will be used later in this work.

Theorem 2.4.11 ([65, Appendix A],[17, Theorem 2.3])**.** *Let $(\mathcal{E}, \mathfrak{D})$ be an unbounded odd Kasparov (A, B)-module. Then, assuming that $\mathfrak{D}$ has a spectral gap around zero, the Kasparov product of $K_1(A)$ with the class of $(\mathcal{E}, \mathfrak{D})$ is represented by*

$$\langle [u], [\mathcal{E}, \mathfrak{D}] \rangle = [\ker PuP] - [\mathrm{coker} PuP] = \mathrm{Ind}(PuP) \in K_0(B), \tag{2.4.2}$$

where P is the non-negative spectral projection for the operator $\mathfrak{D}$.

2.4.3 KK-equivalence, mapping cones and six term exact sequences

Any fixed element $\alpha \in KK_d(A, B)$ determines homomorphisms in K-theory and in K-homology, given by left and right multiplication:

$$\otimes_A \alpha : K_j(A) \to K_{j+d}(B), \quad \alpha_B \otimes : K^j(B) \to K^{j+d}(A). \tag{2.4.3}$$

We say that two C^*-algebras A and B are *KK-equivalent* if there exists $\alpha \in KK_d(B, A)$ and $\beta \in KK_{-d}(B, A)$ such that $\alpha \otimes_B \beta = 1_A$ and $\beta \otimes_A \alpha = 1_B$. The elements α and β are said to be *invertible*.

Morita equivalence implies KK-equivalence, which is implemented by the class of the right B-module $\mathcal{E}$ such that $\mathcal{L}_B(\mathcal{E}) \simeq A$, with $\phi : A \to \mathcal{L}_B(\mathcal{E})$ the isomorphism and $F = 0$. The fact that the modules are full implies that the action of A on E is by compact operators, hence the Kasparov module is well-defined.

As a consequence of KK-equivalence one has Bott periodicity in KK-theory:

Proposition 2.4.12 ([8, Theorem 19.2.1, Corollary 19.2.2])**.** *For any A and B there are isomorphisms $KK^1(A,B) \simeq KK(A,SB) \simeq KK(SA,B)$ and $KK(A,B) \simeq KK^1(A,SB) \simeq KK^1(SA,B) \simeq KK(S^2A,B) \simeq KK(A,S^2B) \simeq KK(SA,SB)$.*

Definition 2.4.13. The *cone* over a C^*-algebra A is the C^*-algebra

$$CA := \{f \in C([0,1),A) \mid f(0)=0\}\,,$$

with point-wise operation and norm the supremum norm.

The *mapping cone* for a morphism $\alpha : A \to B$ is the C^*-algebra

$$C_\alpha := \{a \oplus f \in A \oplus CB \,|f(1)=\alpha(a)\}\,.$$

For every C^*-algebra A, the cone fits in an exact sequence involving the suspension

$$0 \longrightarrow SA \longrightarrow CA \longrightarrow A \longrightarrow 0\,.$$

Similarly, for a morphism $\alpha : A \to B$, the mapping cone C_α is related to A and B via the exact sequence

$$0 \longrightarrow SB \xrightarrow{\iota} C_\alpha \xrightarrow{\pi} A \longrightarrow 0\,, \tag{2.4.4}$$

where $\iota(f\otimes a)(t) := f(t)a$ and π is the projections $\pi(a\oplus f)=a$. The exact sequence (2.4.4) admits a completely positive cross section given by $\phi(a) = (a,(1-t)\alpha(a))$.

In order to construct six term exact sequences in KK-theory, one needs the following important result.

Theorem 2.4.14 ([8, Theorem 19.5.3])**.** *Let A be a C^*-algebra, $I \subseteq A$ an ideal. Suppose that the exact sequence*

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{p} A/I \longrightarrow 0\,,$$

admits a completely positive splitting. Then the exact sequence

$$0 \longrightarrow SI \longrightarrow CA \xrightarrow{\pi} C_p \longrightarrow 0\,, \tag{2.4.5}$$

admits a completely positive splitting as well.

This fact has the following important consequence, which was observed in [26].

Theorem 2.4.15. *Let A a C^*-algebra and $I \subseteq A$ an ideal satisfying the assumptions of the previous lemma. The map $e : I \to C_p$ defined by $e(x) = (x,0)$, seen as an element of $KK(I,C_q)$, is a KK-equivalence.*

The inverse of e is the element $u \in KK(C_p,I) \simeq KK^1(C_p,SI)$ representing the extension (2.4.5).

As a corollary, one obtains six-term exact sequences in KK-theory, of which the exact sequences (1.3.4) and (1.3.5) are particular instances.

Theorem 2.4.16 ([8, Theorem 19.5.7])**.** *Let*

$$0 \longrightarrow I \xrightarrow{\ i\ } A \xrightarrow{\ p\ } A/I \longrightarrow 0\ ,$$

be a short exact sequence of C^*-algebras, admitting a completely positive splitting. Then for any separable C^*-algebra C, the following six-term exact sequence is exact:

$$\begin{array}{ccccc} KK_0(C,I) & \xrightarrow{i_*} & KK_0(C,A) & \xrightarrow{p_*} & KK_0(C,A/I) \\ \partial\uparrow & & & & \downarrow\partial \\ KK_1(C,A/I) & \xleftarrow[p_*]{} & KK_1(C,A) & \xleftarrow[i_*]{} & KK_1(C,I) \end{array} .$$

If A is separable and C sigma unital, then the following six-term exact sequence is exact

$$\begin{array}{ccccc} KK_0(I,C) & \xleftarrow[i^*]{} & KK_0(A,C) & \xleftarrow[p^*]{} & KK_0(A/I,C) \\ \downarrow\partial & & & & \partial\uparrow \\ KK_1(A/I,C) & \xrightarrow{p^*} & KK_1(A,C) & \xrightarrow{i^*} & KK_1(I,C) \end{array}$$

where ∂ is the multiplication with the class in $KK^1(A/I,I)$ corresponding to the extension. Up to the identification $KK^1(A/I,I)$ with $KK(S(A/I),I)$, the extension class corresponds to $\iota^*(u)$, where ι is the natural inclusion of $S(A/I)$ into C_p and $u \in KK(C_p,I)$ is the inverse of $e : I \to C_p$ defined in Theorem 2.4.15.

Part II

Pimsner algebras and principal circle bundles

Chapter 3

Pimsner algebras and generalized crossed products

3.1 Pimsner algebras

In his breakthrough paper [67], starting from a full C^*-correspondence $(\mathcal{E}, \phi)$ such that the left action $\phi : B \to \mathcal{L}_B(\mathcal{E})$ is an isometric $*$-homomorphism, Pimsner constructed two C^*-algebras: these are now referred to as the *Toeplitz algebra* and the *Cuntz-Pimsner algebra* of the C^*-correspondence $(\mathcal{E}, \phi)$, denoted by $\mathcal{T}_{\mathcal{E}}$ and $\mathcal{O}_{\mathcal{E}}$ respectively. The former is actually an extension of the second, and can be thought of as a generalization of the Toeplitz algebra, while the latter encompasses a large class of examples, like Cuntz-Krieger algebras and crossed products by the integers. Both algebras are characterized by universal properties and depend only on the isomorphism class of the C^*-correspondence.

Generalized crossed product algebras, a related notion, were defined in [1] as universal C^*-algebras associated to any Hilbert (B, B)-bimodule $\mathcal{E}$. We will present their definition and recall how, for the case of a self-Morita equivalence bimodule, they are isomorphic to the corresponding Cuntz-Pimsner algebra.

We will conclude with six term exact sequences in KK-theory, naturally associated to any Pimsner algebra.

3.1.1 The Toeplitz algebra of a full C^*-correpsondence

In this section we will work under the following assumption:
Assumption 3.1.1. The image of ϕ is contained in $\mathcal{K}_B(\mathcal{E})$.

Iterating the construction of the interior tensor product module described in Subsection 2.2.1, one considers the the k-fold tensor products

$$\mathcal{E}^{(k)} := \mathcal{E}^{\widehat{\otimes}_\phi^k} \quad k > 0. \tag{3.1.1}$$

Then one builds the infinite direct sum module

$$\mathcal{E}_+ = B \oplus \bigoplus_{k=1}^{\infty} \mathcal{E}^{(k)}. \tag{3.1.2}$$

It is a C^*-correspondence over B, with left action ϕ_+ given, for all $b \in B$, by

$$\phi_+(b)(\xi_1 \otimes \ldots \otimes \xi_k) = \phi(b)\xi_1 \otimes \ldots \otimes \xi_k;$$

for $k \geq 1$ and $\xi_1, \ldots, \xi_k \in \mathcal{E}$ and

$$\phi_+(b)(b') = bb'$$

for $b' \in B$. It is referred to as the *(positive) Fock correspondence* associated to the correspondence $\mathcal{E}_\phi$.

One can naturally associate to any element $\xi \in \mathcal{E}$ a creation and an annihilation operator in $\mathcal{L}_B(\mathcal{E}_+)$; the creation operator is given by

$$T_\xi(\xi_1 \otimes \ldots \otimes \xi_k) = \xi \otimes \xi_1 \otimes \ldots \otimes \xi_k, \qquad T_\xi(b) = \xi b, \tag{3.1.3}$$

and its adjoint is the annihilation operator

$$T_\xi^*(\xi_1 \otimes \ldots \otimes \xi_k) = \phi(\langle \xi, \xi_1 \rangle)\xi_2 \otimes \ldots \otimes \xi_k, \qquad T_\xi^*(b) = 0. \tag{3.1.4}$$

Definition 3.1.2. The Toeplitz algebra $\mathcal{T}_\mathcal{E}$ of the C^*-correspondence $\mathcal{E}_\phi$ is the smallest C^*-subalgebra of $\mathcal{L}_B(\mathcal{E}_+)$ that contains all the T_ξ for $\xi \in \mathcal{E}$.

The algebra is universal in the following sense:

Theorem 3.1.3 ([67, Theorem 3.4])**.** *Let $(\mathcal{E}, \phi)$ be a full C^*-correspondence over B, C any C^*-algebra and $\psi : B \to C$ a $*$-homomorphism with the property that there exist elements $t_\zeta \in C$ for all $\zeta \in \mathcal{E}$ such that*

1. *$\alpha t_\xi + \beta t_\eta = t_{\alpha\xi+\beta\eta}$ for all $\alpha, \beta \in \mathbb{C}$ and $\xi, \eta \in \mathcal{E}$;*
2. *$t_\xi \psi(a) = t_{\xi a}$ and $\psi(a)t_\xi = t_{\phi(a)\xi}$ for all $\xi \in \mathcal{E}$ and $a \in B$;*
3. *$t_\xi^* t_\eta = \psi(\langle \xi, \eta \rangle) \in B$ for all $\xi, \eta \in \mathcal{E}$;*

then there exists a unique extension $\tilde{\psi} : \mathcal{T}_\mathcal{E} \to C$ that maps T_ξ to t_ξ.

3.1.2 The Pimsner Algebra of a full C^*-correspondence

The (Cuntz-)Pismner algebra $\mathcal{O}_\mathcal{E}$ of a full C^*-correspondence $(\mathcal{E}, \phi)$ is a quotient of the Toeplitz algebra. Under Assumption 3.1.1 one has the following:

Definition 3.1.4. The Pimsner algebra $\mathcal{O}_\mathcal{E}$ of the C^*-correspondence $(\mathcal{E}, \phi)$ is the quotient algebra appearing in the exact sequence

$$0 \longrightarrow \mathcal{K}_B(\mathcal{E}_+) \longrightarrow \mathcal{T}_\mathcal{E} \xrightarrow{\pi} \mathcal{O}_\mathcal{E} \longrightarrow 0. \tag{3.1.5}$$

It is easy to check that since $\mathcal{E}$ is full, then $\mathcal{E}_+$ is a full Hilbert module as well; hence $\mathcal{K}_B(\mathcal{E}_+)$ is by definition Morita equivalent to the algebra B.

The image of an element $T_\xi \in \mathcal{T}_\mathcal{E}$ under the quotient map π will be denoted by S_ξ.

Example 3.1. Let $B = \mathbb{C}$ and $\mathcal{E} = \mathbb{C}^n$ and ϕ the left action by multiplication. If one chooses a basis for $\mathbb{C}^n$, then the Toeplitz algebra of $(\mathcal{E}, \phi)$ is generated by n isometries $V_1, \ldots, V_n$ satisfying $\sum_i V_i V_i^* \leq 1$. This is the Toeplitz extension for the Cuntz algebras $\mathcal{O}_n$:

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow C^*(V_1, \ldots, V_n) \longrightarrow \mathcal{O}_n \longrightarrow 0.$$

In particular, for $n = 1$ one gets the classical Toeplitz extension

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow \mathcal{T}_\mathbb{C} \longrightarrow C(S^1) \longrightarrow 0. \tag{3.1.6}$$

Example 3.2. Generalizing the previous example, let $\mathbf{A}$ be a $n \times n$ matrix with entries in $\{0, 1\}$ with no rows or columns equal to zero. Let us consider the finite dimensional commutative C^*-algebra $B = \mathbb{C}^n$, that we identify with the set of $n \times n$ diagonal matrices. B is generated by the minimal projections $\mathbf{p}_i$, for $i = 1, \ldots, n$.

Let e_{ij} be the standard basis of $\mathbb{C}^n \otimes \mathbb{C}^n$. One defines $\mathcal{E}$ to be the vector space

$$\mathcal{E} = \text{span}\{e_{ij} \mid \mathbf{A}_{ij} = 1\} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n. \tag{3.1.7}$$

The left and right module structures are given by the natural conditions

$$\phi(\mathbf{p}_k) e_{ij} = \delta_{ki} e_{ij} \quad e_{ij} \mathbf{p}_k = \delta_{kj} \mathbf{e}_{kj}$$

with inner product

$$\langle e_{ij}, e_{kl} \rangle = \delta_{ik} \delta_{jl} \mathbf{p}_j.$$

This module admits a finite frame $\left\{e_i = \sum_j e_{ij} \mid \mathbf{A}_{ij} = 1, i = 1, \ldots, n\right\}$.

The inner product of elements in the frame has the form

$$\langle e_i, e_j \rangle = \delta_{ij} \sum_k A_{ik} \mathbf{p}_k,$$

while the left action on frame elements is given by

$$\phi(\mathbf{p}_i) = \theta_{e_i, e_i} \in \mathcal{K}_B(\mathcal{E}).$$

It follows that the algebra $\mathcal{O}_\mathcal{E}$ is generated by elements $S_i := S_{e_i}$ that satisfy

$$S_i S_i^* = \mathbf{p}_i, \quad S_i^* S_i = \sum_{j=1}^n \mathbf{A}_{ij} S_j S_j{}^*$$

and hence it coincides with the Cuntz-Krieger algebra $\mathcal{O}_\mathbf{A}$ of [25], described in Subsection 1.4.2.

Example 3.3. More generally, if the module $\mathcal{E}$ is finitely generated projective, the Pimsner algebra of $(\mathcal{E}, \phi)$ can be realized explicitly in terms of generators and relations [47]. Since $\mathcal{E}$ is finitely generated projective, it admits a finite frame $\{\eta_j\}_{j=1}^n$. Then, from the frame reconstruction formula (2.1.9), for any $b \in B$:

$$\phi(b)\eta_j = \sum_{j=1}^{n} \eta_i \langle \eta_i, \phi(b)\eta_j \rangle_B,$$

The C^*-algebra $\mathcal{O}_\mathcal{E}$ is the universal C^*-algebra generated by B together with n operators $S_1, \dots, S_n$, satisfying

$$S_i^* S_j = \langle \eta_i, \eta_j \rangle_B, \quad \sum\nolimits_j S_j S_j^* = 1, \quad \text{and} \quad bS_j = \sum\nolimits_i S_i \langle \eta_i, \phi(b)\eta_j \rangle_B, \tag{3.1.8}$$

for $b \in B$, and $j = 1, \dots, n$. The generators S_i are partial isometries if and only if the frame satisfies $\langle \eta_i, \eta_j \rangle = 0$ for $i \neq j$. For $B = \mathbb{C}$ and $\mathcal{E}$ a Hilbert space of dimension n, one recovers the Cuntz algebra $\mathcal{O}_n$ of Example 3.1.

Example 3.4. Let B be a C^*-algebra and $\alpha : B \to B$ an automorphism of B. Then $\mathcal{E} = B$ can be naturally made into a C^*-correspondence.
The right Hilbert B-module structure is the standard one, with right B-valued inner product $\langle a, b \rangle_B = a^* b$.
The automorphism α is used to define the left action via $a \cdot b = \alpha(a)b$ and left B-valued inner product given by ${}_B\langle a, b \rangle = \alpha(a^* b)$.

Each module $\mathcal{E}^{(k)}$ is isomorphic to B as a right-module, with left action

$$a \cdot (x_1 \otimes \cdots \otimes x_k) = \alpha^k(a)\alpha^{k-1}(x_1) \cdots \alpha(x_{k-1})x_k. \tag{3.1.9}$$

The corresponding Pimsner algebra $\mathcal{O}_\mathcal{E}$ coincides then with the crossed product algebra $B \rtimes_\alpha \mathbb{Z}$, while the Toeplitz algebra $\mathcal{T}_\mathcal{E}$ agrees with the Toeplitz algebra $\mathcal{T}(B, \alpha)$ of [66, Section 2], that appears in the exact sequence

$$0 \longrightarrow B \otimes \mathcal{K}(\mathcal{H}) \longrightarrow \mathcal{T}(B, \alpha) \longrightarrow B \rtimes_\alpha \mathbb{Z} \longrightarrow 0. \tag{3.1.10}$$

Similarly to the Toeplitz algebra $\mathcal{T}_E$ (*cf.* Theorem 3.1.3), the Pimsner algebra $\mathcal{O}_\mathcal{E}$ can be characterized in terms of its universal properties.

Theorem 3.1.5 ([67, Theorem 3.12])**.** *Let $(\mathcal{E}, \phi)$ as above, C a C^*-algebra and $\psi : B \to C$ a $*$-homomorphism. Suppose that there exist elements $s_\zeta \in C$ for all $\zeta \in \mathcal{E}$ such that*

1. $\alpha \cdot s_\xi + \beta \cdot s_\eta = s_{\alpha \cdot \xi + \beta \cdot \eta}$ *for all* $\alpha, \beta \in \mathbb{C}$ *and* $\xi, \eta \in \mathcal{E}$;
2. $s_\xi \cdot \psi(a) = s_{\xi \cdot a}$ *and* $\psi(a) \cdot s_\xi = s_{\phi(a)(\xi)}$ *for all* $\xi \in \mathcal{E}$ *and* $a \in B$;
3. $s_\xi^* s_\eta = \psi(\langle \xi, \eta \rangle)$ *for all* $\xi, \eta \in \mathcal{E}$;
4. $s_\xi s_\eta^* = \psi(\phi^{-1}(\theta_{\xi,\eta}))$ *for all* $\xi, \eta \in \mathcal{E}$;

then there exists a unique $*$-homomorphism $\widehat{\psi} : \mathcal{O}_\mathcal{E} \to C$ with $\widehat{\psi}(S_\xi) = s_\xi$ for all $\xi \in \mathcal{E}$.

3.1.3 The case of a self-Morita equivalence bimodule

In view of our geometrical motivation, which is the study of noncommutative circle and line bundles, we will now restrict our attention to case of self-Morita equivalence bimodules. For this reason, we will work under the following assumption:

Assumption 3.1.6. The map ϕ is an isomorphism onto the compacts $\mathcal{K}_B(\mathcal{E})$.

Note that of all the examples of Hilbert bimodules, with corresponding Pimsner algebras, presented in 3.1.1, only the Toeplitz extension of (3.1.6) and the crossed product algebra are the Pimsner algebras of a self-Morita equivalence bimodule. In the other examples the map ϕ is not surjective.

Given a self-Morita equivalence bimodule $(\mathcal{E}, \phi)$ for the C^*-algebra B, in Section 2.3 we described how to take interior tensor products of self-Morita equivalences and how to turn the dual $\mathcal{E}^*$ into a self-Morita equivalence bimodule $(\mathcal{E}^*, \phi^*)$. In the self-Morita equivalence bimodule case the situation is highly symmetric, and instead of working with a direct sum indexed over $\mathbb{N}$, one can construct a direct sum over the integers. Indeed, for every $k \in \mathbb{Z}$ one defines the module $\mathcal{E}^{(k)}$ as:

$$\mathcal{E}^{(k)} := \begin{cases} \mathcal{E}^{\widehat{\otimes}_\phi^k} & k > 0 \\ B & k = 0 \\ (\mathcal{E}^*)^{\widehat{\otimes}_{\phi^*}^{-k}} & k < 0 \end{cases} .$$

Clearly, $\mathcal{E}^{(1)} = \mathcal{E}$ and $\mathcal{E}^{(-1)} = \mathcal{E}^*$. From the very definition of these Hilbert B-modules, one has isomorphisms

$$\mathcal{K}(\mathcal{E}^{(k)}, \mathcal{E}^{(l)}) \simeq \mathcal{E}^{(l-k)}, \quad \text{for } k, l \in \mathbb{Z}.$$

Out of them, one constructs the Hilbert B-module $\mathcal{E}_\infty$ as a direct sum:

$$\mathcal{E}_\infty := \bigoplus_{k \in \mathbb{Z}} \mathcal{E}^{(k)} , \tag{3.1.11}$$

which is referred to as the *two-sided Fock module* of the self-Morita equivalence bimodule $\mathcal{E}$.

As described in the previous section, one naturally defines creation and annihilation operators on the two-sided Fock module as well. First of all, for each $\xi \in \mathcal{E}$ one has a bounded adjointable operator (a creation operator) $S_\xi : \mathcal{E}_\infty \to \mathcal{E}_\infty$, shifting the degree by $+1$, defined on simple tensors by:

$$\begin{aligned} S_\xi(b) &:= \xi b , && b \in B , \\ S_\xi(\xi_1 \otimes \cdots \otimes \xi_k) &:= \xi \otimes \xi_1 \otimes \cdots \otimes \xi_k , && k > 0 , \\ S_\xi(\lambda_{\xi_1} \otimes \cdots \otimes \lambda_{\xi_{-k}}) &:= \lambda_{\xi_2 \cdot \phi^{-1}(\theta_{\xi_1,\xi})} \otimes \lambda_{\xi_3} \otimes \cdots \otimes \lambda_{\xi_{-k}} , && k < 0 . \end{aligned}$$

The adjoint of S_ξ (an annihilation operator) is easily found to be given by ${S_\xi}^* := S_{\lambda_\xi} : \mathcal{E}_\infty \to \mathcal{E}_\infty$:

$$\begin{aligned} S_{\lambda_\xi}(b) &:= \lambda_\xi b\,, & b \in B\,,\\ S_{\lambda_\xi}(\xi_1 \otimes \ldots \otimes \xi_k) &:= \phi(\langle \xi, \xi_1\rangle)\xi_2 \otimes \xi_3 \otimes \cdots \otimes \xi_k\,, & k > 0\,,\\ S_{\lambda_\xi}(\lambda_{\xi_1} \otimes \ldots \otimes \lambda_{\xi_{-k}}) &:= \lambda_\xi \otimes \lambda_{\xi_1} \otimes \cdots \otimes \lambda_{\xi_{-k}}\,, & k < 0\,; \end{aligned}$$

In particular, $S_\xi(\lambda_{\xi_1}) = S_\xi {S_{\xi_1}}^* = \phi^{-1}(\theta_{\xi,\xi_1}) \in B$ and ${S_\xi}^* S_{\xi_1} = S_{\lambda_\xi}(\xi_1) = \langle \xi, \xi_1\rangle \in B$.

Definition 3.1.7. The *Pimsner algebra* $\mathcal{O}_{\mathcal{E}}$ of the self-Morita equivalence bimodule $(\mathcal{E}, \phi)$ is the smallest C^*-subalgebra of $\mathcal{L}_B(\mathcal{E}_\infty)$ generated by the creation operators S_ξ for all $\xi \in \mathcal{E}$.

Remark 3.1.8. Again the algebra $\mathcal{O}_{\mathcal{E}}$ depends only on the isomorphism class of the correspondence $(\mathcal{E}, \phi)$.

There is an injective $*$-homomorphism $i : B \to \mathcal{O}_{\mathcal{E}}$. This is induced by the injective $*$-homomorphism $\phi : B \to \mathcal{L}_B(\mathcal{E}_\infty)$ defined by

$$\begin{aligned} \phi(b)(b') &:= b \cdot b'\,,\\ \phi(b)(\xi_1 \otimes \cdots \otimes \xi_n) &:= \phi(b)(\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_n\,,\\ \phi(b)(\lambda_{\xi_1} \otimes \cdots \otimes \lambda_{\xi_n}) &:= \phi^*(b)(\lambda_{\xi_1}) \otimes \lambda_{\xi_2} \otimes \cdots \otimes \lambda_{\xi_n}\\ &= \lambda_{\xi_1 \cdot b^*} \otimes \lambda_{\xi_2} \otimes \cdots \otimes \lambda_{\xi_n}\,, \end{aligned}$$

and whose image is in the Pimsner algebra $\mathcal{O}_{\mathcal{E}}$.

3.2 Circle actions and generalized crossed products

Pimsner algebras are endowed with a natural action of the circle $\gamma : \mathbb{S}^1 \to \mathrm{Aut}(\mathcal{O}_E)$, known as the *gauge action*, a feature they have in common with ordinary crossed products by the integers and with Cuntz-Krieger algebras.

In this section we will describe the gauge action on a Pimsner algebra and its properties, and recall the theory of C^*-algebras endowed with a circle action, focusing on their connection with the notion of generalized crossed products.

3.2.1 The gauge action

By the universal properties of Proposition 3.1.5 (with $C = B$, ψ the identity and $s_\xi := zS_\xi$), the map

$$S_\xi \to \sigma_w(S_\xi) := w^* S_\xi, \qquad w \in \mathbb{S}^1,$$

extends to an automorphism of $\mathcal{O}_{\mathcal{E}}$, that we will denote with γ. The action is strongly continuous.

The fixed point algebra $\mathcal{O}_{\mathcal{E}}^{\gamma}$ is in general bigger than the algebra of scalars B. In the case of a self-Morita equivalence bimodule the two algebras agree, as shown by the following theorem.

Theorem 3.2.1. *Let $\mathcal{O}_{\mathcal{E}}$ be the Pimsner algebra of a full C^*-correspondence over B. Suppose that $\mathcal{E}$ is a self-Morita equivalence bimodule, then the fixed point algebra $\mathcal{O}_{\mathcal{E}}^{\gamma}$ agrees with the algebra of scalars B.*

Proof. Inside the fixed point algebra one has all elements of the form $S_i^* S_j$; by the universal property 3 it follows that all inner products of elements in $\mathcal{E}$ are in the fixed point algebra. By fullness of $\mathcal{E}$ one has that $B \hookrightarrow \mathcal{O}_{\mathcal{E}}^{\gamma}$. Moreover, the fixed point algebra also contains all words of the form $S_{i_1} \dots S_{i_k} {S_{j_1}}^* \dots {S_{j_k}}^*$ for all n. Elements of this form can be naturally seen as elements in $\mathcal{K}_B(\mathcal{E}^{(k)})$.

Using the same notation of [67], we denote with $\mathcal{F}_{\mathcal{E}}$ the C^*-algebra generated in $\lim_n \mathcal{L}_B(\mathcal{E}^{(k)})$ by all the algebras $\mathcal{K}_B(\mathcal{E}^{(k)})$, with the convention that $\mathcal{K}_B(\mathcal{E}^{(0)}) = \mathcal{O}_{\mathcal{E}}^{\gamma}$. If $\mathrm{Im}(\phi) \subseteq \mathcal{K}_B(\mathcal{E})$ then $\mathcal{F}_{\mathcal{E}}$ agrees with the direct limit algebra $\lim_k \mathcal{K}_B(\mathcal{E}^{(k)})$.

It is easy to see that the fixed point algebra $\mathcal{O}_{\mathcal{E}}^{\gamma}$ coincides with the algebra $\mathcal{F}_{\mathcal{E}}$. Notice that whenever the image of ϕ contains $\mathcal{K}_B(\mathcal{E})$, then $\mathcal{K}_B(\mathcal{E}^{(k+1)}) \subseteq \mathcal{K}_B(\mathcal{E}^{(k)})$, which implies that $\mathcal{F}_{\mathcal{E}} = B$. In view of Assumptions 3.1.1 and 3.1.6, this happens in particular in the case of a self-Morita equivalence bimodule. □

3.2.2 Algebras and circle actions

Let A be a C^*-algebra endowed with a strongly continuous action $\sigma : \mathbb{S}^1 \to \mathrm{Aut}(A)$. For each $k \in \mathbb{Z}$, one defines the *k-th spectral subspace* for the action σ to be

$$A_k := \left\{ \xi \in A \mid \sigma_w(\xi) = w^{-k}\, \xi \quad \text{for all } w \in \mathbb{S}^1 \right\}.$$

Clearly, the invariant subspace $A_0 \subseteq A$ is a C^*-subalgebra of A, with unit whenever A is unital; this is the *fixed-point subalgebra* A^{σ} for the action.

For every pair of integers $k, l \in \mathbb{Z}$, the subspace $A_k A_l$–meant as the *closed* linear span of the set of products xy with $x \in A_k$ and $y \in A_l$–is contained in A_{k+l}. Thus, the algebra A is $\mathbb{Z}$-graded and the grading is compatible with the involution, that is $A_k^* = A_{-k}$ for all $k \in \mathbb{Z}$.

In particular, for any $k \in \mathbb{Z}$ the space $A_k^* A_k$ is a closed two-sided ideal in A_0. Thus, each spectral subspace A_k has a natural structure of Hilbert A_0-bimodule (not necessarily full) with left and right Hermitian products:

$$_{A_0}\langle x, y \rangle = xy^*, \qquad \langle x, y \rangle_{A_0} = x^* y, \qquad \text{for all } x, y \in A_k. \tag{3.2.1}$$

3.2.3 Generalized crossed products

A somewhat better framework for understanding the relation between Pimsner algebras and algebras endowed with a circle action is that of generalized crossed products. They were introduced in [1] and are naturally associated to Hilbert bimodules via the notion of a covariant representation.

Definition 3.2.2. Let $\mathcal{E}$ be a Hilbert (B, B)-bimodule (not necessarily full). A covariant representation of $\mathcal{E}$ on a C^*-algebra C is a pair $(\pi, \mathcal{T})$ where

1. $\pi : B \to C$ is a $*$-homomorphism of algebras;

2. $\mathcal{T} : \mathcal{E} \to C$ satisfies

$$\mathcal{T}(\xi)\pi(b) = \mathcal{T}(\xi b) \qquad \mathcal{T}(\xi)^*\mathcal{T}(\eta) = \pi(\langle \xi, \eta\rangle_B)$$
$$\pi(b)\mathcal{T}(\xi) = \mathcal{T}(b\xi) \qquad \mathcal{T}(\xi)\mathcal{T}(\eta)^* = \pi({}_B\langle \xi, \eta\rangle)$$

   for all $b \in B$ and $\xi, \eta \in \mathcal{E}$.

Using the theory of hereditary subalgebras in a C^*-algebra, one can show that covariant representations always exist (*cf.* [1, Proposition 2.3]).

Definition 3.2.3. Let $\mathcal{E}$ be a Hilbert (B, B)-bimodule. The *generalized crossed product* $B \rtimes_{\mathcal{E}} \mathbb{Z}$ of B by the Hilbert bimodule $\mathcal{E}$ is the universal C^*-algebra generated by the covariant representations of $\mathcal{E}$.

In [1, Proposition 2.9] the generalized crossed product algebra is realized as a cross-sectional algebra (*à la* Fell-Doran) for a suitable C^*-algebraic bundle over $\mathbb{Z}$.

It is worth stressing that a generalized crossed product need not be a Pimsner algebra in general, since the representation of B giving the left action need not be injective, nor the modules full. However, a self-Morita equivalence bimodule $(\mathcal{E}, \phi)$ is always a Hilbert bimodule, with left B-valued inner product constructed in terms of the map ϕ, as described in (2.2.4), and the bimodule is full. By using the universal properties in Theorem 3.1.5, one shows that for a self-Morita equivalence bimodule the two constructions yield isomorphic algebras.

The advantage of using generalized crossed products is that one can reconstruct a C^*-algebra carrying a circle action as a generalized crossed product if and only if a certain completeness condition is satisfied. As we will see, this condition is intrinsically geometrical, and it is deeply connected to principal actions.

Theorem 3.2.4 ([1, Theorem 3.1]). *Let A be a C^*-algebra with a strongly continuous action of the circle. The algebra A is isomorphic to $A_0 \rtimes_{A_1} \mathbb{Z}$ if and only if A is generated, as a C^*-algebra, by the fixed point algebra A_0 and the first spectral subspace A_1 of the circle action.*

The above condition was introduced in [34] and is referred to as having a *semi-saturated* action. It is fulfilled in a large class of examples, like crossed product by the integers, and noncommutative (or quantum) principal circle bundles, as we shall see quite explicitly in Chapter 4. In fact, this condition encompasses more general non-principal actions, which are however beyond the scope of the present work.

In Theorem 3.2.4 a central character is played by the module A_1. If one assumes that it is a full bimodule, that is if

$$A_1^* A_1 = A_0 = A_1 A_1^*, \tag{3.2.2}$$

the action σ is said to have *large spectral subspaces* (*cf.* [64, Section 2]), a slightly stronger condition than semi-saturatedness (*cf.* [5, Proposition 3.4]). Firstly, the condition above is equivalent to the condition that all bimodules A_k are full, that is $A_k^* A_k = A_0 = A_k A_k^*$ for all $k \in \mathbb{Z}$. When this happens, all bimodules A_k are self-Morita equivalence bimodules for A_0, with isomorphisms $\phi : A_0 \to \mathcal{K}_{A_0}(A_k)$ given by

$$\phi(a)(\xi) := a\,\xi, \qquad \text{for all } a \in A_0,\ \xi \in A_k. \tag{3.2.3}$$

Combining Theorem 3.2.4 with the fact that for a self-Morita equivalence bimodule the generalized crossed product construction and Pimsner's construction yield the same algebra, one obtains the following result.

Theorem 3.2.5. *[5, Theorem 3.5] Let A be a C^*-algebra with a strongly continuous action of the circle. Suppose that the first spectral subspace A_1 is a full and countably generated Hilbert bimodule over A_0. Then the Pimsner algebra $\mathcal{O}_{A_1}$ of the self-Morita equivalence (A_1, ϕ), with ϕ as in is* (3.2.3)*, is isomorphic to A. The isomorphism is given by $S_\xi \mapsto \xi$ for all $\xi \in A_1$.*

Up to completions, all examples considered in this work will fit into the framework of the previous theorem.

3.3 Six term exact sequences

With a Pimsner algebra $\mathcal{O}_{\mathcal{E}}$ come two natural six term exact sequences in KK-theory, which provide an elegant description of the KK-theory groups of $\mathcal{O}_{\mathcal{E}}$.

We will describe the exact sequences for the case of a self-Morita equivalence bimodule: these relate the KK-theory groups of the Pimsner algebra $\mathcal{O}_{\mathcal{E}}$ with that of the C^*-algebra of (the base space) scalars B. The corresponding sequences in K-theory are noncommutative analogues of the Gysin sequence (A.2.2), which in the commutative case relates the K-theories of the total space and of the base space of a principal circle bundle. The classical cup product with the Euler class is replaced, in the noncommutative setting, by a Kasparov product with the identity minus the generating Hilbert C^*-module $\mathcal{E}$.

We begin with a proposition, which is an immediate consequence of Theorem 3.2.1 and [67, Theorem 2.5].

Proposition 3.3.1. *Let $\mathcal{E}$ be a self-Morita equivalence bimodule. The exact sequence*

$$0 \longrightarrow \mathcal{K}_B(\mathcal{E}_+) \longrightarrow \mathcal{T}_{\mathfrak{E}} \xrightarrow{\pi} \mathcal{O}_{\mathfrak{E}} \longrightarrow 0. \tag{3.3.1}$$

admits a completely positive cross section $s : \mathcal{O}_{\mathfrak{E}} \to \mathcal{T}_{\mathfrak{E}}$.

Since the exact sequence has a completely positive cross section, by Theorem 2.4.16 it will induce six term exact sequences in KK-theory.

In the sequences we will use the fact that the algebra $\mathcal{K}_B(\mathcal{E}_+)$ is by definition Morita equivalent to B, and that by [67, Theorem 4.4] the Toeplitz algebra is KK-equivalent to B. We will now describe the maps appearing in the exact sequence.

First of all, in view of Assumption 3.1.6 the following class is well-defined.

Definition 3.3.2. The class in $KK_0(B,B)$ defined by the even Kasparov module $(\mathfrak{E}, \phi, 0)$ (with trivial grading) will be denoted by $[\mathfrak{E}]$.

Next, consider the orthogonal projection $P : \mathcal{E}_\infty \to \mathcal{E}_\infty$ with range

$$\mathrm{Im}(P) = \bigoplus_{k\geq 0}^{\infty} \mathfrak{E}^{(k)} \subseteq \mathcal{E}_\infty\,. \tag{3.3.2}$$

Since $[P, S_\xi] \in \mathcal{K}(\mathcal{E}_\infty)$ for all $\xi \in \mathfrak{E}$, one has $[P, S] \in \mathcal{K}(\mathcal{E}_\infty)$ for all $S \in \mathcal{O}_{\mathfrak{E}}$. Then, let $F := 2P - 1 \in \mathcal{L}(\mathcal{E}_\infty)$ and let $\widehat{\phi} : \mathcal{O}_{\mathfrak{E}} \to \mathcal{L}(\mathcal{E}_\infty)$ be the inclusion.

Definition 3.3.3. The class in $KK_1(\mathcal{O}_{\mathfrak{E}}, B)$ defined by the odd Kasparov module $(\mathcal{E}_\infty, \widehat{\phi}, F)$ will be denoted by $[\partial]$ and it is referred to as the *extension class*.

The class $[\partial]$ admits a natural unbounded representative (*cf.*[18, Section 6]). This is the analogue of the number operator on the Fock space of quantum mechanics. Geometrically, this operator can be interpreted as the infinitesimal generator for the gauge action. It is defined as

$$\mathcal{N} : \mathfrak{Dom}(\mathcal{N}) \to \mathcal{E}_\infty, \qquad \mathcal{N}\left(\sum_{k\in\mathbb{Z}} x_k\right) := \sum_{k\in\mathbb{Z}} k\, x_k \tag{3.3.3}$$

on the dense domain $\mathfrak{Dom}(\mathcal{N}) \subseteq \mathcal{E}_\infty$

$$\mathfrak{Dom}(\mathcal{N}) := \left\{ \xi = \sum_{k\in\mathbb{Z}} \xi_k \in \mathcal{E}_\infty \mid \xi_k \in \mathfrak{E}^{(k)},\ \left\| \sum_{k\in\mathbb{Z}} k^2 \langle \xi_k, \xi_k \rangle \right\| < \infty \right\}. \tag{3.3.4}$$

This defines a self-adjoint and regular operator on $\mathcal{E}_\infty$ (*cf.* [65, Proposition 4.6] and [16, Proposition 2.7]) and it follows from [71, Theorem 3.1] that the pair $(\mathcal{E}_\infty, \mathcal{N})$ yields a class in the odd unbounded Kasparov bivariant K-theory $KK_1(B, \mathcal{O}_{\mathfrak{E}})$ whose bounded transform is the module of Definition 3.3.3.

For any separable C^*-algebra C one then has group homomorphisms

$$[\mathfrak{E}] : KK_*(B,C) \to KK_*(B,C)\,, \quad [\mathfrak{E}] : KK_*(C,B) \to KK_*(C,B)$$

and

$$[\partial] : KK_*(C, \mathcal{O}_{\mathcal{E}}) \to KK_{*+1}(C, B)\,, \quad [\partial] : KK_*(B, C) \to KK_{*+1}(\mathcal{O}_{\mathcal{E}}, C)\,,$$

which are induced by the Kasparov product. With abuse of notation we denote the product with the class with the same symbol as the class itself: care should be taken since the Kasparov products above are sometimes taken on the left and sometimes on the right.

These yield natural six term exact sequences in KK-theory [67, Theorem 4.8].

Theorem 3.3.4. *Let $\mathcal{O}_{\mathcal{E}}$ be the Pimsner algebra of the self-Morita equivalence bimodule $(\mathcal{E}, \phi)$ over the C^*-algebra B. If C is any separable C^*-algebra, there are two exact sequences:*

$$\begin{array}{ccccc} KK_0(C,B) & \xrightarrow{1-[\mathcal{E}]} & KK_0(C,B) & \xrightarrow{i_*} & KK_0(C,\mathcal{O}_{\mathcal{E}}) \\ {\scriptstyle [\partial]}\uparrow & & & & \downarrow{\scriptstyle [\partial]} \\ KK_1(C,\mathcal{O}_{\mathcal{E}}) & \xleftarrow[i_*]{} & KK_1(C,B) & \xleftarrow[1-[\mathcal{E}]]{} & KK_1(C,B) \end{array}$$

and

$$\begin{array}{ccccc} KK_0(B,C) & \xleftarrow[1-[\mathcal{E}]]{} & KK_0(B,C) & \xleftarrow[i^*]{} & KK_0(\mathcal{O}_{\mathcal{E}},C) \\ \downarrow{\scriptstyle [\partial]} & & & & {\scriptstyle [\partial]}\uparrow \\ KK_1(\mathcal{O}_{\mathcal{E}},C) & \xrightarrow{i^*} & KK_1(B,C) & \xrightarrow{1-[\mathcal{E}]} & KK_1(B,C) \end{array}$$

with i^*, i_* the homomorphisms in KK-theory induced by the inclusion $i : B \to \mathcal{O}_{\mathcal{E}}$.

In particular, for $C = \mathbb{C}$ one obtains a six-term exact sequence in K-theory:

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{1-[\mathcal{E}]} & K_0(B) & \xrightarrow{i_*} & K_0(\mathcal{O}_{\mathcal{E}}) \\ {\scriptstyle [\partial]}\uparrow & & & & \downarrow{\scriptstyle [\partial]} \\ K_1(\mathcal{O}_{\mathcal{E}}) & \xleftarrow[i_*]{} & K_1(B) & \xleftarrow[1-[\mathcal{E}]]{} & K_1(B) \end{array} \quad . \tag{3.3.5}$$

with i_* the homomorphism in K-theory induced by the inclusion $i : B \to \mathcal{O}_{\mathcal{E}}$. This could be considered as a generalization of the classical *Gysin sequence* in K-theory of (A.2.2) for the *noncommutative line bundle* $\mathcal{E}$ over the *noncommutative space* B and with the map $1 - [\mathcal{E}]$ having the same rôle as the *Euler class* $\chi(\mathcal{E}) := 1 - [\mathcal{E}]$ of the line bundle $\mathcal{E}$.

Note that whenever $K_1(B) = 0$ one can easily compute the K-theory groups of the Pimsner algebra $\mathcal{O}_{\mathcal{E}}$ as

$$K_0(\mathcal{O}_{\mathcal{E}}) \simeq \operatorname{coker}(1 - [\mathcal{E}]), \qquad K_1(\mathcal{O}_{\mathcal{E}}) \simeq \ker(1 - [\mathcal{E}]). \tag{3.3.6}$$

We will use this fact in Chapter 5 for computing the K-theory groups of quantum lens spaces of arbitrary dimension.

The *dual* sequence would then be an analogue in K-homology:

$$\begin{array}{ccccc} K^0(B) & \xleftarrow[1-[\mathcal{E}]]{} & K^0(B) & \xleftarrow[i^*]{} & K^0(\mathcal{O}_{\mathcal{E}}) \\ \downarrow{[\partial]} & & & & \uparrow{[\partial]} \\ K^1(\mathcal{O}_{\mathcal{E}}) & \xrightarrow{i^*} & K^1(B) & \xrightarrow{1-[\mathcal{E}]} & K^1(B) \end{array} . \tag{3.3.7}$$

where now i^* is the induced homomorphism in K-homology. With the same reasoning as above, whenever $K^1(B) = 0$ one can easily compute the K-homology groups of the Pimsner algebra $\mathcal{O}_{\mathcal{E}}$ as

$$K^0(\mathcal{O}_{\mathcal{E}}) \simeq \ker(1 - [\mathcal{E}]), \qquad K^1(\mathcal{O}_{\mathcal{E}}) \simeq \operatorname{coker}(1 - [\mathcal{E}]). \tag{3.3.8}$$

Example 3.5. As a particular case of the above exact sequences, one has the Pimsner-Voiculescu exact sequence for the K-theory of crossed products by the integers of [66]. This reads

$$\begin{array}{ccccc} K_0(B) & \xrightarrow{1-\alpha_*} & K_0(B) & \xrightarrow{i_*} & K_0(B \rtimes_\alpha \mathbb{Z}) \\ \uparrow{[\partial]} & & & & \downarrow{[\partial]} \\ K_1(B \rtimes_\alpha \mathbb{Z}) & \xleftarrow[i_*]{} & K_1(B) & \xleftarrow[1-\alpha_*]{} & K_1(B) \end{array} , \tag{3.3.9}$$

where Kasparov product with $1 - [\mathcal{E}]$, reduces, in this case, to the map $1 - \alpha_*$. This is induced by the identity on B and, with the same convention of Definition 1.4.1, by the automorphism $\alpha := \alpha_1 : B \to B$.

Chapter 4

Pimsner algebras from principal circle bundles

In this chapter we explore the connections between principal circle bundles–both commutative and noncommutative–, frames for modules as described in Section 2, and $\mathbb{Z}$-graded algebras. This is based on Sections 3 and 4 of [3] and on [4].

We will describe noncommutative principal bundles at the level of coordinate algebras, focusing on the case of circle bundles. These can be naturally given a structure of $\mathbb{Z}$-graded algebras. We will state a necessary and sufficient condition for a $\mathbb{Z}$-graded algebra to be a noncommutative principal bundle.

As we will see, this follows from the fact that the condition of having a principal bundle is equivalent, for an abelian structure group G, to the total space algebra being *strongly graded* over the Pontryagin dual $\hat{G}$.

When completing with natural C^*-norms one has to consider continuous circle actions on a C^*-algebra, and the $\mathbb{Z}$-grading will agree on that given by spectral subspaces, as described in Subsection 3.2.2. We will see that the C^*-algebra of continuous functions on the total space of a noncommutative circle bundle can be realized as a Pimsner algebra over the base space, for the Hilbert module given by the first spectral subspace of the corresponding circle action.

4.1 Noncommutative principal circle bundles

In noncommutative geometry the notion of a circle action is dualized by considering a coaction of the dual group Hopf algebra. Principality of the action is encoded in the notion of Hopf-Galois extension. For more details on the theory of Hopf algebras and their coactions, we refer the reader to [61].

A *noncommutative principal bundle* is a triple $(\mathcal{A}, \mathcal{H}, \mathcal{B})$, where $\mathcal{A}$ is the $*$-algebra of functions on the *total space*, $\mathcal{H}$ is the Hopf-algebra of functions on the *structure group*, with $\mathcal{A}$ being a right $\mathcal{H}$-comodule $*$-algebra, that is there is a right coaction

$$\Delta_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}.$$

The functions on the *base space* are given by the subalgebra of coinvariant elements:

$$\mathcal{B} = \mathcal{A}^{co\mathcal{H}} := \{x \in \mathcal{A} \mid \Delta_R(x) = x \otimes 1\}.$$

We assume the algebra $\mathcal{A}$, and hence $\mathcal{B}$, to be unital. We denote by $\Omega^1_{un}(\mathcal{A})$ and $\Omega^1_{un}(\mathcal{B})$ the bimodules of universal differential forms over $\mathcal{A}$ and $\mathcal{B}$, respectively.

Definition 4.1.1. One says that the datum $\big(\mathcal{A}, \mathcal{H}, \mathcal{B}\big)$ is a *noncommutative (or quantum) principal circle bundle* if and only if the sequence

$$0 \to \mathcal{A}(\Omega^1_{un}(\mathcal{F}))\mathcal{A} \to \Omega^1_{un}(\mathcal{A}) \xrightarrow{\text{ver}} \mathcal{A} \otimes \ker \epsilon_{\mathcal{H}} \to 0 \tag{4.1.1}$$

is exact, with $\epsilon_{\mathcal{H}}$ denoting the counit of the Hopf algebra $\mathcal{H}$. Here the first map is inclusion while the second one, given by

$$\text{ver}(a \otimes b) := (a \otimes 1)\Delta_R(b),$$

generates the analogue of *vertical one-forms* on the bundle.

The above condition is that of having a quantum principal bundle with the universal calculus, in the sense of [14]; this in turn is equivalent, by [40, Proposition 1.6], to the statement that the triple $(\mathcal{A}, \mathcal{H}, \mathcal{B})$ is a *Hopf-Galois extension*.

When $\mathcal{H}$ is cosemisimple and has an invertible antipode, exactness of the sequence (4.1.1) is equivalent to the statement that the canonical map

$$\chi : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}\,, \quad x \otimes y \mapsto x\,\Delta_R(y) \tag{4.1.2}$$

is an isomorphism. By [77, Theorem I], the map χ in (4.1.2) is injective whenever it is surjective, and thus it is enough to check surjectivity .

When studying the structure of noncommutative circle bundles, one needs to consider the unital complex algebra

$$\mathcal{O}(U(1)) := \mathbb{C}[z, z^{-1}]/\langle 1 - zz^{-1}\rangle,$$

where $\langle 1-zz^{-1}\rangle$ is the ideal generated by $1-zz^{-1}$ in the polynomial algebra $\mathbb{C}[z, z^{-1}]$ on two variables. The algebra $\mathcal{O}(U(1))$ is a Hopf algebra by defining, for any $n \in \mathbb{Z}$, the coproduct $\Delta : z^n \mapsto z^n \otimes z^n$, the antipode $S : z^n \mapsto z^{-n}$ and the counit $\epsilon : z^n \mapsto 1$. It is cosemisimple with bijective antipode, so we can rephrase the principality condition:

Proposition 4.1.2. *The datum* $\big(\mathcal{A}, \mathcal{O}(U(1)), \mathcal{B}\big)$ *is a* noncommutative (or quantum) principal circle bundle *if and only if the* canonical map

$$\chi : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}(U(1)),$$

is surjective.

Given $\mathcal{A}$ a right $\mathcal{O}(U(1))$-comodule algebra as above, if one defines

$$\mathcal{A}_k := \{x \in \mathcal{A} \mid \Delta_R(x) = x \otimes z^{-k}\},$$

by its very definition, the fixed point algebra $\mathcal{B} \simeq \mathcal{A}_0$ and $\mathcal{A}_k\mathcal{A}_l \subseteq \mathcal{A}_{k+l}$. This provides $\mathcal{A}$ with a $\mathbb{Z}$-graded algebra structure.

Remark 4.1.3. Note that, since the group is a classical Abelian group, one may as well have considered an action σ of $\mathbb{S}^1$, obtaining the degree k-part as

$$\mathcal{A}_k = \{x \in \mathcal{A} \mid \sigma_w(x) = xw^{-k} \text{ for all } w \in \mathbb{S}^1\}. \tag{4.1.3}$$

Conversely, if $\mathcal{A} = \oplus_{k\in\mathbb{Z}}\mathcal{A}_k$ is a $\mathbb{Z}$-graded unital algebra. The unital algebra homomorphism,

$$\Delta_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}(U(1)), \quad x \mapsto x \otimes z^{-k}\,, \ \text{ for } \ x \in \mathcal{A}_k\,,$$

turns $\mathcal{A}$ into a right comodule algebra over $\mathcal{O}(U(1))$. Clearly the unital subalgebra of coinvariant elements coincides with $\mathcal{A}_0$.

We now translate the condition of principality for the case of $\mathbb{Z}$-graded algebras by presenting a necessary and sufficient condition for the canonical map (4.1.2) to be surjective, as described in [5, Theorem 4.3](see also [80, Lemma 5.1]). This condition is more manageable in general, and in particular it can be usefully applied in examples like the quantum lens spaces as principal circle bundles over quantum weighted projective spaces [5, 30].

Theorem 4.1.4. *The triple* $\big(\mathcal{A}, \mathcal{O}(U(1)), \mathcal{A}_0\big)$ *is a noncommutative principal circle bundle if and only if there exist finite sequences*

$$\{\xi_j\}_{j=1}^N\,,\ \{\beta_i\}_{i=1}^M \ \textit{in } \mathcal{A}_1 \quad \textit{and} \quad \{\eta_j\}_{j=1}^N\,,\ \{\alpha_i\}_{i=1}^M \ \textit{in } \mathcal{A}_{-1}$$

such that one has identities:

$$\sum\nolimits_{j=1}^N \xi_j\eta_j = 1_{\mathcal{A}} = \sum\nolimits_{i=1}^M \alpha_i\beta_i\,. \tag{4.1.4}$$

Proof. Suppose first that $\big(\mathcal{A}, \mathcal{O}(U(1)), \mathcal{A}_0\big)$ is a quantum principal circle bundle. Thus, that the canonical map

$$\chi : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}(U(1))$$

is an isomorphism. For each $k \in \mathbb{Z}$, define the idempotents

$$\begin{aligned} &P_k : \mathcal{O}(U(1)) \to \mathcal{O}(U(1))\,, \quad && P_k : z^m \mapsto \delta_{km}z^m \quad \text{and} \\ &E_k : \mathcal{A} \to \mathcal{A}\,, && E_n : x_m \mapsto \delta_{km}x_m \end{aligned}$$

where $x_m \in \mathcal{A}_m$ and where δ_{km} denotes the Kronecker delta. Clearly,

$$\chi \circ (1 \otimes E_{-k}) = (1 \otimes P_k) \circ \chi : \mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}(U(1))\,. \tag{4.1.5}$$

for all $n \in \mathbb{Z}$. Let us now define the element

$$\gamma := \chi^{-1}(1_{\mathcal{A}} \otimes z) = \sum_{j=1}^{N} \gamma_j^0 \otimes \gamma_j^1 \,.$$

It then follows from (4.1.5) that

$$\gamma = (1 \otimes E_{-1})(\gamma) = \sum_{j=1}^{N} \gamma_j^0 \otimes E_{-1}(\gamma_j^1).$$

To continue, we remark that

$$m(\gamma) = m \circ \chi^{-1}(1_{\mathcal{A}} \otimes z) = (\mathrm{Id} \otimes \epsilon)(1_{\mathcal{A}} \otimes z) = 1_{\mathcal{A}},$$

where $m : \mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A} \to \mathcal{A}$ is the algebra multiplication. And this implies that

$$1_{\mathcal{A}} = \sum_{j=1} \gamma_j^0 \cdot E_{-1}(\gamma_j^1) = \sum_{j=1}^{N} E_1(\gamma_j^0) \cdot E_{-1}(\gamma_j^1) \,.$$

We therefore put,

$$\xi_j := E_1(\gamma_0^j) \quad \text{and} \quad \eta_j := E_{-1}(\gamma_1^j) \,, \quad \text{for all } \ j = 1, \dots, N \,.$$

Next, we define the element

$$\delta := \chi^{-1}(1_{\mathcal{A}} \otimes z^{-1}) = \sum_{i=1}^{M} \delta_i^0 \otimes \delta_i^1 \,.$$

An argument similar to the one before then shows that $\sum_{i=1}^{M} \alpha_i \cdot \beta_i = 1_{\mathcal{A}}$, with

$$\alpha_i := E_{-1}(\delta_i^0) \quad \text{and} \quad \beta_i := E_1(\delta_i^1) \,, \quad \text{for all } \ i = 1, \dots, M \ .$$

This proves the first half of the theorem.

To prove the second half we suppose that there exist sequences $\{\xi_j\}_{j=1}^N$, $\{\beta_i\}_{i=1}^M$ in $\mathcal{A}_1$ and $\{\eta_j\}_{j=1}^N$, $\{\alpha_i\}_{i=1}^M$in $\mathcal{A}_{-1}$ such that $\sum_{j=1}^N \xi_j \eta_j = 1_{\mathcal{A}} = \sum_{i=1}^M \alpha_i \beta_i$.

We define the inverse map $\chi^{-1} : \mathcal{A} \otimes \mathcal{O}(U(1)) \to \mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A}$ by the formula

$$\chi^{-1} : x \otimes z^k \mapsto \begin{cases} \sum_{j_k=1}^{N} \ x\, \xi_{j_1} \cdot \ldots \cdot \xi_{j_k} \otimes \eta_{j_k} \cdot \ldots \cdot \eta_{j_1} \,, & k \geq 0 \\ \\ \sum_{i_k=1}^{M} \ x\, \alpha_{i_1} \cdot \ldots \cdot \alpha_{i_{-k}} \otimes \beta_{i_{-k}} \cdot \ldots \cdot \beta_{i_1} \,. & k \leq 0 \end{cases} \,. \tag{4.1.6}$$

This completes the proof of the theorem. □

Now, (4.1.4) are exactly the frame relations (2.1.7) for $\mathcal{A}_1$ and $\mathcal{A}_{-1}$, which imply that they are finitely generated and projective over $\mathcal{A}_0$.

As already argued at the end of Section 2, with the ξ's and the η's as above, one defines the module homomorphisms

$$\Phi_1 : \mathcal{A}_1 \to (\mathcal{A}_0)^N\,,$$
$$\Phi_1(\zeta) = (\eta_1\zeta\,, \eta_2\,\zeta\,, \cdots\,, \eta_N\,\zeta)^t$$

and

$$\Psi_1 : (\mathcal{A}_0)^N \to \mathcal{A}_1\,,$$
$$\Psi_1(x_1\,, x_2\,, \cdots\,, x_N)^t = \xi_1\,x_1 + \xi_2\,x_2 + \cdots + \xi_N\,x_N\,.$$

It then follows that $\Psi_1\Phi_1 = \mathrm{Id}_{\mathcal{A}_1}$. Thus $\mathbf{e}_1 := \Phi_1\Psi_1$ is an idempotent in $M_N(\mathcal{A}_0)$, and $\mathcal{A}_1 \simeq \mathbf{e}_1(\mathcal{A}_0)^N$. Similarly, with the α's and the β's as above, one defines the module homomorphisms

$$\Phi_{-1} : \mathcal{A}_1 \to (\mathcal{A}_0)^M\,,$$
$$\Phi_{-1}(\zeta) = (\beta_1\zeta\,, \beta_2\,\zeta\,, \cdots\,, \beta_M\,\zeta)^t$$

and

$$\Psi_{-1} : (\mathcal{A}_0)^M \to \mathcal{A}_1\,,$$
$$\Psi_{-1}(x_1\,, x_2\,, \cdots\,, x_M)^t = \alpha_1\,x_1 + \alpha_2\,x_2 + \cdots + \alpha_M\,x_M\,.$$

Now one checks that $\Psi_{-1}\Phi_{-1} = \mathrm{Id}_{\mathcal{A}_{-1}}$. Thus $\mathbf{e}_{-1} := \Phi_{-1}\Psi_{-1}$ is an idempotent in $M_M(\mathcal{A}_0)$, and $\mathcal{A}_{-1} \simeq \mathbf{e}_{-1}(\mathcal{A}_0)^M$.

Note that condition (4.1.4) implies that the modules $\mathcal{A}_{-1}\mathcal{A}_1$ and $\mathcal{A}_1\mathcal{A}_{-1}$ are not only contained in $\mathcal{A}_0$, but they are actually equal to it.

4.1.1 Strongly graded algebras

The relevance of graded algebras for noncommutative principal bundles was already shown in [84]. The condition of having a principal action can be translated into a condition on the corresponding grading, as explained in [61, Chapter 8].

Proposition 4.1.5. *Let $\mathcal{A} := \bigoplus_{k\in\mathbb{Z}} \mathcal{A}_k$ be a $\mathbb{Z}$ graded unital algebra. The following facts are equivalent.*

1. $\mathcal{A}_k\mathcal{A}_l = \mathcal{A}_{k+l}$ *for all* $k, l \in \mathbb{Z}$;
2. $\mathcal{A}_k\mathcal{A}_{-k} = \mathcal{A}_0$ *for all* $k \in \mathbb{Z}$;
3. $\mathcal{A}_1\mathcal{A}_{-1} = \mathcal{A}_0 = \mathcal{A}_{-1}\mathcal{A}_1$.

Proof. Conditions 1–3 are successively stronger, hence it only suffices to show that $3 \Rightarrow 1$.

The fact that $\mathcal{A}_k\mathcal{A}_l \subseteq \mathcal{A}_{k+l}$ is simply the definition of graded algebra. Recall that for any graded unital algebra $\mathcal{A}$ one has that $1_{\mathcal{A}} \in \mathcal{A}_0$. Hence

$$\mathcal{A}_k \subseteq \mathcal{A}_k\mathcal{A}_0, \quad \text{for all } k \in \mathbb{Z}. \tag{4.1.7}$$

Now suppose that $l > 0$. By 4.1.7 and iterating the second half of Condition 3 we get that

$$\mathcal{A}_{k+l} \subseteq \mathcal{A}_{k+l}\mathcal{A}_0 = \mathcal{A}_{k+l}\mathcal{A}_{-1}\mathcal{A}_1 = \mathcal{A}_{k+l-1}\mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_k\mathcal{A}_l.$$

Similarly, for $l < 0$, one performs successive iterations of the first half of Condition 2. □

Definition 4.1.6. A $\mathbb{Z}$-graded algebra that satisfies one of the above conditions is called a *strongly $\mathbb{Z}$-graded algebra.*

In the context of strongly $\mathbb{Z}$-graded algebras, the fact that all right modules $\mathcal{A}_n$ for all $n \in \mathbb{Z}$ are finitely generated projective, as implied by the frame condition (4.1.4), is a consequence of [63, Corollary I.3.3].

We can therefore rephrase Theorem 4.1.3 as follows:

Corollary 4.1.7. *The datum $\big(\mathcal{A}, \mathcal{O}(U(1)), \mathcal{A}_0\big)$ is a noncommutative principal circle bundle if and only if the algebra $\mathcal{A}$ is strongly $\mathbb{Z}$-graded.*

This can be generalized to the case of any (multiplicative) group G with unit e in the following sense: one says that an algebra $\mathcal{A}$ is *G-graded* if its admits a direct sum decomposition labelled by elements of G, that is $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$, with the property that $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$, for all $g, h \in G$. If $\mathcal{H} := \mathbb{C}[G]$ denotes the group algebra, it is well-know that $\mathcal{A}$ is G-graded if and only if $\mathcal{A}$ is a right $\mathcal{H}$-comodule algebra for the coaction $\delta : \mathcal{A} \to \mathcal{A}\otimes\mathcal{H}$ defined on homogeneous elements $a_g \in \mathcal{A}_g$ by $\delta(a_g) = a_g \otimes g$. Clearly, the coinvariants are given by $\mathcal{A}^{co\mathcal{H}} = \mathcal{A}_e$, the identity components. One has then the following result:

Theorem 4.1.8 ([61, Theorem 8.1.7])**.** *The datum $\big(\mathcal{A}, \mathcal{H}, \mathcal{A}_e\big)$ is a noncommutative principal $\mathcal{H}$-bundle for the canonical map*

$$\chi : \mathcal{A} \otimes_{\mathcal{A}_e} \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}\,, \quad a \otimes b \mapsto \textstyle\sum_g ab_g \otimes g\,,$$

if and only if $\mathcal{A}$ is strongly graded*, that is $\mathcal{A}_g\mathcal{A}_h = \mathcal{A}_{gh}$, for all $g, h \in G$.*

Proof. First note that $\mathcal{A}$ being strongly graded is equivalent to $\mathcal{A}_g\mathcal{A}_{g^{-1}} = \mathcal{A}_e$, for all $g \in G$. Then one proceeds in constructing an inverse of the canonical map pretty much as in (4.1.6). Since, for each $g \in G$, the unit $1_\mathcal{A} \in \mathcal{A}_e = \mathcal{A}_{g^{-1}}\mathcal{A}_g$, there exists $\xi_{g^{-1},j}$ in $\mathcal{A}_g$ and $\eta_{g,j} \in \mathcal{A}_{g^{-1}}$, such that $\sum_j \eta_{g,j}\xi_{g^{-1},j} = 1_\mathcal{A}$. Then, $\chi^{-1} : \mathcal{A} \otimes \mathcal{H} \to \mathcal{A} \otimes_{\mathcal{A}_e} \mathcal{A}$, is given by

$$\chi^{-1} : a \otimes g \mapsto \textstyle\sum_j a\,\xi_{g^{-1},j} \otimes \eta_{g,j}.$$

□

4.1.2 Tensor powers of line bundles

In the context of the previous sections, the modules $\mathcal{A}_1$ and $\mathcal{A}_{-1}$ emerge as a central character.

They are actually *line bundles*, being self-Morita equivalence bimodules (in the algebraic sense) over the algebra $\mathcal{A}_0$. In the same vein, by condition 2 in Proposition 4.1.5, all modules $\mathcal{A}_k$ for $k \in \mathbb{Z}$ are line bundles as well.

Given any natural number d, one is therefore led to consider the direct sum

$$\mathcal{A}^{\mathbb{Z}_d} := \oplus_{k\in\mathbb{Z}} \mathcal{A}_{kd}. \tag{4.1.8}$$

This vector space turns out to be isomorphic to the fixed point algebra for an action of $\mathbb{Z}_d := \mathbb{Z}/d\mathbb{Z}$ on the starting algebra $\mathcal{A}$:

Proposition 4.1.9. *Let $\mathcal{A} \simeq \oplus_{k\in\mathbb{Z}} \mathcal{A}_k$ be a $\mathbb{Z}$-graded unital algebra, and denote with σ the corresponding circle action. The vector space $\mathcal{A}^{\mathbb{Z}_d} := \oplus_{n\in\mathbb{Z}} \mathcal{A}_{dn}$ is a *-algebra made of all elements of $\mathcal{A}$ that are invariant under the action $\sigma^{1/d} : \mathbb{Z}_d \to \mathrm{Aut}(\mathcal{A})$ obtained by restricting the circle action.*

Proof. By (4.1.3), the action $\sigma^{1/d}$ is defined on the generator of $\mathbb{Z}_d$ by the condition

$$\mathrm{e}^{2\pi i/d} \cdot x = \mathrm{e}^{2\pi i k/d} x, \quad \text{for } x \in \mathcal{A}_k. \tag{4.1.9}$$

It is straightforward to see that the algebra of invariant elements with respect to (4.1.9) is precisely $\mathcal{A}^{\mathbb{Z}_d}$. □

As a corollary of Theorem 4.1.4 one gets the following:

Proposition 4.1.10 ([5, Proposition 4.6])**.** *Suppose $\left(\mathcal{A}, \mathcal{O}(U(1)), \mathcal{A}_0\right)$ is a noncommutative principal circle bundle. Then, for all $d \in \mathbb{N}$, the datum $\left(\mathcal{A}^{\mathbb{Z}_d}, \mathcal{O}(U(1)), \mathcal{A}_0\right)$ is a noncommutative principal circle bundle as well.*

The proof of this result goes along the line of Theorem 4.1.4 and shows also that the right modules $\mathcal{A}_d$ and $\mathcal{A}_{-d}$ are finitely generated projective over $\mathcal{A}_0$ for all $d \in \mathbb{N}$. Indeed, let the finite sequences $\{\xi_j\}_{j=1}^N$, $\{\beta_i\}_{i=1}^M$ in $\mathcal{A}_1$ and $\{\eta_j\}_{j=1}^N$, $\{\alpha_i\}_{i=1}^M$ in $\mathcal{A}_{-1}$ be as in Theorem 4.1.4. Then, for each multi-index $J \in \{1, \dots, N\}^d$ and each multi-index $I \in \{4.6, \dots, M\}^d$ the elements

$$\begin{aligned} &\xi_J := \xi_{j_1} \cdot \ldots \cdot \xi_{j_d}\,, \quad \beta_I := \beta_{i_d} \cdot \ldots \cdot \beta_{i_1} \in \mathcal{A}_d \quad \text{and} \\ &\eta_J := \eta_{j_d} \cdot \ldots \cdot \eta_{j_1}\,, \quad \alpha_I := \alpha_{i_1} \cdot \ldots \cdot \alpha_{i_d} \in \mathcal{A}_{-d}\,, \end{aligned}$$

are clearly such that

$$\sum_{J\in\{1,\dots,N\}^d} \xi_J\, \eta_J = 1_{\mathcal{A}^{\mathbb{Z}_d}} = \sum_{I\in\{1,\dots,M\}^d} \alpha_I\, \beta_I\ .$$

These allow one on one hand to apply Theorem 4.1.4 to show principality and on the other hand to construct idempotents $\mathbf{e}_{(-d)}$ and $\mathbf{e}_{(d)}$, thus showing the finite projectivity of the right modules $\mathcal{A}_d$ and $\mathcal{A}_{-d}$ for all $d \in \mathbb{N}$.

4.2 Pimsner algebras from principal circle bundles

4.2.1 An algebraic Pimsner construction

We present here an algebraic version of Pimsner construction, showing how to construct a strongly graded algebra starting from two bimodules.

Let $\mathcal{A}_0$ be an associative complex algebra, and $\mathcal{A}_1, \mathcal{A}_{-1}$ two bimodules, together with two maps

$$\psi : \mathcal{A}_1 \otimes_{\mathcal{A}_0} \mathcal{A}_{-1} \to \mathcal{A}_0 \quad \text{and} \quad \phi : \mathcal{A}_{-1} \otimes_{\mathcal{A}_0} \mathcal{A}_1 \to \mathcal{A}_0 .$$

For every $n \geq 1$ we set

$$\mathcal{A}_n := \underbrace{\mathcal{A}_1 \otimes_{\mathcal{A}_0} \mathcal{A}_1 \otimes_{\mathcal{A}_0} \cdots \otimes_{\mathcal{A}_0} \mathcal{A}_1}_{n\text{-times}} \qquad \mathcal{A}_{-n} := \underbrace{\mathcal{A}_{-1} \otimes_{\mathcal{A}_0} \mathcal{A}_{-1} \otimes_{\mathcal{A}_0} \cdots \otimes_{\mathcal{A}_0} \mathcal{A}_{-1}}_{n\text{-times}} .$$

Then we define two algebras $\mathcal{A}_+ := \oplus_{n\geq 1} \mathcal{A}_n$ and $\mathcal{A}_- := \oplus_{n \geq 1} \mathcal{A}_{-n}$.

Finally, the algebra

$$\mathcal{A} := \mathcal{A}_- \oplus \mathcal{A}_0 \oplus \mathcal{A}_+$$

is a unital associative $\mathbb{Z}$-graded algebra if and only if the following compatibility conditions are satisfied

$$m_L^1 \circ (\psi \otimes \mathrm{Id}) = m_R^1 \circ (\mathrm{Id} \otimes \phi), \qquad m_L^{-1} \circ (\phi \otimes \mathrm{Id}) = m_R^{-1} \circ (\mathrm{Id} \otimes \psi),$$

where $m_L^i : \mathcal{A}_0 \otimes_{\mathcal{A}_0} \mathcal{A}_i \to \mathcal{A}_i$ and $m_R^i : \mathcal{A}_i \otimes_{\mathcal{A}_0} \mathcal{A}_0 \to \mathcal{A}_i$ are the left and right multiplications with elements of $\mathcal{A}_0$, for $i = 1, -1$.

Proposition 4.2.1. *The maps ψ and ϕ are surjective, if and only if they are injective.*

Proof. This fact is a consequence of the above compatibility conditions. □

Note that the maps ϕ, ψ are bijective if and only if the modules $\mathcal{A}_i$'s are self-Morita equivalence bimodules (in the algebraic sense) and they are inverse to one another. In particular, this gives the following:

Corollary 4.2.2. *The algebra $\mathcal{A}$ is strongly $\mathbb{Z}$-graded if and only if the modules $\mathcal{A}_i$'s are self-Morita equivalence bimodules.*

A somewhat related construction was already described in [19] for the case of rings. The starting point there is the notion of R-system for a ring R, i.e a triple

(P, Q, ψ), where P and Q are R-bimodules and $\psi : P \otimes_R Q \to R$ is a R-bimodule homomorphism. Out of these ingredients one constructs a ring $\mathcal{T}_{(P,Q,\psi)}$, which plays the rôle of the Toeplitz algebra, called the *Toeplitz ring*, and a quotient of the latter, $\mathcal{O}_{(P,Q,\psi)}$, the analogue of the Cuntz-Pimsner algebra, called the *Cuntz-Pimsner* ring.

4.2.2 C^*-algebras and C^*-completions of $\mathbb{Z}$-graded algebras

If one compares condition (3.2.2) with Theorem 4.1.4, it is clear that a C^*-algebra A carrying a circle action with large spectral subspaces is strongly $\mathbb{Z}$-graded. One is then naturally led to consider the corresponding Pimsner algebra.

For commutative algebras this connection was already established in [36, Proposition 5.8] with the following result:

Proposition 4.2.3. *Let A be a unital, commutative C^*-algebra carrying a circle action. Suppose that the first spectral subspace $\mathcal{E} = A_1$ is finitely generated projective over $B = A_0$. Suppose that $\mathcal{E}$ generates A as a C^*-algebra. Then the following facts hold:*

1. *$B = C(X)$ for some compact space X,*
2. *$\mathcal{E}$ is the module of sections of some line bundle $L \to X$,*
3. *$A = C(P)$, where $P \to X$ is the principal $\mathbb{S}^1$-bundle over X associated with the line bundle L, and the $\mathbb{S}^1$ action on A is the one induced by the principal $\mathbb{S}^1$-action on P.*

More generally, let us start with $\mathcal{A} = \oplus_{k\in\mathbb{Z}}\mathcal{A}_k$ a graded $*$-algebra. Denote by σ the circle action coming from the grading, i.e. given by

$$\sigma_w(a) = w^{-k}a \quad \forall a \in \mathcal{A}_k. \tag{4.2.1}$$

In addition, suppose there is a C^*-norm on $\mathcal{A}$, and that σ is isometric with respect to this norm:

$$\|\sigma_w(a)\| \leq \|a\|, \quad \forall w \in \mathbb{S}^1, \ a \in \mathcal{A}. \tag{4.2.2}$$

Denoting by A the completion of $\mathcal{A}$, one has the following [5, Subsection 3.6]:

Lemma 4.2.4. *The action $\{\sigma_w\}_{w\in\mathbb{S}^1}$ extends by continuity to a strongly continuous action of $\mathbb{S}^1$ on A. Furthermore, each spectral subspace A_k for the extended action agrees with the closure of $\mathcal{A}_k \subseteq A$.*

Proof. Once $\mathcal{A}_k$ is shown to be dense in A_k the rest follows from standard arguments. Thus, for $k \in \mathbb{Z}$, define the bounded operator $E_{(k)} : A \to A_k$ by

$$E_{(k)} : a \mapsto \int_{\mathbb{S}^1} w^n \, \sigma_w(a) \, \mathrm{d}w \,,$$

where the integration is carried out with respect to the Haar-measure on $\mathbb{S}^1$. Such operators are named *conditional expectations.* We have that $E_{(k)}(a) = a$ for all $a \in A_k$ and then that $\|E_{(k)}\| \leq 1$. This implies that $\mathcal{A}_k \subseteq A_k$ is dense. □

The left and right A_0-valued Hermitian products as in (3.2.1) will make each spectral subspace A_k a (not necessarily full) Hilbert C^*-module over A_0. These become full exactly when $\mathcal{A}$ is strongly graded. Theorem 3.2.5 leads then to:

Proposition 4.2.5. *Let $\mathcal{A} = \oplus_{k\in\mathbb{Z}}\mathcal{A}_k$ be a strongly graded $*$-algebra, endowed with a *-norm such that the induced circle action is isometric. Then its C^*-closure A is generated, as a C^*-algebra, by A_1, and it is isomorphic to the Pimsner algebra $\mathcal{O}_{A_1}$ over A_0.*

We conclude this section by investigating what happens when the C^*-norm on $\mathcal{A} = \oplus_{k\in\mathbb{Z}}\mathcal{A}_k$ is changed. Thus, let $\|\cdot\|' : \mathcal{A} \to [0,\infty)$ be another C^*-norm on $\mathcal{A}$ satisfying

$$\|\sigma_w(x)\|' \leq \|a\|', \quad \text{for all } w \in \mathbb{S}^1 \text{ and } a \in \mathcal{A}\,.$$

The corresponding completion A' will carry an induced circle action $\{\sigma'_w\}_{w\in\mathbb{S}^1}$.

Theorem 4.2.6 ([5, Theorem 3.10])**.** *Suppose that $\|a\| = \|a\|'$ for all $a \in \mathcal{A}_0$. Then $\{\sigma_w\}_{w\in\mathbb{S}^1}$ has large spectral subspaces if and only if $\{\sigma'_w\}_{w\in\mathbb{S}^1}$ has large spectral subspaces. In this case, the identity map $\mathcal{A} \to \mathcal{A}$ induces an isomorphism $A \to A'$ of C^*-algebras. In particular, we have that $\|a\| = \|a\|'$ for all $a \in \mathcal{A}$.*

Proof. Remark first that the identity map $\mathcal{A}_k \to \mathcal{A}_k$ induces an isometric isomorphism of Hilbert C^*-modules $A_k \to A'_k$ for all $k \in \mathbb{Z}$. This is a consequence of the identity $\|a\| = \|a\|'$ for all $a \in \mathcal{A}_0$. But then we also have isomorphisms

$$A_1^{(k)} \simeq A_1'^{(k)} \quad \text{and} \quad A_{-1}^{(k)} \simeq A_{-1}'^{(k)} \quad \forall\, k \in \mathbb{N},$$

with the above modules defined as in (3.1.1), for the right action of A_0 given by multiplication. These observations imply that $\{\sigma_w\}_{w\in\mathbb{S}^1}$ has large spectral subspaces if and only if $\{\sigma'_w\}_{w\in\mathbb{S}^1}$ has large spectral subspaces. But it then follows from Theorem 3.2.5 that

$$A \simeq \mathcal{O}_{A_1} \simeq \mathcal{O}_{A'_1} \simeq A'\,,$$

with corresponding isomorphism $A \simeq A'$ induced by the identity map $\mathcal{A} \to \mathcal{A}$. □

The previous result can be seen as a manifestation of the gauge-invariant uniqueness theorem, [51, Theorem 6.2 and Theorem 6.4]. This property was indirectly used already in [67, Theorem 3.12] for the proof of the universal properties of Pimsner algebras.

4.3 Twistings of graded algebras

A rich class of examples comes from starting from the $\mathbb{Z}$-graded algebra of a given principal circle bundle and twisting the product in the algebra by a given automorphism.

Let $\mathcal{A} = \oplus_{k\in\mathbb{Z}}\mathcal{A}_k$ be a $\mathbb{Z}$-graded unital $*$-algebra. In the examples of Subsection 4.3.1 the algebra $\mathcal{A}$ will be the commutative algebra of coordinates on a classical principal circle bundle, but this construction works for any unital $*$-algebra $\mathcal{A}$, commutative or not.

Definition 4.3.1. Let γ be a graded unital $*$-automorphism of $\mathcal{A}$. We define a new unital graded $*$-algebra $(\mathcal{A}, \star_\gamma) =: \mathcal{B} = \bigoplus_{k\in\mathbb{Z}}\mathcal{B}_n$ as follows: for all k we set $\mathcal{B}_k := \mathcal{A}_k$ as vector spaces (but not as modules), the involution is unchanged, and the product on $\mathcal{B}$ is defined by:

$$a \star_\gamma b = \gamma^k(a)\gamma^{-l}(b) , \qquad \text{for all } \ a \in \mathcal{B}_l,\ b \in \mathcal{B}_k, \tag{4.3.1}$$

where the product on the right hand side is the one in $\mathcal{A}$.

It is indeed straightforward to check that the new product satisfies

i) associativity: for all $a \in \mathcal{A}_k$, $b \in \mathcal{A}_l$, $c \in \mathcal{A}_m$ it holds that

$$(a \star_\gamma b) \star_\gamma c = a \star_\gamma (b \star_\gamma c) = \gamma^{l+m}(a)\gamma^{m-k}(b)\gamma^{-k-l}(c),$$

ii) $(a \star_\gamma b)^* = b^* \star_\gamma a^*$, for all a, b.

Furthermore, the unit is preserved, that is: $1 \star_\gamma a = a \star_\gamma 1 = a$ for all a, and the degree zero subalgebra has undeformed product: $\mathcal{B}_0 = \mathcal{A}_0$. Finally,

$$a \star_\gamma \xi = \gamma^k(a)\xi , \qquad \xi \star_\gamma a = \xi\gamma^{-k}(a) , \qquad \text{for all } a \in \mathcal{B}_0,\ \xi \in \mathcal{B}_k.$$

Thus the left $\mathcal{B}_0$-module structure of $\mathcal{B}_k$ is the one of $\mathcal{A}_k$ twisted with γ^k, and the right $\mathcal{B}_0$-module structure is the one of $\mathcal{A}_k$ twisted with γ^{-k}.

We write this as $\mathcal{B}_n = {}_{\gamma^k}(\mathcal{A}_k)_{\gamma^{-k}}$.

Remark 4.3.2. For the particular case when $\mathcal{A}$ is commutative, from the deformed product (4.3.1) one gets commutation rules:

$$a \star_\gamma b = \gamma^{-2l}(b) \star_\gamma \gamma^{2k}(a) , \tag{4.3.2}$$

for all $a \in \mathcal{B}_l$, $b \in \mathcal{B}_k$.

Theorem 4.3.3. *Assume the datum $\left(\mathcal{A}, \mathcal{O}(U(1)), \mathcal{A}_0\right)$ is a noncommutative principal circle bundle. Then, the datum $\left(\mathcal{B}, \mathcal{O}(U(1)), \mathcal{A}_0\right)$ is a noncommutative principal circle bundle as well.*

Proof. With the notation of Theorem 4.1.4, denoting $\alpha_i^\gamma = \gamma^{-1}(\alpha_i)$, $\beta_i^\gamma = \gamma^{-1}(\beta_i)$, $\xi_i^\gamma = \gamma(\xi_i)$ and $\eta_i^\gamma = \gamma(\eta_i)$, the collections $\{\xi_i^\gamma\}_{i=1}^N, \{\beta_i^\gamma\}_{i=1}^M$ in B_1 and $\{\eta_i^\gamma\}_{i=1}^N, \{\alpha_i^\gamma\}_{i=1}^M$

in $\mathcal{B}_{-1}$ are such that:

$$\sum_{i=1}^{N} \xi_i^\gamma \star_\gamma \eta_i^\gamma = \sum_{i=1}^{N} \xi_i \eta_i = 1 \ , \quad \sum_{i=1}^{M} \alpha_i^\gamma \star_\gamma \beta_i^\gamma = \sum_{i=1}^{M} \alpha_i \beta_i = 1 \ .$$

Hence the thesis, when applying Theorem 4.1.4. □

Remark 4.3.4. There is an isomorphism of bimodules ${}_{\gamma^k}(\mathcal{A}_k)_{\gamma^{-k}} \simeq {}_{\gamma^{2k}}(\mathcal{A}_k)_{\mathrm{Id}}$, implemented by the map $a \mapsto \gamma^k(a)$, for $a \in \mathcal{A}_k$. This map intertwines the deformed product $\star_\gamma$ with a new product

$$a \star'_\gamma b = \gamma^{2k}(a)b \ , \qquad \text{for all } a \in \mathcal{B}_l,\ b \in \mathcal{B}_k,$$

and the undeformed involution with a new involution:

$$a^\dagger = \gamma^{-2k}(a^*), \qquad \text{for all } a \in \mathcal{B}_k.$$

By construction $(\mathcal{A}, \star_\gamma)$ is isomorphic to $(\mathcal{A}, \star'_\gamma)$ with deformed involution.

Suppose that $\mathcal{A}$ is dense in a graded C^*–algebra A and that the action γ extends to a C^*-automorphism. Then the completion $\mathcal{E}_k$ of $\mathcal{A}_k$ becomes a self-Morita equivalence A_0-bimodule in the sense of Definition 2.3.1 (with $\phi = \gamma^{2k}$), and the completion of $\mathcal{B}$ is the Pimsner algebra over A_0 associated to the C^*-correspondence $(\mathcal{E}_1, \gamma^2)$.

The above example shows some similarity with the construction described in Example 3.4. To make this analogy more precise, let us assume the algebra $\mathcal{A}$ (and hence $\mathcal{A}_0$) to be commutative; what we are doing here is taking the commutative line bundle $\mathcal{E}_1$, the completion of $\mathcal{A}_1$, which is an element in the classical Picard group $\mathrm{CPic}(A_0)$, and twisting it with an automorphism of the base space algebra $\mathcal{A}_0$ to obtain the C^*-correspondence $(\mathcal{E}_1, \gamma^2)$, which is an element of $\mathrm{Pic}(A_0)$. This is consistent with the description of the Picard group of a commutative algebra that we gave in Theorem 2.3.3.

4.3.1 Examples

Examples of the above construction are the irrational rotation algebra introduced by Rieffel in [73] and isospectral deformations in the sense of [22], in particular θ-deformed spheres and lens spaces.

The irrational rotation algebra

Being a crossed product, the irrational rotation algebra $C(\mathbb{T}^2_\theta) \simeq C(\mathbb{S}^1) \rtimes_\alpha \mathbb{Z}$ can be naturally seen as a Pimsner algebra over $C(\mathbb{S}^1)$. The automorphism α of $C(\mathbb{S}^1)$ is the one induced by the $\mathbb{Z}$-action generated by a rotation by $2\pi i\theta$ on $\mathbb{S}^1$. We will now show how the coordinate algebra of the noncommutative torus emerges from the deformed construction outlined above.

Let $\mathcal{A} = \mathcal{A}(\mathbb{T}^2)$ be the commutative unital $*$-algebra generated by two unitary elements u and v. This is graded by assigning to u, v degree $+1$ and to their adjoints degree -1. The degree zero part is $\mathcal{A}_0 \simeq \mathcal{A}(\mathbb{S}^1)$, generated by the unitary $u^* v$. Let $\theta \in \mathbb{R}$ and γ be the graded $*$-automorphism given by

$$\gamma_\theta(u) = e^{2\pi i\theta} u \ , \qquad \gamma_\theta(v) = v \ .$$

From (4.3.2) we get

$$u \star_{\gamma_\theta} v = e^{2\pi i\theta} v \star_{\gamma_\theta} u \ ,$$

plus the relations

$$u \star_{\gamma_\theta} u^* = u^* \star_{\gamma_\theta} u = 1 \ , \qquad v \star_{\gamma_\theta} v^* = v^* \star_{\gamma_\theta} v = 1 \ .$$

Thus the deformed algebra $\mathcal{B} := (\mathcal{A}, \star_{\gamma_\theta}) = \mathcal{A}(\mathbb{T}^2_\theta)$ is the coordinate algebra of the noncommutative torus.

The corresponding C^*-completion $B \simeq A_\theta$, the irrational rotation algebra of Example 1.5, which can be seen as a Pimsner algebra over $C(\mathbb{S}^1)$ for the self-Morita equivalence bimodule $(\mathcal{E}_1, \gamma^2)$, as described at the end of Remark 4.3.4.

$\boldsymbol{\theta}$-deformed spheres and lens spaces

Let $\mathcal{A} = \mathcal{A}(\mathbb{S}^{2n+1})$ be the commutative unital $*$-algebra generated by elements $z_0, \dots, z_n$ and their adjoints, with relation

$$\sum_{i=0}^{n} z_i^* z_i = 1.$$

This is graded by assigning to $z_0, \dots, z_n$ degree $+1$ and to their adjoints degree -1. For this grading the degree zero part is $\mathcal{A}_0 \simeq \mathcal{A}(\mathbb{CP}^n)$.

Any matrix $\mathbf{u} = \{\mathbf{u}_{ij}\} \in U(n+1)$ defines a graded $*$-automorphism γ by

$$\gamma_{\mathbf{u}}(z_i) = \sum\nolimits_{j=0}^{n} \mathbf{u}_{ij} z_j \ , \qquad i = 0, \dots, n.$$

Since a unitary matrix can be diagonalized by a unitary transformation, one can assume that $\mathbf{u}$ is diagonal. If one defines $\lambda_{ij} := \mathbf{u}_{ii}^2 \bar{\mathbf{u}}_{jj}^2$, then from (4.3.2) one gets

$$z_i \star_{\gamma_{\mathbf{u}}} z_j = \lambda_{ij} \, z_j \star_{\gamma_{\mathbf{u}}} z_i \ , \qquad z_i \star_{\gamma_{\mathbf{u}}} z_j^* = \bar{\lambda}_{ij} \, z_j^* \star_{\gamma_{\mathbf{u}}} z_i \ , \qquad \text{for all } i, j,$$

(and each z_i is normal for the deformed product, since $\lambda_{ii} = 1$), together with the conjugated relations, and a spherical relation

$$\sum\nolimits_{i=0}^{n} z_i^* \star_\gamma z_i = 1 \ .$$

To use a more common notation, consider the matrix $\mathbf{\Theta} = \{\theta_{ij}\}$ defined by $\lambda_{ij} = e^{2\pi i \theta_{ij}}$. It is real (since we have $\lambda_{ij}\bar{\lambda}_{ij} = 1$), and antisymmetric (since $\bar{\lambda}_{ij} = \lambda_{ji}$). We shall then denote by $\mathcal{A}(\mathbb{S}^{2n+1}_{\mathbf{\Theta}})$ the algebra $\mathcal{A}(\mathbb{S}^{2n+1})$ with deformed product $\star_\gamma$.

Proposition 4.3.5. *The datum* $\Big(\mathcal{A}(\mathbb{S}^{2n+1}_{\mathbf{\Theta}}), \mathcal{O}(U(1)), \mathcal{A}(\mathbb{CP}^n)\Big)$ *is a noncommutative principal circle bundle.*

With the same decomposition and notation as in (4.1.8), for any natural number d, consider the algebra

$$\mathcal{A}(L^{2n+1}_{\mathbf{\Theta}}(d;\underline{1})) := \mathcal{A}(\mathbb{S}^{2n+1}_{\mathbf{\Theta}})^{\mathbb{Z}_d} = \oplus_{n\in\mathbb{Z}}\Big(\mathcal{A}(\mathbb{S}^{2n+1}_{\mathbf{\Theta}})\Big)_{dn},$$

which we think of as the coordinate algebra of the $\mathbf{\Theta}$-deformed lens spaces. Clearly, for $d = 1$ we get back the algebra $\mathcal{A}(\mathbb{S}^{2n+1}_{\mathbf{\Theta}})$.

Proposition 4.3.6. *The datum* $\Big(\mathcal{A}(L^{2n+1}_{\mathbf{\Theta}}(d;\underline{1})), \mathcal{O}(U(1)), \mathcal{A}(\mathbb{CP}^n)\Big)$ *is a noncommutative principal circle bundle.*

Let use denote by $C(\mathbb{CP}^n)$, $C(\mathbb{S}^{2n+1}_{\mathbf{\Theta}})$ and $C(L^{2n+1}_{\mathbf{\Theta}}(d;\underline{1}))$ the universal enveloping C^*-algebras for each of the coordinate algebra and by $\mathcal{E}_1$ the completion of the module $\mathcal{B}_1$. Since the circle action extended to $C(\mathbb{S}^{2n+1}_{\mathbf{\Theta}})$ has large spectral subspaces, the d-th spectral subspace $\mathcal{E}_d$ agrees with $\mathcal{E}_1^{(d)}$.

Proposition 4.3.7. *For all integers $d \geq 1$, the C^*-algebra $C(L^{2n+1}_{\mathbf{\Theta}}(d;\underline{1}))$ is a Pimsner algebra over $C(\mathbb{CP}^n)$ for the Hilbert bimodule $\mathcal{E}_d$.*

4.4 Quantum weighted projective and lens spaces

We describe here another class of examples of noncommutative principal circle bundles that can be realized as Pimsner algebras. These belong to the so-called *q-deformations* and they come from quantum groups.

In Sections 5.1 and 6.1 we will give a detailed descriptions of the two classes of algebras appearing in the recent works [3, 5], both at the coordinate and as the C^*-level.

4.4.1 The algebras

The quantum spheres $\mathbb{S}^{2n+1}_q$ were defined in [85] as homogeneous spaces for the quantum group $SU_q(n)$. We recall their definition here.

Definition 4.4.1. Let $0 < q < 1$. The coordinate algebra of the unit quantum sphere $\mathbb{S}^{2n+1}_q$ is the unital $*$-algebra $\mathcal{A}(\mathbb{S}^{2n+1}_q)$ generated by $2n + 2$ elements $\{z_i, z_i^*\}_{i=0,\dots,n}$ subject to the relations:

$$z_i z_j = q^{-1} z_j z_i \qquad 0 \leq i < j \leq n \;, \tag{4.4.1}$$
$$z_i^* z_j = q z_j z_i^* \qquad i \neq j \;, \tag{4.4.2}$$
$$[z_n^*, z_n] = 0, \qquad [z_i^*, z_i] = (1 - q^2) \sum_{j=i+1}^{n} z_j z_j^* \qquad i = 0, \ldots, n-1 \;, \tag{4.4.3}$$

together with the sphere relation:

$$z_0 z_0^* + z_1 z_1^* + \ldots + z_n z_n^* = 1 \;. \tag{4.4.4}$$

The original notation of [85] is obtained by setting $q = e^{h/2}$.

Irreducible representation for the quantum spheres $\mathbb{S}_q^{2n+1}$ were constructed in [43]. If one denotes by $|p_1, \ldots, p_n\rangle$ the standard orthonormal basis of $\ell^2(\mathbb{N}^n)$, one has a faithful $*$-representation $\pi : \mathcal{A}(\mathbb{S}_q^{2n+1}) \to \mathcal{B}(\ell^2(\mathbb{N}^n))$ given by

$$\begin{aligned} \pi(z_i)|p_1, \ldots, p_n\rangle &= q^{p_1 + \cdots + p_{i-1}} (1 - q^{2(p_1+1)})^{1/2} |p_1, \ldots, p_i + 1, \ldots, p_n\rangle, \quad 0 \leq i < n \\ \pi(z_n)|p_1, \ldots, p_n\rangle &= q^{p_1 + \cdots + p_n} |p_1, \ldots p_n\rangle. \end{aligned} \tag{4.4.5}$$

One can define a weighted circle action on the sphere algebras $\mathcal{A}(\mathbb{S}_q^{2n+1})$. A *weight vector* $\underline{\ell} = (\ell_0, \ldots, \ell_n)$ is a finite sequence of positive integers, called *weights*. A weight vector is said to be *coprime* if $\gcd(\ell_0, \ldots, \ell_n) = 1$; and it is *pairwise coprime* if $\gcd(\ell_i, \ell_j) = 1$, for all $i \neq j$.

For any coprime weight vector $\underline{\ell} = (\ell_0, \ldots, \ell_n)$, we define an action of the circle $\mathbb{S}^1$ on the algebra $\mathcal{A}(\mathbb{S}_q^{2n+1})$ by setting

$$\sigma_w^{\underline{\ell}} : (z_0, z_1, \ldots, z_n) \mapsto (w^{\ell_0} z_0, w^{\ell_1} z_1, \ldots, w^{\ell_n} z_n), \qquad w \in \mathbb{S}^1. \tag{4.4.6}$$

Definition 4.4.2. The coordinate algebra of the *quantum n-dimensional weighted projective space* associated with the weight vector $\underline{\ell}$ is the fixed point algebra for the action (4.4.6), and it is denoted by $\mathcal{A}(\mathbb{WP}_q^n(\underline{\ell}))$.

Equivalently, the circle action induces a $\mathbb{Z}$-grading on the coordinate algebra $\mathcal{A}(\mathbb{S}_q^{2n+1})$ given by

$$\mathcal{A}_k \simeq \{a \in \mathcal{A}(\mathbb{S}_q^{2n+1}) \mid \sigma_w^{\underline{\ell}}(a) = a w^{-k} \quad \forall w \in \mathbb{S}^1\}, \tag{4.4.7}$$

which is equivalent to that obtained by declaring each z_i to be of degree ℓ_i and each z_i^* of degree $-\ell_i$. The algebra $\mathcal{A}(\mathbb{WP}_q^n(\underline{\ell}))$ is the degree zero part of $\mathcal{A}(\mathbb{S}_q^{2n+1})$ with respect to the above grading.

Example 4.1. For $\underline{\ell} = (1, \ldots, 1)$ one gets the coordinate algebra $\mathcal{A}(\mathbb{CP}_q^n)$ of the unweighted quantum projective space $\mathbb{CP}_q^n$. This is the $*$-subalgebra of $\mathcal{A}(\mathbb{S}_q^{2n+1})$ generated by the elements $p_{ij} := z_i^* z_j$ for $i, j = 0, 1, \ldots, n$. We will describe this algebra in more detail in Section 5.1.

Definition 4.4.3. For a fixed positive integer N, one defines the coordinate algebra of the quantum lens space $\mathcal{A}(L_q^{2n+1}(N; \underline{\ell}))$ as

$$\mathcal{A}(L_q^{2n+1}(N;\underline{\ell})) := \mathcal{A}(\mathbb{S}_q^{2n+1})^{\mathbb{Z}_N}, \tag{4.4.8}$$

with the same decomposition and notation as in (4.1.8). Equivalently, $\mathcal{A}(L_q^{2n+1}(N;\underline{\ell}))$ is the invariant subalgebra of $\mathcal{A}(\mathbb{S}_q^{2n+1})$ with respect to the cyclic action obtained by restricting (4.4.6) to the finite cyclic group $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z} \subseteq \mathbb{S}^1$.

The following result was already present in [30], although it was not stated in full generality. Partial results in this direction can be found in [3, 5].

Proposition 4.4.4 (*cf.* [13, Proposition 4.2])**.** *For all weight vectors $\underline{\ell}$, denote by $N_{\underline{\ell}} := \prod_i \ell_i$. The triple $\big(\mathcal{A}(L_q^{2n+1}(N_{\underline{\ell}};\underline{\ell}), \mathcal{O}(U(1)), \mathcal{A}(\mathbb{WP}_q^n(\underline{\ell}))\big)$ is a noncommutative principal circle bundle.*

Proof. Along the lines of Theorem 4.1.4 one shows that there exist finite sequences of elements satisfying (4.1.4). This fact relies on the proof of [30, Proposition 7.1] were one can find polynomials A_i and B_i in $\mathcal{A}(\mathbb{WP}_q^n(\underline{\ell}))$, for $i = 0, \ldots, n$, with the property that

$$\sum_{i=1}^n A_i z_i^{\ell_i} z_i^{*\ell_i} = 1 = \sum_{i=1}^n B_i z_i^{*\ell_i} z_i^{\ell_i}.$$

The statement follows from Theorem 4.1.4, after observing that the $z_i^{\ell_i}$ and hence the $A_i z_i^{\ell_i}$ have degree $+1$, while the z_i^{ℓ} and hence the $B_i z_i^{*\ell_i}$ have degree -1. ☐

Example 4.2. A class of examples of the above construction is that of multidimensional quantum teardrops studied in [13], that are obtained for the weight vector $\underline{\ell} = (1, \ldots, 1, m)$ having all but the last entry equal to 1.

Fix an integer $d \geq 1$. From Proposition 4.1.10 applied to the lens space algebra $\mathcal{A}(L_q^{2n+1}(dN_{\underline{\ell}};\underline{\ell})) := A(L_q^{2n+1}(N_{\underline{\ell}};\underline{\ell}))^{\mathbb{Z}_d}$ one gets the following:

Proposition 4.4.5. *The triple $\Big(\mathcal{A}(L_q^{2n+1}(dN_{\underline{\ell}};\underline{\ell}))\,,\, \mathcal{O}(U(1))\,,\, \mathcal{A}(\mathbb{WP}_q^n(\underline{\ell}))\Big)$ is a noncommutative principal circle bundle for all integers $d \geq 1$.*

For $\underline{\ell} = (1, \ldots, 1)$ one gets the unweighted quantum lens spaces $\mathcal{A}(L_q^{2n+1}(d;\underline{1}))$, that will be denoted by $\mathcal{A}(L_q^{2n+1}(d))$ in order to lighten notation.

4.4.2 C^*-completions and Pimsner algebras

Let $C(\mathbb{S}_q^{2n+1})$ be the C^*-completion of $\mathcal{A}(\mathbb{S}_q^{2n+1})$ in the universal C^*-norm. By universality, one extends the weighted circle action to the C^*-algebra and defines $C(\mathbb{WP}_q^n(\underline{\ell}))$ and $C(L_q^{2n+1}(N;\underline{\ell}))$ as the fixed point C^*-subalgebras of $C(\mathbb{S}_q^{2n+1})$ for the circle action and for the action of the subgroup $\mathbb{Z}_N \subset \mathbb{S}^1$, respectively.

At least for $n = 1$, the fixed point C^*-subalgebras coincide with the corresponding universal enveloping C^*-algebras, as we will describe in Subsection 6.1.3.

Let us now denote with B the algebra $C(\mathbb{WP}_q^n(\underline{\ell}))$ and with $\mathcal{E}$ the first spectral subspace of $C(\mathbb{S}_q^{2n+1})$ for the weighted action of $\mathbb{S}^1$, and let $\phi : B \to \mathcal{L}_B(\mathcal{E})$ be the

*-homomorphism obtained by left multiplication. The symbol $\mathcal{E}^{(d)}$ will denote the d-th interior tensor power of $\mathcal{E}$ with itself over the map ϕ, as in (3.1.1).

Proposition 4.4.6. *Let $d \geq 1$. There is an isomorphism of C^*-algebras*

$$\mathcal{O}_{\mathcal{E}^{(d)}} \simeq C(L_q^{2n+1}(dN_{\underline{\ell}}; \underline{\ell})).$$

Proof. The result is a direct consequence of Theorem 3.2.5, once one proves that $\mathcal{E}$ is full. But this follows at once from Proposition 4.4.5, since the modules are finitely generated projective. ☐

Particular examples that are of interest for the present work are unweighted quantum lens spaces of any (odd) dimension and weighted lens spaces over teardrops (n=1). In the next two sections we will describe these two classes of examples in detail, focusing on the bundle structures, on the resulting Pimsner algebras obtained as C^*-algebraic completions and on the corresponding exact sequences.

Part III

Gysin sequences for quantum lens spaces

Chapter 5

A Gysin sequence in K-theory for quantum lens spaces

This chapter is based on the material contained in the paper [3]. There we constructed a Gysin sequence in K-theory that allows one to compute the K-groups of quantum lens spaces, as kernel and cokernel of a suitable $\mathbb{Z}$-module map, as described in (3.3.6). This work was later put in the more general framework of Pimsner algebras, using the approach described in the previous chapters.

Here we will obtain a Gysin sequence in K-theory as a particular case of the six-term exact sequence naturally associated to Pimsner algebras.

We will explicitely describe the maps appearing in the exact sequence in terms of representatives of K-theory classes for the quantum projective space $C(\mathbb{CP}^n_q)$. This explicit description will allow us to construct, via pull-back, representatives of the K-theory of the quantum lens space $C(L_q^{2n+1}(d))$.

5.1 Quantum lens and projective spaces

5.1.1 Complex projective spaces

The coordinate algebra $\mathcal{A}(\mathbb{S}_q^{2n+1})$ of the quantum sphere is defined as the $*$-algebra generated by $2n+2$ elements $\{z_i, z_i^*\}_{i=0,\ldots,n}$ subject to the relations (4.4.1-4.4.4).

The unweighted circle action on $\mathcal{A}(\mathbb{S}_q^{2n+1})$, obtained by setting $\underline{\ell} = (1, \ldots, 1)$ in (4.4.6) is given on generators by

$$\sigma_w : (z_0, z_1, \ldots, z_n) \mapsto (wz_0, wz_1, \ldots, wz_n), \qquad w \in \mathbb{S}^1. \tag{5.1.1}$$

To the best of our knowledge, the coordinate algebra $\mathcal{A}(\mathbb{CP}^n_q)$ of the quantum projective space $\mathbb{CP}^n_q$ first appeared in [89]. It is defined as the fixed point algebra $\mathcal{A}(\mathbb{CP}^n_q) \subset \mathcal{A}(\mathbb{S}_q^{2n+1})$ with respect to the circle action (5.1.1).

The algebra $\mathcal{A}(\mathbb{CP}^n_q)$ admits the following elegant characterization:

Theorem 5.1.1. *1. The algebraic quantum projective space $\mathcal{A}(\mathbb{CP}^n_q)$ agrees with the unital *-subalgebra of $\mathcal{A}(\mathbb{S}^{2n+1}_q)$ generated by the elements $p_{ij} := z_i^* z_j$ for $i, j = 1, \ldots, n$.*

2. The elements p_{ij} satisfy

$$\begin{aligned}
p_{ij}p_{kl} &= q^{\mathrm{sign}(k-i)+\mathrm{sign}(j-l)}\, p_{kl}p_{ij} && \text{if } i \neq l \text{ and } j \neq k\ , \\
p_{ij}p_{jk} &= q^{\mathrm{sign}(j-i)+\mathrm{sign}(j-k)+1}\, p_{jk}p_{ij} - (1-q^2)\textstyle\sum_{l>j} p_{il}p_{lk} && \text{if } i \neq k\ , \\
p_{ij}p_{ji} &= q^{2\mathrm{sign}(j-i)} p_{ji}p_{ij} + (1-q^2)\left(\textstyle\sum_{l>i} q^{2\mathrm{sign}(j-i)} p_{jl}p_{lj} - \sum_{l>j} p_{il}p_{li}\right) && \text{if } i \neq j\ ,
\end{aligned} \tag{5.1.2}$$

with $\mathrm{sign}(0) := 0$.

*3. $\mathcal{A}(\mathbb{CP}^n_q)$ agrees with the universal unital *-algebra with generators p_{ij} subject to the relations* (5.1.2)*.*

Equivalently, the circle action induces a $\mathbb{Z}$ grading on $\mathcal{A}(\mathbb{S}^{2n+1}_q)$, given by the vector space decomposition

$$\mathcal{A}(\mathbb{S}^{2n+1}_q) = \bigoplus_{k\in\mathbb{Z}} \mathcal{A}_k,$$

where the degree k-part is the module of equivariant sections

$$\mathcal{A}_k \simeq \{a \in \mathcal{A}(\mathbb{S}^{2n+1}_q) \mid \sigma_w(a) = aw^{-k} \quad \forall w \in \mathbb{S}^1\}. \tag{5.1.3}$$

The *-algebra $\mathcal{A}(\mathbb{CP}^n_q)$ can be realized as the degree zero part of $\mathcal{A}(\mathbb{S}^{2n+1}_q)$ with respect to the above grading.

The modules $\mathcal{A}_k$ are projective of finite type. In order to see this, one needs to recall some notation. The q-*analogue* of an integer $n \in \mathbb{Z}$ is given by

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}}\ ;$$

it is defined for $q \neq 1$ and is equal to n in the limit $q \to 1$. For any $n \geq 0$, one defines the factorial of the q-number $[n]$ by setting $[0]! := 1$ and then $[n]! := [n][n-1]\cdots[1]$. The q-multinomial coefficients are in turn defined by

$$[j_0, \ldots, j_n]! := \frac{[j_0 + \ldots + j_n]!}{[j_0]! \ldots [j_n]!}\ .$$

For $k \in \mathbb{Z}$, we define $\Psi_k := (\psi^k_{j_0,\ldots,j_n})$ to be the vector-valued function on $\mathbb{S}^{2n+1}_q$ with components

$$\psi^k_{j_0,\ldots,j_n} := \begin{cases} [j_0, \ldots, j_n]!^{\frac{1}{2}} q^{-\frac{1}{2}\sum_{r<s} j_r j_s} (z_0^{j_0})^* \ldots (z_n^{j_n})^* & \text{for} \quad k \geq 0\,, \\[2ex] [j_0, \ldots, j_n]!^{\frac{1}{2}} q^{\frac{1}{2}\sum_{r<s} j_r j_s + \sum_{r=0}^n r j_r} z_0^{j_0} \ldots z_n^{j_n} & \text{for} \quad k \leq 0\,, \end{cases} \tag{5.1.4}$$

with $j_0 + \ldots + j_n = |k|$. Then $\Psi_k^* \Psi_k = 1$ and the range projection lives in a matrix algebra of a certain size:

$$\mathbf{p}_k := \Psi_k \Psi_k^* \in \mathrm{M}_{\mathrm{d}_k}(\mathcal{A}(\mathbb{CP}^n_q)), \qquad \mathrm{d}_k := \binom{|k|+n}{n}. \tag{5.1.5}$$

This fact was proven in [27], generalizing the special case where $n = 2$ of [28]. By construction, the entries of the matrix $\mathbf{p}_k$ are $\mathbb{S}^1$-invariant and so they are indeed elements of the algebra $\mathcal{A}(\mathbb{CP}^n_q)$.

To each projection $\mathbf{p}_k$ there corresponds a module of sections of a line bundle, as we will now describe.

The column vector Ψ_k has d_k entries, all of which are elements of $\mathcal{A}(\mathbb{S}^{2n+1}_q)$. We consider the collection

$$\mathcal{A}_k := \left\{ \varphi_k := v \cdot \Psi_k = \sum_{j_0+\ldots+j_n=k} v_{j_0,\ldots,j_n} \, \psi^k_{j_0,\ldots,j_n} \right\}, \tag{5.1.6}$$

where $v = (v_{j_0,\ldots,j_n}) \in (\mathcal{A}(\mathbb{CP}^n_q))^{\mathrm{d}_k}$.

Each $\mathcal{A}_k$ obtained in this way turns out to be isomorphic to the module of equivariant sections described in (5.1.3), whence the abuse of notation.

An argument as in [29, Prop. 3.3] yields the following:

Proposition 5.1.2. *There are left $\mathcal{A}(\mathbb{CP}^n_q)$-module isomorphisms*

$$\mathcal{A}_k \simeq (\mathcal{A}(\mathbb{CP}^n_q))^{\mathrm{d}_k} \mathbf{p}_k$$

and right $\mathcal{A}(\mathbb{CP}^n_q)$-module isomorphisms

$$\mathcal{A}_k \simeq \mathbf{p}_{-k} (\mathcal{A}(\mathbb{CP}^n_q))^{\mathrm{d}_k}.$$

Since $\mathcal{A}_k$ is a bimodule, we have to make a choice: we always use the right $\mathcal{A}(\mathbb{CP}^n_q)$-module identification, in order to be consistent with the convention used in Part I. By Proposition 4.4.4 the algebra $\mathcal{A}(\mathbb{S}^{2n+1}_q)$ is strongly graded, hence a quantum principal bundle.

5.1.2 Quantum lens spaces

Recall from Definition 4.4.3 that the coordinate algebra of the quantum lens space $L^{2n+1}_q(d)$ is constructed, as a vector space, in terms of the graded parts of $\mathcal{A}(\mathbb{S}^{2n+1}_q)$:

$$\mathcal{A}(L^{2n+1}_q(d)) := \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_{dk} \,. \tag{5.1.7}$$

By Proposition 4.4.5, the coordinate algebra of the quantum lens space is also strongly graded, hence the inclusion $\mathcal{A}(\mathbb{CP}_q^n) \hookrightarrow \mathcal{A}(L_q^{2n+1}(d))$ is a quantum principal bundle as well.

The above definition is consistent with the classical construction of lens spaces as orbit spaces for an action of the finite cyclic group $\mathbb{Z}_d$ on odd-dimensional spheres. In particular, by Proposition 4.1.9, the algebra $\mathcal{A}(L_q^{2n+1}(d))$ is made of all elements of $\mathcal{A}(\mathbb{S}_q^{2n+1})$ which are invariant under the action of the cyclic group $\mathbb{Z}_d$. This was the content of [3, Proposition 4.1].

5.1.3 C*-completions and Pimsner algebra structure

The C^*-algebra $C(\mathbb{S}_q^{2n+1})$ of continuos function on the quantum sphere $\mathbb{S}_q^{2n+1}$ is defined as the universal enveloping C^*-algebra of the coordinate algebra $\mathcal{A}(\mathbb{S}_q^{2n+1})$. The algebra of continuous functions on the quantum projective space $\mathbb{CP}_q^n$ is then defined to be the C^*-subalgbera $C(\mathbb{CP}_q^n) \subset C(\mathbb{S}_q^{2n+1})$ of invariant elements with respect to the circle action on $C(\mathbb{S}_q^{2n+1})$ obtained by extending the action (5.1.1).

It was shown in [45, Subsection 4.3] that both algebras are actually isomorphic to the Cuntz-Krieger algebra of a graph. This fact can be used in particular for computing the K-theory and K-homology groups of the algebra $\mathbb{CP}_q^n$, as kernels and cokernels of certain $\mathbb{Z}$-module maps.

Theorem 5.1.3. *We have isomorphisms of groups*

$$K_0(C(\mathbb{CP}_q^n)) \simeq K^0(C(\mathbb{CP}_q^n)) \simeq \mathbb{Z}^{n+1} \quad \text{and} \quad K_1(C(\mathbb{CP}_q^n)) \simeq K^1(C(\mathbb{CP}_q^n)) \simeq 0. \tag{5.1.8}$$

Similarly, the C^*-algebra of continuous functions on the quantum lens space $C(L_q^{2n+1}(d))$ is the invariant C^*-subalgebra of $C(\mathbb{S}_q^{2n+1})$ with respect to the $\mathbb{Z}_d$ action obtained by restricting the $\mathbb{S}^1$ action. It was shown in [46, Section 2] that this algebra is isomorphic to the C^*-algebra of a graph.

Here we are interested in another feature of the algebra $C(L_q^{2n+1}(d))$, that of being a Pimsner algebra over the C^*-algebra of continuous functions over the quantum projective space, that we will denote by $B := C(\mathbb{CP}_q^n)$, in analogy with the notation of Chapter 3.

Furthermore, $\mathcal{E}$ will denote the Hilbert C^*-module over B obtained as the closure of $\mathcal{A}_{-1}$ in the C^*-norm of $C(\mathbb{S}_q^{2n+1})$; it is nothing but the (minus)-first spectral subspace for the circle action on $C(\mathbb{S}_q^{2n+1})$.

As usual, $\phi : B \to \mathcal{L}_B(\mathcal{E})$ will denote the *-homomorphism obtained by left multiplication. We denote with $\mathcal{E}^{(d)}$, the d-th interior tensor power of $\mathcal{E}$ with itself over the map ϕ, in analogy with (3.1.1).

Theorem 5.1.4. *For all $d \in \mathbb{N}$, there is an isomorphism of C^*-algebras*

$$\mathcal{O}_{\mathcal{E}^{(d)}} \simeq C(L_q^{2n+1}(d)).$$

We will now use this result to construct six term exact sequence in KK-theory, following the construction described in 3.3.

5.2 Gysin sequences

For each $d \in \mathbb{N}$, let $[\mathcal{E}^{(d)}] \in KK(C(\mathbb{CP}_q), C(\mathbb{CP}_q))$ denote the class of the Hilbert C^*-module $\mathcal{E}^{(d)}$ as above. Then, given any separable C^*-algebra C, by Theorem 3.3.4 we obtain two six term exact sequences:

$$\begin{array}{ccccc} KK_0(C, C(\mathbb{CP}^n_q)) & \xrightarrow{1-[\mathcal{E}^{(d)}]} & KK_0(C, C(\mathbb{CP}^n_q)) & \xrightarrow{j_*} & KK_0\Big(C, C(L^{2n+1}_q(d))\Big) \\ {\scriptstyle [\partial]}\uparrow & & & & \downarrow{\scriptstyle [\partial]} \\ KK_1(C, C(L^{2n+1}_q(d))) & \xleftarrow[j_*]{} & KK_1(C, C(\mathbb{CP}^n_q)) & \xleftarrow[1-[\mathcal{E}^{(d)}]]{} & KK_1(C, C(\mathbb{CP}^n_q)) \end{array} \tag{5.2.1}$$

and

$$\begin{array}{ccccc} KK_0(C(\mathbb{CP}^n_q), C) & \xleftarrow[1-[\mathcal{E}^{(d)}]]{} & KK_0(C(\mathbb{CP}^n_q), C) & \xleftarrow[j^*]{} & KK_0\Big(C(L^{2n+1}_q(d)), C\Big) \\ \downarrow{\scriptstyle [\partial]} & & & & {\scriptstyle [\partial]}\uparrow \\ KK_1\Big(C(L^{2n+1}_q(d)), C\Big) & \xrightarrow{j^*} & KK_1(C(\mathbb{CP}^n_q), C) & \xrightarrow{1-[\mathcal{E}^{(d)}]} & KK_1(C(\mathbb{CP}^n_q), C) \end{array} , \tag{5.2.2}$$

where j_* and j^* are the maps in KK-theory induced by $j : C(\mathbb{CP}^n_q) \hookrightarrow C(L^{2n+1}_q(d))$.

We will refer to these two sequences as the *Gysin sequences* (in KK-theory) for the quantum lens space $L^{2n+1}_q(d)$.

5.2.1 The Gysin sequence in K-theory

The above sequence (5.2.1) reduces, for $C = \mathbb{C}$, to a six term exact sequence in K-theory. We will now describe its features and the properties of the maps at play.

Fist of all we observe that, in virtue of (5.1.8), the sequence reduces to

$$0 \longrightarrow K_1(C(L^{2n+1}_q(d)) \xrightarrow{[\partial]} K_0(C(\mathbb{CP}^n_q)) \xrightarrow{1-[\mathcal{E}^{(d)}]} K_0(C(\mathbb{CP}^n_q)) \to \tag{5.2.3}$$

$$\xrightarrow{j_*} K_0(C(L^{2n+1}_q(d))) \longrightarrow 0\,,$$

This has the important consequence that, as described at the end of Section 3.3, we obtain a description of the K-groups of $C(\mathbb{CP}^n_q)$ as

$$K_0(C(L_q^{2n+1}(d))) \simeq \operatorname{coker}(1-[\mathcal{E}^{(d)}]), \qquad K_1(C(L_q^{2n+1}(d))) \simeq \ker(1-[\mathcal{E}^{(d)}]). \quad (5.2.4)$$

By fixing a basis of representatives of classes in $K_0(C(\mathbb{CP}_q^n))$, we will construct explicit representatives for K-theory classes of $C(L_q^{2n+1}(d))$.

Remark 5.2.1. The map $[\partial] : K_1(C(L_q^{2n+1}(d)) \to K_0(C(\mathbb{CP}_q^n))$ was given in [3] as the Kasparov product with an unbounded Kasparov module. The Kasparov module defined there agrees with the one described in (3.3.3).

By its very definition the operator $\mathcal{N}$ has a spectral gap around zero. Hence, by 2.4.11 the map $[\partial]$ is given explicitly on classes $[\mathbf{u}] \in K_1(C(L_q^{2n+1}(d))$ as an index

$$\langle [\mathbf{u}], [\partial] \rangle := [\operatorname{Ker} P\mathbf{u}P] - [\operatorname{Coker} P\mathbf{u}P] \in K_0(C(\mathbb{CP}_q^n)),$$

where P denotes the spectral projection for the self-adjoint operator $\mathcal{N}$ and associated to the positive Fock space defined in (3.3.2).

K-theory and K-homology of the quantum projective space

We will now describe explicit representatives for the K-theory of the quantum projective space, and their parings with K-homology classes, following [29].

We let $[\mathbf{p}_k]$ denote the class in $K_0(\mathcal{A}(\mathbb{CP}_q^n))$ of the projection $\mathbf{p}_k$ defined in (5.1.5). The inclusion $\mathcal{A}(\mathbb{CP}_q^n) \hookrightarrow C(\mathbb{CP}_q^n)$ induces an isomorphism of K-theory groups

$$K_0(C(\mathbb{CP}_q^n)) \simeq K_0(\mathcal{A}(\mathbb{CP}_q^n)). \quad (5.2.5)$$

This fact is proven by considering pairings with even Fredholm modules that are generators of the homology group $K^0(C(\mathbb{CP}_q^n))$. These are given explicitly in the same work ([29]) and have the form

$$\mathcal{F}_m = (\mathcal{A}(\mathbb{CP}_q^n),\, \mathcal{H}_{(m)},\, \pi^{(m)},\, \gamma_{(m)},\, F_{(m)}), \quad \text{for} \quad 0 \le m \le n\,. \quad (5.2.6)$$

We shall not dwell on the explicit description of the Fredholm modules here, since they are beyond the scope of the present work. However, we are interested in their pairings with K-theory, which were computed in Propositions 4 and 5 of [29], leading to the following result.

Proposition 5.2.2. *For all $k \in \mathbb{N}$ and for all $0 \le m \le n$ it holds that*

$$\langle [\mathcal{F}_m], [\mathbf{p}_{-k}] \rangle := \operatorname{Tr}_{\mathcal{H}_m}(\gamma_{(m)}(\pi^{(m)}(\operatorname{Tr} \mathbf{p}_{-k})) = \binom{k}{m}\,,$$

with $\binom{k}{m} := 0$ when $m > k$. Moreover, the elements $[\mathcal{F}_0], \dots, [\mathcal{F}_n]$ are generators of $K^0(C(\mathbb{CP}_q^n))$, and the elements $[\mathbf{p}_0], \dots, [\mathbf{p}_{-n}]$ are generators of $K_0(C(\mathbb{CP}_q^n))$.

Indeed, the matrix of couplings $\mathbf{m} \in \mathrm{M}_{n+1}(\mathbb{Z})$ with $\mathbf{m}_{ij} := \langle [\mathcal{F}_i], [\mathbf{p}_{-j}] \rangle = \binom{j}{i}$, for $i, j = 0, 1, \dots, n$, has inverse with integer entries $(\mathbf{m}^{-1})_{ij} = (-1)^{i+j}\binom{j}{i}$. Thus the aforementioned elements are a basis of $\mathbb{Z}^{n+1}$ as a $\mathbb{Z}$-module, which is equivalent to saying that they generate $\mathbb{Z}^{n+1}$ as an Abelian group.

This in particular implies that the projections $\mathbf{p}_k$ are not only generators of $K_0(\mathcal{A}(\mathbb{CP}^n_q))$ but also of $K_0(C(\mathbb{CP}^n_q))$, hence yielding the isomorphism (5.2.5).

Motivated from Proposition 5.1.2 we will denote the class of the projection $\mathbf{p}_k$ by the class $[\mathcal{A}_k]$ of the corresponding right-module $\mathcal{A}_k$, seen as an element of the group $K_0(C(\mathbb{CP}^n_q))$. For each $k \in \mathbb{Z}$ the module $\mathcal{A}_k$ describes a line bundle, in the sense that its *rank* (as computed by pairing with $[\mathcal{F}_0]$) is equal to 1. It is completely characterized by its *first Chern number* (as computed by pairing with the class $[\mathcal{F}_1]$). Indeed, using an argument similar to that of the proof of Proposition 5.2.2 one shows the following.

Proposition 5.2.3. *For all $k \in \mathbb{Z}$ it holds that*

$$\langle [\mathcal{F}_0], [\mathcal{A}_k] \rangle = 1 \quad \textit{and} \quad \langle [\mathcal{F}_1], [\mathcal{A}_k] \rangle = -k \,.$$

From the above discussion, it is clear why the line bundle $\mathcal{A}_{-1}$ emerges as a central character: its only non-vanishing charges are $\langle [\mathcal{F}_0], [\mathcal{A}_{-1}] \rangle = 1$ and $\langle [\mathcal{F}_1], [\mathcal{A}_{-1}] \rangle = 1$. It is the noncommutative module of sections of the *tautological line bundle* for the quantum projective space $\mathbb{CP}^n_q$.

Now consider the element in $K_0(C(\mathbb{CP}^n_q))$ given by

$$u := 1 - [\mathcal{A}_{-1}] \,, \tag{5.2.7}$$

of which we can take powers using the fact that the grading is strong. For $j \geq 0$, as elements in K-theory, one has then

$$u^j = (1 - [\mathcal{A}_{-1}])^j \simeq \sum_{k=0}^{j} (-1)^k \binom{j}{k} [\mathcal{A}_{-k}] \,. \tag{5.2.8}$$

Proposition 5.2.4. *For $0 \leq j \leq n$ and for $0 \leq m \leq n$, it holds that*

$$\langle [\mathcal{F}_m], u^j \rangle = \begin{cases} 0 & \textit{for} \quad j \neq m \\ (-1)^j & \textit{for} \quad j = m \end{cases} \,, \tag{5.2.9}$$

while for all $0 \leq m \leq n$ it holds that

$$\langle [\mathcal{F}_m], u^{n+1} \rangle = 0 \,. \tag{5.2.10}$$

Proof. Denoting as before by $[\mathcal{A}_{-k}]$ the class of the projection $\mathbf{p}_{-k}$ and setting $\binom{k}{m} := 0$ when $k > m$, we compute, using Proposition 5.2.2

$$\langle [\mathcal{F}_m], u^j \rangle = \sum_{k=0}^{j} (-1)^k \binom{j}{k} \langle [\mathcal{F}_m], [\mathcal{A}_{-k}] \rangle = \sum_{k=m}^{j} (-1)^k \binom{j}{k} \binom{k}{m} \,.$$

If $m > j$ this vanishes again due to $\binom{k}{m} := 0$ for $m > k$. If $m \leq j$, it is

$$\langle [\mathcal{F}_m], u^j \rangle = \frac{j!}{m!} \sum_{k=m}^{j} \frac{(-1)^k}{(j-k)!(k-m)!}$$

and a direct computation yields (5.2.9). Similarly, one computes that

$$\langle [\mathcal{F}_m], u^{n+1} \rangle = \frac{(n+1)!}{m!} \sum_{k=m}^{n+1} \frac{(-1)^k}{(n+1-k)!(k-m)!} = 0\,,$$

thus completing the proof. □

The element $u = \chi([\mathcal{A}_{-1}]) := 1 - [\mathcal{A}_{-1}]$ shall be named the *Euler class* of the line bundle $\mathcal{A}_{-1}$, in analogy with the classical case (*cf.* [49, IV.1.13]).

Since for $0 \leq m \leq n$ the elements $[\mathcal{F}_m]$ are generators of $K^0(C(\mathbb{CP}^n_q))$, the fact that $\langle [\mathcal{F}_m], u^{n+1} \rangle = 0$ for $0 \leq m \leq n$ amounts to saying that $u^{n+1} = 0$ in $K_0(C(\mathbb{CP}^n_q))$. On the other hand, since the elements $[\mathcal{A}_{-k}]$ for $0 \leq k \leq n$ are generators of $K_0(C(\mathbb{CP}^n_q))$, the results in (5.2.9) say that the elements $[\mathcal{F}_m]$ and $(-u)^j$ for $0 \leq m, j \leq n$ form dual bases. These two facts, combined with the isomorphism (5.2.5), lead to the following analogue of the classical result (*cf.* [49, Corollary IV.2.11]).

Proposition 5.2.5. *It holds that*

$$K_0(C(\mathbb{CP}^n_q)) \simeq \mathbb{Z}[\mathcal{A}_{-1}]/(1 - [\mathcal{A}_{-1}])^{n+1} \simeq \mathbb{Z}[u]/u^{n+1}$$

where $u = \chi([\mathcal{A}_{-1}]) := 1 - [\mathcal{A}_{-1}]$ *is the Euler class of the line bundle* $\mathcal{A}_{-1}$.

Pulling back line bundles

The K-theory map $j_* : K_0(C(\mathbb{CP}^n_q)) \to K_0(C(L^{2n+1}_q(d)))$ induced by the inclusion $j : C(\mathbb{CP}^n_q) \to C(L^{2n+1}_q(d))$ admits a simpler description at the level of coordinate algebras.

Definition 5.2.6. For each $\mathcal{A}(\mathbb{CP}^n_q)$-bimodule $\mathcal{A}_k$ as in (5.1.6) (a line bundle over $\mathbb{CP}^n_q$), its *pull-back* to $L^{2n+1}_q(d)$ is the $\mathcal{A}(L^{2n+1}_q(d))$-bimodule

$$j_*(\mathcal{A}_k) := \left\{ \widetilde{\varphi}_k = v \cdot \Psi_k = \sum_{j_0+\ldots+j_n=k} v_{j_0,\ldots,j_n}\, \psi^k_{j_0,\ldots,j_n} \right\}, \tag{5.2.11}$$

for $v = (v_{j_0,\ldots,j_n}) \in (\mathcal{A}(L^{(n,r)}_q))^{\mathrm{d}_k}$. We shall often use the shorthand $j_*(\mathcal{A}_k) := \widetilde{\mathcal{A}}_k$.

By embedding the cyclic group $\mathbb{Z}_d$ into $\mathbb{S}^1$ via the d-th roots of unity, each $\widetilde{\mathcal{A}}_k$ is made of elements of $\mathcal{A}(\mathbb{S}^{2n+1}_q)$ which transform as $\widetilde{\varphi}_k \mapsto \widetilde{\varphi}_k\, \mathrm{e}^{-2\pi i\, k/d}$ under the circle action. By its very definition, $\widetilde{\mathcal{A}}_k$ is an $\mathcal{A}(L^{2n+1}_q(d))$-bimodule. Once again, arguments like those of [29, Proposition 3.3] for the $\widetilde{\mathcal{A}}_k$ yield the following.

Proposition 5.2.7. *There are left* $\mathcal{A}(L^{2n+1}_q(d))$*-module isomorphisms*

$$\widetilde{\mathcal{A}}_k \simeq (\mathcal{A}(L^{2n+1}_q(d)))^{\mathrm{d}_k} \mathbf{p}_k$$

and right $\mathcal{A}(L_q^{(n,r)})$-module isomorphisms

$$\widetilde{\mathcal{A}}_k \simeq \mathbf{p}_{-k}(\mathcal{A}(L_q^{2n+1}(d)))^{\mathrm{d}_k} .$$

We stress that the projections $\mathbf{p}_k$ here are those constructed before, in (5.1.4) and (5.1.5), taken now as elements of the group $K_0(C(L_q^{2n+1}(d)))$. Just as for the modules $\mathcal{A}_k$, we need to make a choice of representatives: we will use the right $\mathcal{A}(L_q^{(n,r)})$-module identification and denote by $[\widetilde{\mathcal{A}}_k]$ the class of the projection $\mathbf{p}_k$ as an element in $K_0(C(L_q^{2n+1}(d)))$. Thus, the pull-back of line bundles induces a map

$$j_* : K_0(C(\mathbb{CP}_q^n)) \to K_0(C(L_q^{2n+1}(d))) . \tag{5.2.12}$$

The pull-back of line bundles from $\mathbb{CP}_q^n$ to $L_q^{2n+1}(d)$ could be depicted as

$$\begin{array}{ccc} \widetilde{\mathcal{A}}_k & \xleftarrow[j_*]{\hspace{3em}} & \mathcal{A}_k \\ \uparrow & & \uparrow \\ \mathcal{A}(L_q^{2n+1}(d)) & \xleftarrow[j]{} & \mathcal{A}(\mathbb{CP}_q^n) \end{array} , \tag{5.2.13}$$

where the vertical arrows are given by the module structure. This means, in particular, that for each $\mathcal{A}(\mathbb{CP}_q^n)$-module $\mathcal{A}_k$ its pull-back $\widetilde{\mathcal{A}}_k$ is the $\mathcal{A}(L_q^{2n+1}(d))$-module

$$\widetilde{\mathcal{A}}_k = \mathcal{A}_k \otimes_{\mathcal{A}(\mathbb{CP}_q^n)} \mathcal{A}(L_q^{2n+1}(d)) .$$

From this it follows that $\widetilde{\mathcal{A}}_{-d} = \mathcal{A}_{-d} \otimes_{\mathcal{A}(\mathbb{CP}_q^n)} \mathcal{A}(L_q^{2n+1}(d)) \simeq \mathcal{A}(L_q^{2n+1}(d)) = \widetilde{\mathcal{A}}_0$, thus showing that the module $\widetilde{\mathcal{A}}_{-d}$ is free. The above construction is the dual to the classical one of defining the pull-back as a fibered product (*cf.* [8, Example 1.3.1], [33, Section 16.2]).

While we could have taken this product as defining the pull-back map, we rather prefer the definition in (5.2.6) due to the central character to be played by the partial isometries Ψ_k's of (5.1.4) later on.

We finish this section by underlining the difference between the module $\mathcal{A}_k$ and its pull-back $\widetilde{\mathcal{A}}_k$. While each $\mathcal{A}_k$ is not free when $k \neq 0$ (as a consequence of Proposition 5.2.3), this need not be the case for $\widetilde{\mathcal{A}}_k$, that is the projection $\mathbf{p}_k$ could be trivial (i.e. equivalent to 1) in $K_0(C(L_q^{2n+1}(d)))$. Indeed, the pull-back $\widetilde{\mathcal{A}}_{-d}$ of the line bundle $\mathcal{A}_{-d}$ from the projective space $\mathbb{CP}_q^n$ to the lens space $L_q^{2n+1}(d)$ is free: recall that the corresponding projection is $\mathbf{p}_{-d} := \Psi_{-d}\Psi_{-d}^*$ and here the vector-valued function Ψ_{-d} has entries in the algebra $\mathcal{A}(L_q^{2n+1}(d))$ itself. Thus the condition $\Psi_{-d}^*\Psi_{-d} = 1$ implies that the projector $\mathbf{p}_{-d}$ is equivalent to 1, that is to say, the class of the module $\widetilde{\mathcal{A}}_{-d}$ is that of the trivial bundle. It follows that $(\widetilde{\mathcal{A}}_{-d})^{\otimes k} \simeq \widetilde{\mathcal{A}}_{-kd}$ also has trivial class for any $k \in \mathbb{Z}$, the tensor product being taken over $\mathcal{A}(L_q^{2n+1}(d))$. We will use this fact to show that certain linear combinations of pulled-back line bundles $\widetilde{\mathcal{A}}_{-k}$ define *torsion classes* and, as we shall see later on, they generate the group $K_0(C(L_q^{2n+1}(d)))$.

The Kasparov product with the Euler class

We will now represent the Kasparov product with the Euler class of the C^*-correspondenc $\mathcal{E}^{(d)}$ in $KK(C(\mathbb{CP}^n_q), C(\mathbb{CP}^n_q))$ in terms of an $(n+1) \times (n+1)$ matrix $\mathbf{A}$ with respect to the $\mathbb{Z}$-module basis $\{1, u, \ldots, u^n\}$ of $K^0(C(\mathbb{CP}^n_q)) \simeq \mathbb{Z}^{n+1}$.

Indeed, one has that, for all $i = 0, \ldots, n$ the Kasparov product of the class $[u^i] \in K_0(C(\mathbb{CP}^n_q))$ with the class of the module $[\mathcal{E}^{(d)}] \in KK(C(\mathbb{CP}^n_q), C(\mathbb{CP}^n_q))$ is

$$\left[u^i\right] \otimes_{C(\mathbb{CP}^n_q)} [\mathcal{E}^{(d)}] = [u^i \otimes_{C(\mathbb{CP}^n_q)} \mathcal{A}^{\otimes d}_{-1}] = \\ = [u^i \otimes_{C(\mathbb{CP}^n_q)} (1-u)^{\otimes d}] = [u^i \cdot (1-u)^d].$$

Using the condition $u^{n+1} = 0$ one has

$$(1-u)^d = \sum_{j=0}^{\min(d,n)} (-1)^j \binom{r}{j} u^j \,.$$

Using the $\mathbb{Z}$ module basis $\{1, u, \ldots, u^n\}$. of $K_0(\mathbb{CP}^n)$ it is easy to see that the Kasparov product with $1 - [\mathcal{E}^{(d)}]$ is a $\mathbb{Z}$-module homomorphism represented by an $(n+1) \times (n+1)$ strictly lower triangular matrix with entries on the j-th sub-diagonal equal to $(-1)^{j+1}\binom{d}{j}$ for $j \leq \min(d, n)$ and zero otherwise:

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \cdots\cdots & 0 \\ d & 0 & 0 \cdots\cdots & 0 \\ -\binom{d}{2} & d & 0 \cdots\cdots & 0 \\ \binom{d}{3} & -\binom{d}{2} & d & 0 \\ \vdots & & \ddots & \vdots\ \vdots \\ & & & \\ 0 & 0 & 0 \cdots\cdots & d\ 0 \end{pmatrix} . \tag{5.2.14}$$

5.3 Computing the K-theory of quantum lens spaces

We now use the Gysin sequence (5.2.3) to compute the K-theory of our quantum lens spaces.

Remark 5.3.1. A first computation of the K-theory of the quantum lens spaces dates back to [46], where these spaces are realized as graph algebras, and their K-theory groups are computed as kernel and cokernels of the map $1 - \mathbf{B}^t : \mathbb{Z}^n \to \mathbb{Z}^n$, where $\mathbf{B}$ is the *incidence matrix* of the graph.

It is worth stressing that our construction is structurally different from the one in [46], the only point of contact being that the K-theory is obtained out of a matrix. First, our matrix is different from the incidence matrix of [46]. Second, and more importantly, the novelty of our construction lies in the fact that it allow us to give *geometric* generators of the K_0-group as (combinations) of pulled-back line bundles.

Equation (5.2.14) leads to

$$K_1(C(L_q^{2n+1}(d))) \simeq \ker(\mathbf{A}), \qquad K_0(C(L_q^{2n+1}(d))) \simeq \operatorname{coker}(\mathbf{A})\,, \tag{5.3.1}$$

as $\mathbb{Z}$-module identifications via the surjective map j_*.

We shall obtain explicit generators as classes of *line bundles*, generically torsion ones. This fact will be illustrated by working out some explicit examples. Since the map j_* in (5.2.3) is surjective, the group $K_0(C(L_q^{2n+1}(d)))$ can be obtained by *pulling back* classes from $K_0(C(\mathbb{CP}_q^n))$. The following is then immediate.

Proposition 5.3.2. *The* $(n+1)\times(n+1)$ *matrix* $\mathbf{A}$ *has rank* n*, whence*

$$K_1(C(L_q^{2n+1}(d))) \simeq \mathbb{Z}\,.$$

On the other hand, the structure of the cokernel of the matrix $\mathbf{A}$ depends on the divisibility properties of the integer d. Since $\operatorname{coker}(\mathbf{A}) \simeq \mathbb{Z}^{n+1}/\operatorname{Im}(\mathbf{A})$ and $\operatorname{Im}(\mathbf{A})$ being generated by the columns of $\mathbf{A}$, the vanishing of these columns yields conditions on the generators making them torsion classes in general. Indeed, upon pulling back to the lens space, the vanishing of the the j-th column is just the condition that the pulled back line bundles satisfy $\tilde{\mathcal{A}}_{-(d+j)} = \tilde{\mathcal{A}}_{-j}$; thus this vanishing contains geometric information.

In order to quickly determine $\operatorname{coker}(\mathbf{A})$ (although not directly its generators) one can use the *Smith normal form* of [81] for matrices over a principal ideal domain, such as $\mathbb{Z}$. Thus (*cf.* [58, Theorem 26.2 and Theorem 27.1]) there exist invertible matrices $\mathbf{P}$ and $\mathbf{Q}$ having integer entries which transform $\mathbf{A}$ to a diagonal matrix

$$\operatorname{Sm}(\mathbf{A}) := \mathbf{PAQ} = \operatorname{diag}(\alpha_1, \cdots, \alpha_n, 0)\,, \tag{5.3.2}$$

with integer entries $\alpha_i \geq 1$, ordered in such a way that $\alpha_i \mid \alpha_{i+1}$ for $1 \leq i \leq n$. These integers are algorithmically computed and explicitly given by

$$\alpha_1 = d_1(A)\,, \qquad \alpha_i = d_i(A)/d_{i-1}(A)\,, \quad \text{for each} \quad 2 \leq i \leq n\,,$$

where $d_i(A)$ is the greatest common divisor of the non-zero minors of order i of the matrix A. The above leads directly to the following.

Proposition 5.3.3. *It holds that*

$$\operatorname{coker}(\mathbf{A}) \simeq \operatorname{coker}(\operatorname{Sm}(\mathbf{A})) = \mathbb{Z} \oplus \mathbb{Z}/\alpha_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\alpha_n\mathbb{Z}\,.$$

As a consequence,

$$K_0(C(L_q^{2n+1}(d))) \simeq \mathbb{Z} \oplus \mathbb{Z}_{\alpha_1} \oplus \cdots \oplus \mathbb{Z}_{\alpha_n}\,,$$

with the convention that $\mathbb{Z}_1 = \mathbb{Z}/1\mathbb{Z}$ *is the trivial group.*

As already mentioned, the merit of our construction is not in the computation of the K-theory groups as abstract groups – these are found for instance by using graph algebras as in [45]. Owing to the explicit diagonalization as in (5.3.2) and to Proposition 5.2.5, we also obtain explicit generators as integral combinations of powers of the pull-back to the lens space $L_q^{2n+1}(d)$ of the generator $u := 1 - [\mathcal{A}_{-1}]$. We show how this works and compute $K_0(C(L_q^{2n+1}(d)))$ in some examples.

Example 5.1. If $d = 2$ one computes $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 1$ and $\alpha_n = 2^n$. Hence for $L_q^{2n+1}(d) = \mathbb{S}_q^{2n+1}/\mathbb{Z}_2 = \mathbb{RP}_q^{2n+1}$, the quantum real projective space, we get

$$K_0(C(\mathbb{RP}_q^{2n+1})) = \mathbb{Z} \oplus \mathbb{Z}_{2^n} ,$$

in agreement with [45, Subsection 4.2] (with a shift $n \to n + 1$ from there to here). Moreover, we can construct explicitly the generator of the torsion part of the K-theory group. We claim this is given by $1 - [\tilde{\mathcal{A}}_{-1}]$. First of all, owing to $\tilde{\mathcal{A}}_{-2} \simeq \tilde{\mathcal{A}}_0$ one has

$$(1 - [\tilde{\mathcal{A}}_{-1}])^2 = 2(1 - [\tilde{\mathcal{A}}_{-1}]),$$

and iterating:

$$(1 - [\tilde{\mathcal{A}}_{-1}])^k = 2^{k-1}(1 - [\tilde{\mathcal{A}}_{-1}]).$$

Thus, in a sense one can switch from multiplicative to additive notation. Furthermore, from Proposition 5.2.5 we know that $u^{n+1} = 0$, with $u = 1 - [\mathcal{A}_{-1}]$. When pulled back to the lens space, owing to $\tilde{\mathcal{A}}_{2j} \simeq \tilde{\mathcal{A}}_0$ and $\tilde{\mathcal{A}}_{2j+1} \simeq \tilde{\mathcal{A}}_{-1}$, for all $j \in \mathbb{Z}$, one has

$$0 = (1 - [\tilde{\mathcal{A}}_{-1}])^{n+1} = 2^n(1 - [\tilde{\mathcal{A}}_{-1}]).$$

This amounts to saying that the generator $1 - [\tilde{\mathcal{A}}_{-1}]$ is cyclic of order 2^n.

Example 5.2. For $n = 1$ there is only one $\alpha_1 = d$. Then in this case one has

$$K_0(C(L_q^3(d))) = \mathbb{Z} \oplus \mathbb{Z}_d .$$

From its very definition $[\widetilde{\mathcal{A}}_{-d}] = 1$, hence $\widetilde{\mathcal{A}}_{-1}$ generates the torsion part. Alternatively, from $u^2 = 0$ it follows that $\mathcal{A}_{-j} = -(j-1) + j\mathcal{A}_{-1}$ for all $j > 0$; upon lifting to $L_q^3(d; 1)$, for $j = d$ this yields $d(1 - [\widetilde{\mathcal{A}}_{-1}]) = 0$, i.e. $1 - [\widetilde{\mathcal{A}}_{-1}]$ is cyclic of order d.

Example 5.3. For $n = 2$ there are two cases, according to whether d is even or odd. For the α's in Proposition 5.3.3 one finds:

$$(\alpha_1, \alpha_2) = \begin{cases} (d/2, 2d) & \text{if } \ d \ \text{ even} \\ (d, d) & \text{if } \ d \ \text{ odd} \end{cases} .$$

As a consequence one has that

$$K_0(C(L_q^5(d))) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_{\frac{d}{2}} \oplus \mathbb{Z}_{2d} & \text{if } \ d \ \text{ even} \\ \mathbb{Z} \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d & \text{if } \ d \ \text{ odd} \end{cases} .$$

This is in agreement with [46, Proposition 2.3] (once again with a shift $n \to n + 1$). In particular, for $d = 2$ we get back the case of Example 5.1. In order to identify generators in the two cases, we start from $[\widetilde{\mathcal{A}}_{-d}] = 1$. Direct computations from the conditions $[\widetilde{\mathcal{A}}_{-(d+j)}] = [\widetilde{\mathcal{A}}_{-j}]$ for $j = 0, \cdots, d-1$ lead to

$$\tfrac{1}{2}d(d-1)\,\tilde{u}^2 - d\,\tilde{u} = 0 \quad \text{and} \quad d\,\tilde{u}^2 = 0\,, \tag{5.3.3}$$

where $\tilde{u} = 1 - [\widetilde{\mathcal{A}}_{-1}]$. Indeed these are just the lifts to the lens space $L^5_q(d;1)$ of the non-vanishing columns of the corresponding matrix $\mathbf{A}$ in (5.2.14).

When $d = 2k$ is even, we have conditions coming from $(\widetilde{\mathcal{A}}_{-2})^k \simeq \widetilde{\mathcal{A}}_0$. In fact, due to $[\widetilde{\mathcal{A}}_{-2k}] = 1$, one has $(1 - [\widetilde{\mathcal{A}}_{-k}])^2 = 2(1 - [\widetilde{\mathcal{A}}_{-k}])$, leading to

$$0 = (1 - [\widetilde{\mathcal{A}}_{-k}])^3 = 4(1 - [\widetilde{\mathcal{A}}_{-k}]) = 4k\,\tilde{u} - 2k(k-1)\,\tilde{u}^2.$$

Together with the conditions (5.3.3) this yields

$$\tfrac{1}{2}d\,(\tilde{u}^2 + 2\,\tilde{u}) = 0 \quad \text{and} \quad 2d\,\tilde{u} = 0,$$

that is $\tilde{u}^2 + 2\,\tilde{u}$ is of order $d/2$ while $\tilde{u}$ is of order $2d$ (again, for $d = 2$ this is consistent with the result of Example 5.1, the first ‘generator’ collapsing to the condition $\tilde{u}^2 + 2\,\tilde{u} = 0$).

When $d = 2k+1$ is odd, the conditions (5.3.3) just say that $\tilde{u}$ and $\tilde{u}^2$ are cyclic of order d:

$$d\,\tilde{u} = 0 \quad \text{and} \quad d\,\tilde{u}^2 = 0.$$

Example 5.4. When $n = 3$ the selection of generators for the torsion groups is more involved but still ‘doable’. As before we denote $\tilde{u} = 1 - [\widetilde{\mathcal{A}}_{-1}]$. There are now four possibilities. For the α’s in Proposition 5.3.3 one finds:

	$6 \mid d$	$2 \mid d, 3 \nmid d$	$2 \nmid d, 3 \mid d$	$2 \nmid d, 3 \nmid d$
α_1	$d/6$	$d/2$	$d/3$	d
α_2	$d/2$	$d/2$	d	d
α_3	$12d$	$4d$	$3d$	d

.

As a consequence one has the following cases:

Case $r \equiv 0 \pmod 6$:

$$K_0(C(L^7_q(d))) = \mathbb{Z} \oplus \mathbb{Z}_{\frac{d}{6}} \oplus \mathbb{Z}_{\frac{d}{2}} \oplus \mathbb{Z}_{12d}$$

with generators

$$\tilde{u}^3 + 12\,\tilde{u}\,, \quad \tilde{u}^2 + 6\,\tilde{u}\,, \quad \tilde{u}\,,$$

of order $d/6$, $d/2$ and $12d$, respectively. For the particular case $d = 6$, the first torsion part is absent, one has $\tilde{u}^3 + 12\,\tilde{u} = 0$, and

$$K_0(C(L^7_q(6))) = \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{72}.$$

Case $d \equiv 2, 4 \pmod 6$:

$$K_0(C(L_q^7(d))) = \mathbb{Z} \oplus \mathbb{Z}_{\frac{d}{2}} \oplus \mathbb{Z}_{\frac{d}{2}} \oplus \mathbb{Z}_{4d}$$

with generators

$$\tilde{u}^3 + 2\,\tilde{u}^2\,, \quad \tilde{u}^2 + 2\,\tilde{u}\,, \quad \tilde{u}\,,$$

of order $d/2$, $d/2$ and $4d$, respectively. The particular case $d = 2$ goes back to Example 5.1 with the first and second torsion parts absent and the condition $\tilde{u}^2 + 2\,\tilde{u} = 0$ as in there.

Case $d \equiv 3 \pmod 6$:

$$K_0(C(L_q^7(d))) = \mathbb{Z} \oplus \mathbb{Z}_{\frac{d}{3}} \oplus \mathbb{Z}_d \oplus \mathbb{Z}_{3d}$$

with generators

$$\tilde{u}^3 + 3\,\tilde{u}\,, \quad \tilde{u}^2\,, \quad \tilde{u}\,,$$

of order $d/3$, d and $3d$, respectively. For the particular case $d = 3$ the first torsion part is absent, one has $\tilde{u}^3 + 3\,\tilde{u} = 0$, and

$$K_0(C(L_q^7(3))) = \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9.$$

Case $d \equiv 1, 5 \pmod 6$:

$$K_0(C(L_q^7(d))) = \mathbb{Z} \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d \oplus \mathbb{Z}_d$$

with the three generators of order d given by

$$\tilde{u}^3\,, \quad \tilde{u}^2\,, \quad \tilde{u}\,.$$

To further illustrate the construction, we mention the next case of dimension $n = 4$, for which we list the K-theory groups.

Example 5.5. When $n = 4$ there are 8 possibilities. For the α's in Prop. 5.3.3 one finds:

	$24 \mid d$	$12 \mid d; 8 \nmid d$	$8 \mid d; 6 \nmid d,$	$6 \mid d; 4 \nmid d,$	$4 \mid d; 3, 8 \nmid d$
α_1	$d/24$	$d/12$	$d/8$	$d/6$	$d/4$
α_2	$d/6$	$d/12$	$d/4$	$d/6$	$d/4$
α_3	$6d$	$12d$	$4d$	$4d$	$2d$
α_4	$24d$	$12d$	$8d$	$12d$	$8d$

....

....

$3 \mid d; 2 \nmid d$	$2 \mid d; 3, 4 \nmid d$	$2 \nmid d; 3 \nmid d$
$d/3$	$d/2$	d
$d/3$	$d/2$	d
d	$d/2$	d
$9d$	$8d$	d

.

As a consequence,

$$K_0(C(L_q^9(d))) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_{\frac{d}{24}} \oplus \mathbb{Z}_{\frac{d}{6}} \oplus \mathbb{Z}_{6d} \oplus \mathbb{Z}_{24d} & d \equiv 0 \pmod{24} \\ \mathbb{Z} \oplus \mathbb{Z}_{\frac{d}{12}} \oplus \mathbb{Z}_{\frac{d}{12}} \oplus \mathbb{Z}_{12d} \oplus \mathbb{Z}_{12d} & d \equiv 12 \pmod{24} \\ \mathbb{Z} \oplus \mathbb{Z}_{\frac{d}{8}} \oplus \mathbb{Z}_{\frac{d}{4}} \oplus \mathbb{Z}_{4d} \oplus \mathbb{Z}_{8d} & d \equiv 8, 16 \pmod{24} \\ \mathbb{Z} \oplus \mathbb{Z}_{\frac{d}{6}} \oplus \mathbb{Z}_{\frac{d}{6}} \oplus \mathbb{Z}_{4d} \oplus \mathbb{Z}_{12d} & d \equiv 6 \pmod{12} \\ \mathbb{Z} \oplus \mathbb{Z}_{\frac{d}{4}} \oplus \mathbb{Z}_{\frac{d}{4}} \oplus \mathbb{Z}_{2d} \oplus \mathbb{Z}_{8d} & d \equiv 4, 20 \pmod{24} \\ \mathbb{Z} \oplus \mathbb{Z}_{\frac{d}{3}} \oplus \mathbb{Z}_{\frac{d}{3}} \oplus \mathbb{Z}_{d} \oplus \mathbb{Z}_{9d} & d \equiv 3, 9 \pmod{12} \\ \mathbb{Z} \oplus \mathbb{Z}_{\frac{d}{2}} \oplus \mathbb{Z}_{\frac{d}{2}} \oplus \mathbb{Z}_{\frac{d}{2}} \oplus \mathbb{Z}_{8d} & d \equiv 2 \pmod{12} \\ \mathbb{Z} \oplus \mathbb{Z}_{d} \oplus \mathbb{Z}_{d} \oplus \mathbb{Z}_{d} \oplus \mathbb{Z}_{d} & d \equiv 1, 5 \pmod{6} \end{cases}.$$

Chapter 6

Gysin Sequences in KK-theory for weighted quantum lens spaces

The material covered in this chapter can be found in the paper [5]. There we defined the coordinate algebras of quantum weighted projective lines as fixed point algebras for the weighted circle action on the coordinate algebra $\mathcal{A}(\mathbb{S}^3_q)$ of the quantum 3 sphere. At the C^*-algebra level the lens spaces are given as Pimsner algebras over the C^*-algebra of the continuous functions over the weighted projective spaces.

Using the six term exact sequences of Theorem 3.3.4, we explicitly compute the KK-theory of these spaces for general weights. A central character in this computation is played by an integer matrix whose entries are index pairings. These are in turn computed by pairing the corresponding Chern-Connes characters in cyclic theory. The computation of the KK-theory of our class of q-deformed lens spaces is, to the best of our knowledge, a novel one. Also, it is worth emphasizing that the quantum lens spaces and weighted projective spaces are in general not KK-equivalent to their commutative counterparts.

6.1 Weighted quantum lens spaces in dimension three and quantum teardrops

We recall from Definitions 4.4.2 and 4.4.3 that the quantum weighted projective and lens spaces $\mathbb{WP}^n_q(\underline{\ell})$ and $L^{2n+1}_q(dN_{\underline{\ell}}, \ell)$ are defined, at the coordinate algebra level, as fixed point algebras of the quantum spheres $\mathbb{S}^{2n+1}_q$ for weighted actions of the circle and of a finite cyclic group.

Here we will focus on the $n = 1$ case. From now on, we will denote the weight vector (ℓ_0, ℓ_1) with (m, l) and assume that the entries are coprime.

6.1.1 Weighted projective lines

In order to describe the coordinate algebras of the weighted projective lines, we will use the following characterization.

Theorem 6.1.1 ([12, Theorem. 2.1]). *1. The algebraic quantum projective line $\mathcal{A}(\mathbb{WP}_q(m,l))$ agrees with the unital $*$-subalgebra of $\mathcal{A}(\mathbb{S}^3_q)$ generated by the elements $a = z_0^l(z_1^*)^m$ and $b = z_1 z_1^*$.*

2. The elements a and b satisfy the relations

$$
\begin{aligned}
&b^* = b\ , \quad ba = q^{-2l}\ ab\ , \\
&aa^* = q^{2ml}\ b^m \prod_{j=0}^{l-1}(1-q^{2j}b)\ , \quad a^*a = b^k \prod_{j=1}^{l}(1-q^{-2j}b)\ .
\end{aligned}
\tag{6.1.1}
$$

3. $\mathcal{A}(\mathbb{WP}_q(m,l))$ agrees with the universal unital $*$-algebra with generators a, b, subject to the relations (6.1.1).

In particular $\mathcal{A}(\mathbb{WP}_q(1,1)) = \mathcal{A}(\mathbb{CP}^1_q)$, while $\mathcal{A}(\mathbb{WP}_q(1,l))$ was named *quantum teardrop* in [12] using the very same name given by Thurston in [83] to the corresponding orbifold.

Irreducible representations for the algebra $\mathcal{A}(\mathbb{WP}_q(m,l))$ were given in [12, Proposition 2.2]. These can, up to unitary equivalence, be grouped in two classes:

1. A one dimensional representation defined by

$$\pi_0 : a \mapsto 0, \quad b \mapsto 0. \tag{6.1.2}$$

2. Infinite dimensional representations $\pi_s : \mathcal{A}(\mathbb{WP}_q(m,l)) \mapsto \mathcal{B}(\mathcal{H}_s)$ labeled by $s = 1, 2, \ldots, l$, all on the same separable Hilbert space $\mathcal{H}_s = \ell^2(\mathbb{N}_0)$ with orthonormal basis $|p\rangle, p \in \mathbb{N}_0$, given by

$$
\begin{aligned}
&\pi_s(a)|p\rangle = q^{m(lp+s)} \prod_{r=1}^{l} \left(1 - q^{2(lp+s-r)}\right)^{1/2} |p-1\rangle,\ p \geq 1 \quad \pi_s(a)|0\rangle = 0, \\
&\pi_s(b)|p\rangle = q^{2(lp+s)}|p\rangle.
\end{aligned}
\tag{6.1.3}
$$

By [12, Proposition 2.3], all infinite dimensional representations π_s are faithful.

It is natural, at this point, to wonder whether the representations of $\mathcal{A}(\mathbb{WP}_q(m,l))$ are related to those of the quantum sphere $\mathcal{A}(\mathbb{S}^3_q)$. This is indeed the case; more precisely, let $\pi : \mathcal{A}(\mathbb{S}^3_q) \to \mathcal{B}(\mathcal{H})$ be the representation on $\mathcal{H} = \ell^2(\mathbb{N}_0)$ described in (4.4.5). This is given, on the orthonormal basis $|p\rangle$ of $\mathcal{H}$, by

$$\pi(z_0)|p\rangle = (1-q^{2p})^{1/2}|p-1\rangle, \quad \pi(z_1)|p\rangle = q^{p+1}|p\rangle$$

Let $\theta : \mathcal{A}(\mathbb{WP}_q(m,l)) \to \mathcal{A}(\mathbb{S}^3_q)$ be the inclusion map, given on generators by $a \mapsto z_0^l(z_1^*)^m$, $b \mapsto z_1 z_1^*$.

Proposition 6.1.2 ([12, Proposition 2.4]). *There exists an algebra isomorphism $\phi : \mathcal{B}\Big(\bigoplus_{s=1}^{l} \mathcal{H}_s\Big) \to \mathcal{B}(\mathcal{H})$ that makes the following diagram commutative*

$$\begin{array}{ccc} \mathcal{A}(\mathbb{WP}_q(m,l)) & \xrightarrow{\theta} & \mathcal{A}(\mathbb{S}^3_q) \\ \big\downarrow{\scriptstyle \oplus\pi_s} & & \big\downarrow{\scriptstyle \pi} \\ \mathcal{B}(\bigoplus_{s=1}^{l}\mathcal{H}_s) & \xrightarrow{\phi} & \mathcal{B}(\mathcal{H}) \end{array} .$$

6.1.2 Weighted lens spaces

Recall from Definition 4.4.3 that in order to construct the coordinate algebra of the weighted quantum lens space, we define an action of the cyclic group $\mathbb{Z}_{dlm}$ on the quantum sphere algebra $\mathcal{A}(\mathbb{S}^3_q)$, by restricting the weighted circle action (4.4.6). This is given on generators by

$$\mathrm{e}^{2\pi i/dlm} \cdot (z_0, z_1) \mapsto (\mathrm{e}^{2\pi i/dl} z_0, \mathrm{e}^{2\pi i/dm} z_1)\,. \tag{6.1.4}$$

The coordinate algebra for the *quantum lens space* $L_q(dlm; m, l)$ is the fixed point algebra of the action (6.1.4); it will be denoted by $\mathcal{A}\Big(L_q(dlm; m, l)\Big)$. Equivalently, the algebra can be obtained as a direct sum of line bundles, starting from the $\mathbb{Z}$-graded algebra structure of $\mathcal{A}(\mathbb{S}^3_q)$.

The elements $z_0^l(z_1^*)^m$ and $z_1 z_1^*$, generating the weighted projective space algebra $\mathcal{A}(\mathbb{WP}_q(k, l))$, are clearly invariant leading to an algebra inclusion

$$\mathcal{A}(\mathbb{WP}_q(m,l)) \hookrightarrow \mathcal{A}\Big(L_q(dlm; m, l)\Big) \quad \forall d \in \mathbb{N}.$$

Next, for each $k \in \mathbb{N}_0$, consider the subspaces of $\mathcal{A}(\mathbb{S}^3_q)$ given by

$$\begin{aligned} \mathcal{A}_k(m,l) &:= \sum_{j=0}^{k} (z_0^*)^{lj}(z_1^*)^{m(n-j)} \cdot \mathcal{A}(\mathbb{WP}_q(m,l))\,, \\ \mathcal{A}_{-k}(m,l) &:= \sum_{j=0}^{k} (z_0)^{lj}(z_1)^{m(n-j)} \cdot \mathcal{A}(\mathbb{WP}_q(m,l))\,. \end{aligned} \tag{6.1.5}$$

By construction these subspaces are in fact right-modules over $\mathcal{A}(\mathbb{WP}_q(m,l))$.

Recall that by [91] the algebra $\mathcal{A}(\mathbb{S}^3_q)$ admits a vector space basis given by the vectors $\{e_{p,r,s} \mid p \in \mathbb{Z},\ r, s \in \mathbb{N}_0\}$, where

$$e_{p,r,s} = \begin{cases} z_0^p z_1^r (z_1^*)^s & \text{for} \quad p \geq 0 \\ (z_0^*)^{-p} z_1^r (z_1^*)^s & \text{for} \quad p \leq 0 \end{cases} .$$

Lemma 6.1.3. *Let $k \in \mathbb{Z}$. It holds that*

$$\begin{aligned} e_{p,r,s} \in \mathcal{A}_k(m,l) &\iff pm + (r-s)l = -kml \\ &\iff \sigma_w^{m,l}(e_{p,r,s}) = w^{-kml} e_{p,r,s}\,,\ \forall\, w \in \mathbb{S}^1\,. \end{aligned}$$

As a consequence, it holds that

$$x \in \mathcal{A}_k(n,l) \iff \sigma_w^{m,l}(x) = w^{-kml} x\,,\ \forall\, w \in \mathbb{S}^1\,.$$

Proposition 6.1.4. *The subspaces* $\left\{\mathcal{A}_{dk}(m,l)\right\}_{k\in\mathbb{Z}}$ *give* $\mathcal{A}(L_q(dlm;m,l))$ *the structure of a* $\mathbb{Z}$*-graded unital* $*$*-algebra.*

Corollary 6.1.5. *The algebra* $\mathcal{A}(L_q(ml;m,l))$ *is strongly* $\mathbb{Z}$*-graded, hence the triple* $(\mathcal{A}(L_q(ml;m,l)), \mathcal{O}(U(1)), \mathcal{A}(\mathbb{WP}_q(m,l)))$ *is a quantum principal circle bundle.*

The result follows immediately from Proposition 4.4.4. We show here explicitly how to construct elements

$$\xi_1, \xi_2, \beta_1, \beta_2 \in \mathcal{A}_1(m,l) \quad \text{and} \quad \eta_1, \eta_2, \alpha_1, \alpha_2 \in \mathcal{A}_{-1}(m,l)$$

that satisfy

$$\xi_1\eta_1 + \xi_2\eta_2 = 1 = \alpha_1\beta_1 + \alpha_2\beta_2,$$

along the lines of Theorem 4.1.4.

Firstly, a repeated use of the defining relations of the algebra $\mathcal{A}(\mathbb{S}^3_q)$ leads to

$$(z_0^*)^l z_0^l = \prod_{j=1}^{l} (1 - q^{-2j} z_1 z_1^*)\,.$$

Then, define the polynomial $F \in \mathbb{C}[X]$ by the formula

$$F(X) := \Big(1 - \prod_{j=1}^{l} (1 - q^{-2j} X)\Big)/X\,.$$

Since $z_1 z_1^* = z_1^* z_1$ one has that

$$(z_0^*)^l z_0^l + z_1^*\ F(z_1 z_1^*)\ z_1 = 1\,.$$

In particular, this implies that

$$\begin{aligned}
1 = \Big((z_0^*)^l z_0^l + z_1^*\ F(z_1z_1^*)\ z_1\Big)^m &= \sum_{j=0}^{m} \binom{m}{j} \Big((z_0^*)^l z_0^l\Big)^j\ \Big(z_1^*\ F(z_1z_1^*)\ z_1\Big)^{m-j} \\
&= (z_1^*)^m \Big(F(z_1z_1^*)\Big)^m z_1^m + \sum_{j=1}^{m} \binom{m}{j} \Big((z_0^*)^l z_0^l\Big)^j\ \Big(1 - (z_0^*)^l z_0^l\Big)^{m-j} \\
&= (z_1^*)^m \Big(F(z_1z_1^*)\Big)^m z_1^m + (z_0^*)^l \left\{ \sum_{j=1}^{m} \binom{m}{j} \Big(z_0^l (z_0^*)^l\Big)^{j-1} \Big(1 - z_0^l (z_0^*)^l\Big)^{m-j} \right\} z_0^l\,.
\end{aligned}$$

Define now the polynomial $G \in \mathbb{C}[X]$ by the formula

$$G(X) := (1 - (1-X)^m)/X = \sum_{j=1}^{m} \binom{m}{j} X^{j-1} (1-X)^{m-j}\,, \tag{6.1.6}$$

so that
$$\sum_{j=1}^{m}\binom{m}{j}\Big(z_0^l(z_0^*)^l\Big)^{j-1}\Big(1-z_0^l(z_0^*)^l\Big)^{m-j}=G\Big(z_0^l(z_0^*)^l\Big)\,.$$
And this enables us to write the above identities as
$$1=(z_1^*)^m\Big(F(z_1z_1^*)\Big)^m z_1^m+(z_0^*)^l G\Big(z_0^l(z_0^*)^l\Big)z_0^l\,. \tag{6.1.7}$$

Note that both $F(z_1z_1^*)$ and $G\Big(z_0^l(z_0^*)^l\Big)$ belong to $\mathcal{A}(\mathbb{WP}_q(m,l))$. We thus define
$$\begin{aligned}\xi_1 &:= (z_1^*)^m\Big(F(z_1z_1^*)\Big)^m\,, &\qquad \eta_1 &:= z_1^m\,,\\ \xi_2 &:= (z_0^*)^l\, G\Big(z_0^l(z_0^*)^l\Big)\,, &\qquad \eta_2 &:= z_0^l\,,\end{aligned}$$
and this proves the first half of the proposition.

To prove the second half, we consider instead the identity
$$z_0^l(z_0^*)^l=\prod_{m=0}^{l-1}(1-q^{2m}z_1^*z_1)\,,$$
which again follows by a repeated use of the defining identities for $\mathcal{A}(\mathbb{S}_q^3)$.

The polynomial $\widehat{F}\in\mathbb{C}[X]$ is now given by the formula
$$\widehat{F}(X):=\Big(1-\prod_{m=0}^{l-1}(1-q^{2m}X)\Big)/X\,.$$
and we obtain that
$$z_0^l(z_0^*)^l+z_1\tilde{F}(z_1z_1^*)z_1^*=1\,.$$
By taking m-th powers and computing as above, this yields that
$$1=z_1^m\Big(\tilde{F}(z_1z_1^*)\Big)^m(z_1^*)^m+z_0^l\left\{\sum_{j=1}^{m}\binom{m}{j}\Big((z_0^*)^l z_0^l\Big)^{j-1}\Big(1-(z_0^*)^l z_0^l\Big)^{m-j}\right\}(z_0^*)^l\,.$$
This identity may be rewritten as
$$1=z_1^m\Big(\widehat{F}(z_1z_1^*)\Big)^m(z_1^*)^m+z_0^l G\Big((z_0^*)^l z_0^l\Big)(z_0^*)^l\,,$$
where $G\in\mathbb{C}[X]$ is again the one defined by (6.1.6).
Since both $\widehat{F}(z_1z_1^*)$ and $G\Big((z_0^*)^l z_0^l\Big)$ belong to $\mathcal{A}(\mathbb{WP}_q(m,l))$ we define
$$\begin{aligned}\alpha_1 &:= z_1^m\Big(\widehat{F}(z_1z_1^*)\Big)^m\,, &\qquad \beta_1 &:= (z_1^*)^m\,,\\ \alpha_2 &:= z_0^l\, G\Big((z_0^*)^l z_0^l\Big)\,, &\qquad \beta_2 &:= (z_0^*)^l\,.\end{aligned}$$

Again, the right-modules $\mathcal{A}_1(m,l)$ and $\mathcal{A}_{-1}(m,l)$ play a central rôle, and their completions will enter in the construction of the corresponding Pimsner algebra.

6.1.3 C^*-completions and Pimsner algebra structure

We are now ready to describe the C^*-algebras of continuous function on the weighted projective line and on the weighted lens space.

Definition 6.1.6. The algebra of continuous functions on the *quantum weighted projective line* $\mathbb{WP}_q(m,l)$ is the universal enveloping C^*-algebra, denoted $C(\mathbb{WP}_q(m,l))$, of the coordinate algebra $\mathcal{A}(\mathbb{WP}_q(m,l))$.

It was shown in [12] that this C^*-algebra is isomorphic to a direct sum of compact operators, with unit adjoined.

More precisely, let $\mathcal{H}_s = \ell^2(\mathbb{N}_0)$ and let $\mathcal{K}_s$ denote the algebra of compact operators on the Hilbert space $\mathcal{H}_s$.Then there is a split exact sequence of C^*-algebras

$$0 \longrightarrow \bigoplus_{s=1}^{l} \mathcal{K}_s \longrightarrow C(\mathbb{WP}_q(m,l)) \longrightarrow \mathbb{C} \longrightarrow 0 \,. \tag{6.1.8}$$

By split exactness of the sequence, $C(\mathbb{WP}_q(m,l))$ is isomorphic to a direct sum of l algebras of compact operators with $\mathbb{C}$, i.e. to the unital C^*-algebra

$$\widehat{\oplus_{s=1}^{l}\mathcal{K}_s} \subseteq \mathcal{B}\Big(\oplus_{s=1}^{l} \mathcal{H}_s \Big) \,,$$

where $\widehat{\cdot}$ denotes the unitalization functor. Note that, as a consequence of (6.1.8), the C^*-algebra $C(\mathbb{WP}_q(m,l))$ does not depend on m.

Let $\mathcal{L}_s^1 := \mathcal{L}^1(\mathcal{H}_s)$ denote the trace class operators on the Hilbert space $\mathcal{H}_s$.

Lemma 6.1.7. *The $*$-homomorphism $\pi := \oplus_{s=1}^{l}\pi_s : \mathcal{A}(\mathbb{WP}_q(m,l)) \to \widehat{\oplus_{s=1}^{l}\mathcal{K}_s}$ factorizes through the unital $*$-subalgebra $\widehat{\oplus_{s=1}^{l}\mathcal{L}_s^1}$.*

Proof. Let $s \in \{1,\dots,l\}$. We only need to show that $\pi_s(a), \pi_s(b) \in \mathcal{L}_s^1$.

The operator $\pi_s(b) : \mathcal{H}_s \to \mathcal{H}_s$ is positive and diagonal with eigenvalues $\{q^{2(s+lp)}\}_{p=0}^{\infty}$ each of multiplicity 1.

It is immediate that $\pi_s(b)^{1/2} \in \mathcal{L}_s^1$. Indeed, from (6.1.3),

$$\mathrm{Tr}(\pi_s(b)^{1/2}) = \sum_{p=0}^{\infty} q^s \, q^{lp} = q^s \, (1-q^l)^{-1} < \infty \,,$$

having restricted the deformation parameter to $0 < q < 1$. From $\pi_s(b)^{1/2} \in \mathcal{L}_s^1$ the inclusion $\pi_s(b) \in \mathcal{L}_s^1$ follows as well.

To obtain that $\pi_s(a) \in \mathcal{L}_s^1$ we need to verify that $|\pi_s(a)| \in \mathcal{L}_s^1$. Now, recall that

$$a^*a = b^m \cdot \prod_{j=1}^{l} (1 - q^{-2j} b) \,.$$

Using this relation, we may compute the absolute value:

$$|\pi_s(a)| = \pi_s(b)^{m/2} \cdot \Big(\prod_{j=1}^{l} (1 - q^{-2j}\pi_s(b)) \Big)^{1/2} .$$

Since $\mathcal{L}_s^1$ is an ideal in $\mathcal{B}(\mathcal{H}_s)$ we may thus conclude that $|\pi_s(a)| \in \mathcal{L}_s^1$. □

We now define the algebra of continuous functions on the quantum sphere $\mathbb{S}_q^3$ and show, by using arguments presented in Section 4.2.2, that the fixed point algebra with respect to the extended action agrees with the universal C^*-algebra for $\mathbb{WP}_q(m,l)$.

Definition 6.1.8. The algebra of continuous functions on the *quantum* 3*-sphere* $\mathbb{S}_q^3$ is the universal enveloping C^*-algebra, $C(\mathbb{S}_q^3)$, of the coordinate algebra $\mathcal{A}(\mathbb{S}_q^3)$.

The (weighted) circle action $\Big\{\sigma_w^{(m,l)}\Big\}_{w\in\mathbb{S}^1}$ on $\mathcal{A}(\mathbb{S}_q^3)$ will be denoted simply by $\{\sigma_w\}_{w\in\mathbb{S}^1}$. It induces a strongly continuous circle action on $C(\mathbb{S}_q^3)$. We let $C(\mathbb{S}_q^3)_{(0)}$ denote the fixed point algebra of this action.

Lemma 6.1.9. *The inclusion $\mathcal{A}(\mathbb{WP}_q(m,l)) \hookrightarrow \mathcal{A}(\mathbb{S}_q^3)$ induces an isomorphism of unital C^*-algebras,*

$$i : C(\mathbb{WP}_q(m,l)) \to C(\mathbb{S}_q^3)_{(0)} .$$

Proof. Clearly, one has $\mathrm{Im}(i) \subseteq C(\mathbb{S}_q^3)_{(0)}$ and $\mathrm{Im}(i)$ is shown to be dense by the use of a conditional expectation, like in the proof of Lemma 4.2.4.

It therefore suffices to show that $i : C(\mathbb{WP}_q(m,l)) \to C(\mathbb{S}_q^3)$ is injective. To this end, consider the $*$-homomorphism $\pi := \oplus_{s=1}^{l}\pi_s$ as above. Then, by Proposition 6.1.2 there exist a $*$-homomorphism $\rho : \mathcal{A}(\mathbb{S}_q^3) \to \mathcal{B}(\mathcal{H})$ and an isomorphism $\phi : \mathcal{B}\Big(\oplus_{s=1}^{l} \mathcal{H}_s\Big) \to \mathcal{B}(\mathcal{H})$ such that

$$\phi \circ \pi = \rho \circ i : \mathcal{A}(\mathbb{WP}_q(m,l)) \to \mathcal{B}(\ell^2(\mathbb{N}_0)) .$$

Let now $x \in \mathcal{A}(\mathbb{WP}_q(m,l))$. It follows from the above, that

$$\|x\| = \|\pi(x)\| = \|(\phi\circ\pi)(x)\| = \|(\rho\circ i)(x)\| \le \|i(x)\| .$$

This proves that $i : C(\mathbb{WP}_q(m,l)) \to C(\mathbb{S}_q^3)_{(0)}$ is an isometry and it is therefore injective. □

We now fix $m,l \in \mathbb{N}$ to be coprime positive integers. Let $d \in \mathbb{N}$. With $C(\mathbb{S}_q^3)$ the C^*-algebra of continuous functions on the quantum sphere $\mathbb{S}_q^3$, the action of the cyclic group $\mathbb{Z}_{dlm}$ given on generators in (6.1.4) results into an action $\alpha^{1/d}$ on $C(\mathbb{S}_q^3)$.

Definition 6.1.10. The C^*-algebra of continuous functions on the *quantum lens space* $L_q(dlm;m,l)$ is the fixed point algebra of this action. It is denoted by $C(\mathbb{S}_q^3)^{1/d}$. Thus

$$C(\mathbb{S}_q^3)^{1/d} := \Big\{ x \in C(\mathbb{S}_q^3) \mid \alpha^{1/d}(1,x) = x \Big\} .$$

Lemma 6.1.11. *The C^*-quantum lens space $C(\mathbb{S}_q^3)^{1/d}$ agrees with the closure of the algebraic quantum lens space $\mathcal{A}(L_q(dlm;m,l))$ with respect to the universal C^*-norm on $\mathcal{A}(\mathbb{S}_q^3)$.*

Proof. This follows by applying the bounded operator $E_{1/d} : C(\mathbb{S}^3_q) \to C(\mathbb{S}^3_q)^{1/d}$,

$$E_{1/d} : x \mapsto \frac{1}{dlm} \sum_{i=1}^{dlm} \alpha^{1/d}([i], x) ,$$

with $[i]$ denoting the residual class in $\mathbb{Z}_{dlm}$ of the integer i. □

Alternatively, and in parallel with Definition 6.1.6, we could define the C^*-quantum lens space as the universal enveloping C^*-algebra of the algebraic quantum lens space $\mathcal{A}(L_q(dlm; m, l))$. We will denote this C^*-algebra by $C(L_q(dlm; m, l))$.

Lemma 6.1.12. *For all $d \in \mathbb{N}$, the identity map $\mathcal{A}(L_q(dlm; m, l)) \to \mathcal{A}(L_q(dlm; m, l))$ induces an isomorphisms of C^*-algebras,*

$$C(\mathbb{S}^3_q)^{1/d} \simeq C(L_q(dlm; m, l)) .$$

Proof. To prove the claim we use Theorem 4.2.6. Indeed, let $d \in N$ and let $\| \cdot \| : \mathcal{A}(\mathbb{S}^3_q) \to [0, \infty)$ and $\| \cdot \|' : \mathcal{A}(L_q(dlm; m, l)) \to [0, \infty)$ denote the universal C^*-norms of the two different unital $*$-algebras in question. We then have $\|x\| \leq \|x\|'$ for all $x \in \mathcal{A}(L_q(dlm; m, l))$ since the inclusion $\mathcal{A}(L_q(dlm; m, l)) \to \mathcal{A}(\mathbb{S}^3_q)$ induce a $*$-homomorphism $C(L_q(dlm; m, l)) \to C(\mathbb{S}^3_q)^{1/d}$. But we also have $\|x\|' \leq \|x\|$ since the restriction $\| \cdot \| : \mathcal{A}(\mathbb{WP}_q(m, l)) \to [0, \infty)$ is the maximal C^*-norm on $\mathcal{A}(\mathbb{WP}_q(m, l))$ by Lemma 6.1.9. □

We are ready to realize the C^*-quantum lens spaces as Pimsner algebras. In order to be consistent with Chapter 3 and to lighten the notation, we let $B := C(\mathbb{WP}_q) := C(\mathbb{WP}_q(m, l))$ and $C(L_q(d)) := C(L_q(dlm; m, l))$. As before $\mathcal{E}$ will denote the Hilbert C^*-module over B obtained as the closure of $\mathcal{A}_1(m, l)$ in $C(\mathbb{S}^3_q)$. The $*$-homomorphism $\phi : B \to \mathcal{L}_B(\mathcal{E})$ is induced by the product in $C(\mathbb{S}^3_q)$.

Theorem 6.1.13 ([5, Theorem 6.9])**.** *For all $d \in N$, there is an isomorphism of C^*-algebras,*

$$\mathcal{O}_{\mathcal{E}^{(d)}} \simeq C(\mathbb{S}^3_q)^{1/d} .$$

Proof. Recall from Proposition 6.1.4 that, for all $d \in N$

$$\mathcal{A}(L_q(dlm; m, l)) \simeq \oplus_{n \in \mathbb{Z}} \mathcal{A}_{(dn)}(m, l) .$$

Let us denote by $\{\rho_w\}_{w \in \mathbb{S}^1}$ the associated circle action on $\mathcal{A}(L_q(dlm; m, l))$. Then, we have $\|\rho_w(x)\| \leq \|x\|$ for all $x \in \mathcal{A}(L_q(dlm; m, l))$ and all $w \in \mathbb{S}^1$, where $\| \cdot \|$ is the norm on $C(\mathbb{S}^3_q)^{1/d}$ (the restriction of the maximal C^*-norm on $C(\mathbb{S}^3_q)$). To see this, choose a $z \in \mathbb{S}^1$ such that $z^{dlm} = w$. Then $\sigma^{(m,l)}_z(x) = \rho_w(x)$, where the weighted circle action $\sigma^{(m,l)}$ on $C(\mathbb{S}^3_q)$ is the one defined by extending the action (4.4.6).

An application of Theorem 4.2.5 now shows that $\mathcal{O}_{\mathcal{E}^{(d)}} \simeq C(\mathbb{S}^3_q)^{1/d}$ for all $d \in N$, provided that $\{\rho_w\}_{w \in \mathbb{S}^1}$ satisfies the conditions in 4.2.2. To this end, taking into account the analysis of the coordinate algebra $\mathcal{A}(L_q(lm; m, l))$ provided in Subsection 6.1.2, the only non-trivial thing to check is that the collections

$$\langle \mathcal{E}, \mathcal{E}\rangle := \mathrm{Span}\Big\{\xi^*\eta \mid \xi,\eta \in \mathcal{E}\Big\} \quad \text{and} \quad \langle \mathcal{E}^*, \mathcal{E}^*\rangle := \mathrm{Span}\Big\{\xi\eta^* \mid \xi,\eta \in \mathcal{E}\Big\}$$

are dense in $C(\mathbb{WP}_q(m,l))$. But this follows at once from the fact that the modules are finitely generated projective. ☐

6.2 Gysin sequences

For each $d \in \mathbb{N}$, let $[\mathcal{E}^{(d)}] \in KK(C(\mathbb{WP}_q), C(\mathbb{WP}_q))$ denote the class of the self-Morita equivalence bimodule $\mathcal{E}^{(d)}$ as described in Definition 3.3.2.

Then, given any separable C^*-algebra C, by Theorem 3.3.4 we obtain two six term exact sequences in KK-theory.

$$\begin{array}{ccccc} KK_0(C, C(\mathbb{WP}_q)) & \xrightarrow{1-[\mathcal{E}^{(d)}]} & KK_0(C, C(\mathbb{WP}_q)) & \xrightarrow{j_*} & KK_0\Big(C, C(L_q(d))\Big) \\ {\scriptstyle [\partial]}\Big\uparrow & & & & \Big\downarrow{\scriptstyle [\partial]} \\ KK_1(C, C(L_q(d))) & \xleftarrow[j_*]{} & KK_1(C, C(\mathbb{WP}_q)) & \xleftarrow[1-[\mathcal{E}^{(d)}]]{} & KK_1(B, C(\mathbb{WP}_q)) \end{array} \tag{6.2.1}$$

and

$$\begin{array}{ccccc} KK_0(C(\mathbb{WP}_q), C) & \xleftarrow[1-[\mathcal{E}^{(d)}]]{} & KK_0(C(\mathbb{WP}_q), C) & \xleftarrow[j^*]{} & KK_0\Big(C(L_q(d)), C\Big) \\ \Big\downarrow{\scriptstyle [\partial]} & & & & {\scriptstyle [\partial]}\Big\uparrow \\ KK_1\Big(C(L_q(d)), C\Big) & \xrightarrow{j^*} & KK_1(C(\mathbb{WP}_q), C) & \xrightarrow{1-[\mathcal{E}^{(d)}]} & KK_1(C(\mathbb{WP}_q), C) \end{array}, \tag{6.2.2}$$

where j_* and j^* are the maps in KK-theory induced by $j : C(\mathbb{WP}^n_q) \hookrightarrow C(L_q(d))$.

We will refer to these two sequences as the *Gysin sequences* (in KK-theory) for the quantum lens space $L_q(dlm; m, l)$.

6.2.1 K-theory and K-homology for weighted quantum projective lines

As a direct consequence of the extension (6.1.8), one has the following corollary

Corollary 6.2.1 ([12, Corollary 5.3])**.** . *The K-groups of $C(\mathbb{WP}_q(m,l))$ are:*

$$K_0(C(\mathbb{WP}_q(m,l))) = \mathbb{Z}^{l+1}\,, \quad K_1(C(\mathbb{WP}_q(m,l))) = 0\,.$$

Recall the representations π_s of $C(\mathbb{WP}_q(m,l))$ given in (6.1.3). For each $r \in \{1,\dots,l\}$, let $\mathbf{p}_r \in C(\mathbb{WP}_q(m,l))$ denote the orthogonal projection defined by

$$\pi_s(\mathbf{p}_r) = \begin{cases} \mathbf{e}_{00} \text{ for } s = r \\ \ 0 \ \text{ for } s \neq r \end{cases}, \tag{6.2.3}$$

where $\mathbf{e}_{00} : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ denotes the orthogonal projection onto the closed subspace $\mathbb{C}|0\rangle \subseteq \ell^2(\mathbb{N}_0)$. For $r = 0$, let $\mathbf{p}_0 = 1 \in C(\mathbb{WP}_q(m,l))$. The classes of these $l+1$ projections $\{\mathbf{p}_r,\ r = 0,1,\ldots,l\}$ form a basis for the group $K_0(C(\mathbb{WP}_q(m,l)))$.

Similarly for the K-homology, we have

$$K^0(C(\mathbb{WP}_q(m,l))) = \mathbb{Z}^{l+1}, \quad K^1(C(\mathbb{WP}_q(m,l))) = 0\,.$$

In order to construct Fredholm modules, we let $\mathcal{E}$ denote, as before, the Hilbert C^*-module over the quantum weighted projective line $C(\mathbb{WP}_q(m,l))$ which is obtained as the closure of $\mathcal{A}_1(m,l)$ in $C(\mathbb{S}^3_q)$.

The two polynomials in $\mathcal{A}(\mathbb{WP}_q(m,l))$ at the end of Subsection 6.1.2, written as

$$\begin{aligned}(F(z_1z_1^*))^m &= \Big(\big(1-(z_0^*)^l z_0^l\big)/(z_1z_1^*)\Big)^m \quad \text{and} \\ G\big(z_0^l(z_0^*)^l\big) &= \big(1-(1-z_0^l(z_0^*)^l)^m\big)/(z_0^l(z_0^*)^l)\,,\end{aligned}$$

are manifestly positive, since $\|z_1z_1^*\| \le 1$ and thus also $\|z_0^l(z_0^*)^l\|, \|(z_0^*)^l z_0^l\| \le 1$ in $C(\mathbb{WP}_q(m,l))$. It therefore makes sense to take their square roots:

$$\begin{aligned}\xi_1 &:= F(z_1z_1^*)^{m/2} = \Big(\big(1-(z_0^*)^l z_0^l\big)/(z_1z_1^*)\Big)^{m/2} \in C(\mathbb{WP}_q(m,l)) \quad \text{and} \\ \xi_0 &:= G\big(z_0^l(z_0^*)^l\big)^{1/2} = \Big(\big(1-(1-z_0^l(z_0^*)^l)^m\big)/(z_0^l(z_0^*)^l)\Big)^{1/2} \in C(\mathbb{WP}_q(m,l))\,.\end{aligned}$$

Next, define the morphism of Hilbert C^*-modules $\Psi : \mathcal{E} \to C(\mathbb{WP}_q(m,l))^2$ by

$$\Psi : \eta \mapsto \begin{pmatrix} \xi_1 z_1^m\ \eta \\ \xi_0 z_0^l\ \eta \end{pmatrix},$$

whose adjoint $\Psi^* : C(\mathbb{WP}_q(m,l))^2 \to \mathcal{E}$ is given by

$$\Psi^* : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto (z_1^*)^m \xi_1\ x + (z_0^*)^l \xi_0\ y\,.$$

It then follows from (6.1.7) that $\Psi^*\Psi = \mathrm{Id}_{\mathcal{E}}$. The associated orthogonal projection is

$$\mathbf{p} := \Psi\Psi^* = \begin{pmatrix} \xi_1\ (z_1z_1^*)^m\ \xi_1 & \xi_1\ z_1^m(z_0^*)^l\ \xi_0 \\ \xi_0\ z_0^l(z_1^*)^m\ \xi_1 & \xi_0\ z_0^l(z_0^*)^l\ \xi_0 \end{pmatrix} \in M_2(C(\mathbb{WP}_q(m,l)))\,. \tag{6.2.4}$$

Let $\mathcal{H} := \ell^2(\mathbb{N}_0) \otimes \mathbb{C}^2$. We use the subscripts "+" and "−" to indicate that the corresponding spaces are thought of as being even or odd respectively, for a suitable $\mathbb{Z}_2$-grading γ. Then $\mathcal{H}_\pm$ will be two copies of $\mathcal{H}$. For each $s \in \{1,\ldots,l\}$, with the $*$-representations π_0 and π_s given in (6.1.2) and (6.1.3), define the even $*$-homomorphism

$$\rho_s : \mathcal{A}(\mathbb{WP}_q(m,l)) \to \mathcal{B}\Big(\mathcal{H}_+ \oplus \mathcal{H}_-\Big)\,, \quad \rho_s : x \mapsto \begin{pmatrix} \pi_s(\Psi x \Psi^*) & 0 \\ 0 & \pi_0(\Psi x \Psi^*) \end{pmatrix}\,.$$

We are slightly abusing notation here: the element $\Psi x \Psi^*$ is a 2×2 matrix, hence π_s and π_0 have to be applied component-wise. Next, define

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{6.2.5}$$

Lemma 6.2.2. *The datum* $\mathcal{F}_s := \Big(\mathcal{H}_+ \oplus \mathcal{H}_-, \rho_s, F, \gamma\Big)$, *defines an even* 1*-summable Fredholm module over the coordinate algebra* $\mathcal{A}(\mathbb{WP}_q(m,l))$.

Proof. It is enough to check that $\pi_s(\Psi z_1 z_1^* \Psi^*), \pi_s(\Psi z_0^l (z_1^*)^m \Psi^*) \in \mathcal{L}^1(\mathcal{H})$ and furthermore that $\pi_s(\mathbf{p}) - \pi_0(\mathbf{p}) \in \mathcal{L}^1(\mathcal{H})$, for $\mathbf{p}$ the projection in (6.2.4).

That the two operators involving the generators $z_1 z_1^*$ and $z_0^l (z_1^*)^m$ lie in $\mathcal{L}^1(\mathcal{H})$ follows easily from Lemma 6.1.7. To see that $\pi_s(\mathbf{p}) - \pi_0(\mathbf{p}) \in \mathcal{L}^1(\mathcal{H})$ note that

$$\pi_0(\mathbf{p}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\,.$$

The desired follows again from Lemma 6.1.7, since the operators $\pi_s(z_1 z_1^*)^m$, $\pi_s(z_0^l (z_1^*)^m)$, and $\pi_s(1 - z_0^l (z_0^*)^l)$ are of trace class. ☐

For $s = 0$, we take

$$\rho_0 := \begin{pmatrix} \pi_0 & 0 \\ 0 & 0 \end{pmatrix} : C(\mathbb{WP}_q(m,l)) \to \mathcal{L}(\mathbb{C} \oplus \mathbb{C})$$

and define the even 1-summable Fredholm module

$$\mathcal{F}_0 := \Big(\mathbb{C}_+ \oplus \mathbb{C}_-, \rho_0, F, \gamma\Big)\,.$$

Remark 6.2.3. The 1-summable $l+1$ Fredholm modules over $\mathcal{A}(\mathbb{WP}_q(m,l))$ we have defined are different from the 1-summable Fredholm modules defined in [12, Section 4]. The present Fredholm modules are obtained by *twisting* the Fredholm modules in [12] with the Hilbert C^*-module $\mathfrak{E}$.

6.2.2 Index pairings

We have the classes in the K-homology group $K^0(C(\mathbb{WP}_q(m,l)))$ represented by the even 1-summable Fredholm modules $\mathcal{F}_s$, $s = 0, \ldots, l$, which we described in the previous paragraph. We are interested in computing the index pairings

$$\langle [\mathcal{F}_s], [\mathbf{p}_r] \rangle := \frac{1}{2}\,\mathrm{Tr}\Big(\gamma F [F, \rho_s(\mathbf{p}_r)]\Big) \in \mathbb{Z}\,, \quad \text{for } r, s \in \{0, \ldots, l\}\,.$$

Proposition 6.2.4. *It holds that:*

$$\langle [\mathcal{F}_s], [\mathbf{p}_r] \rangle = \begin{cases} 1 \textit{ for} \quad s = r \\ 1 \textit{ for} \quad r = 0 \\ 0 \qquad \textit{otherwise} \end{cases} .$$

Proof. Suppose first that $r, s \in \{1, \ldots, l\}$. We then have:

$$\langle [\mathcal{F}_s], [\mathbf{p}_r] \rangle = \mathrm{Tr}\Big(\pi_s(\Psi \mathbf{p}_r \Psi^*)\Big) ,$$

and the above operator trace is well-defined since $\pi_s(\Psi \mathbf{p}_r \Psi^*)$ is an orthogonal projection in $\mathrm{M}_2(\mathcal{K})$ and it is therefore of trace class. We may then compute as follows:

$$\begin{aligned} \mathrm{Tr}\Big(\pi_s(\Psi \mathbf{p}_r \Psi^*)\Big) &= \mathrm{Tr}\Big(\pi_s(\xi_1 z_1^m \mathbf{p}_r (z_1^*)^m \xi_1)\Big) + \mathrm{Tr}\Big(\pi_s(\xi_0 z_0^l \mathbf{p}_r (z_0^*)^l \xi_0)\Big) \\ &= \mathrm{Tr}\Big(\pi_s(\mathbf{p}_r (z_1^*)^m \xi_1^2 z_1^m)\Big) + \mathrm{Tr}\Big(\pi_s(\mathbf{p}_r (z_0^*)^l \xi_0^2 z_0^l)\Big) \\ &= \mathrm{Tr}\Big(\pi_s(\mathbf{p}_r)\Big) = \delta_{sr} , \end{aligned}$$

where the second identity follows from [79, Corollary 3.8] and δ_{sr} denotes the Kronecker delta.

If $r \in \{1, \ldots, l\}$ and $s = 0$, then $\rho_0(\mathbf{p}_r) = 0$ and thus

$$\langle [\mathcal{F}_0], [\mathbf{p}_r] \rangle = 0.$$

Next, suppose that $r = s = 0$. Then

$$\langle [\mathcal{F}_0], [\mathbf{p}_0] \rangle = \mathrm{Tr} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1 .$$

Finally, suppose that $r = 0$ and $s \in \{1, \ldots, l\}$. We then compute

$$\begin{aligned} \langle [\mathcal{F}_s], [\mathbf{p}_0] \rangle &= \mathrm{Tr}\Big(\pi_s(\mathbf{p}) - \pi_0(\mathbf{p})\Big) = \mathrm{Tr}\Big(\pi_s(\xi_1^2 (z_1 z_1^*)^m)\Big) + \mathrm{Tr}\Big(\pi_s(\xi_0 z_0^l (z_0^*)^l \xi_0) - 1\Big) \\ &= \mathrm{Tr}\Big(\pi_s(1 - (z_0^*)^l z_0^l)^m\Big) - \mathrm{Tr}\Big(\pi_s(1 - z_0^l (z_0^*)^l)^m\Big) . \end{aligned}$$

We will prove in the next lemma that this quantity is equal to 1. This will complete the proof of the present proposition. □

Lemma 6.2.5. *It holds that:*

$$\mathrm{Tr}\Big(\pi_s(1 - (z_0^*)^l z_0^l)^m\Big) - \mathrm{Tr}\Big(\pi_s(1 - z_0^l (z_0^*)^l)^m\Big) = \mathrm{Tr}\Big(\pi_s([z_0^l, (z_0^*)^l])\Big) = 1 .$$

Proof. Notice firstly that $\pi_s\Big(1 - (z_0^*)^l z_0^l\Big), \pi_s\Big(1 - z_0^l (z_0^*)^l\Big) \in \mathcal{L}^1(\ell^2(\mathbb{N}_0))$ by Lemma 6.1.7. It then follows by induction that

$$\mathrm{Tr}\Big(\pi_s(1 - (z_0^*)^l z_0^l)^m\Big) - \mathrm{Tr}\Big(\pi_s(1 - z_0^l (z_0^*)^l)^m\Big) = \mathrm{Tr}\Big(\pi_s([z_0^l, (z_0^*)^l])\Big) .$$

Indeed, with $x := z_0^l$, for all $j \in \{2, 3, \ldots\}$, one has that,

$$\begin{aligned}
&\mathrm{Tr}\Big(\pi_s(1-x^*x)^j\Big) - \mathrm{Tr}\Big(\pi_s(1-xx^*)^j\Big) = \\
&\quad = \mathrm{Tr}\Big(\pi_s(1-x^*x)^{j-1}\Big) - \mathrm{Tr}\Big(\pi_s(xx^*(1-xx^*)^{j-1})\Big) - \mathrm{Tr}\Big(\pi_s(1-xx^*)^j\Big) = \\
&\quad = \mathrm{Tr}\Big(\pi_s(1-x^*x)^{j-1}\Big) - \mathrm{Tr}\Big(\pi_s(1-xx^*)^{j-1}\Big) .
\end{aligned}$$

It therefore suffices to show that $\mathrm{Tr}\Big(\pi_s([z_0^l,(z_0^*)^l])\Big) = 1$. Now, one has:

$$[z_0^l,(z_0^*)^l] = \sum_{j=0}^{l} (-1)^j q^{j(j-1)} \binom{l}{j}_{q^2} (1-q^{-2jl})\ (z_1 z_1^*)^j$$

where the notation $\binom{l}{j}_{q^2}$ refers to the q^2-binomial coefficient, defined by the identity

$$\prod_{j=1}^{l} (1+q^{2(j-1)}Y) = \sum_{j=0}^{l} q^{m(m-1)} \binom{l}{j}_{q^2} Y^j$$

in the polynomial algebra $\mathbb{C}[Y]$. Then, as in [12, Proposition 4.3] one computes:

$$\begin{aligned}
\mathrm{Tr}\Big(\pi_s([z_0^l,(z_0^*)^l])\Big) &= \sum_{j=1}^{l} (-1)^j q^{j(j-1)} \binom{l}{j}_{q^2} (1-q^{-2jl}) \frac{q^{2js}}{1-q^{2jl}} = \\
&= 1 - \sum_{j=0}^{l} (-1)^j q^{j(j-1)} \binom{l}{j}_{q^2} q^{2j(s-l)} = \\
&= 1 - \prod_{j=1}^{l} (1-q^{2(s-j)}) = 1 ,
\end{aligned}$$

since, due to $s \in \{1,\dots,l\}$ one of the factors in the product must vanish. □

Remark 6.2.6. The non-vanishing of the pairings in Proposition 6.2.4 for $r=0$ means that the class of the projection $\mathbf{p}$ in (6.2.4) is non-trivial in $K_0(C(\mathbb{WP}_q(m,l)))$. (In this case the pairings are computing the couplings of the Fredholm modules of [12, Section 4] with the projection $\mathbf{p}$.) Geometrically this means that the line bundle $\mathcal{A}_{(1)}(m,l)$ over $\mathcal{A}(\mathbb{WP}_q(m,l))$ and then the quantum principal circle bundles $\mathcal{A}(\mathbb{WP}_q(m,l))) \hookrightarrow \mathcal{A}(L_q(dlm);m,l)$ are non-trivial.

6.3 Computing the KK-theory of quantum lens spaces

As a direct consequence of (6.1.8), since the C^*-algebra $C(\mathbb{WP}_q(m,l))$ is isomorphic to $\widehat{\bigoplus_{s=1}^{l} \mathcal{K}_s}$, we have that that $C(\mathbb{WP}_q(m,l))$ is KK-equivalent to $\mathbb{C}^{l+1}$.

To show this equivalence explicitly, for each $s \in \{0,\dots,l\}$ we define a KK-class $[\Pi_s] \in KK(C(\mathbb{WP}_q(m,l)),\mathbb{C})$ via the Kasparov module $\Pi_s \in \mathbb{E}(C(\mathbb{WP}_q(m,l)),\mathbb{C})$ given by:

$$\mathit{\Pi}_s := \Big(\ell^2(\mathbb{N}_0)_+ \oplus \ell^2(\mathbb{N}_0)_-, \widetilde{\pi}_s, F, \gamma\Big) \quad \text{for } s \neq 0,$$
$$\mathit{\Pi}_0 := (\mathbb{C}, \pi_0, 0) \quad \text{for } s = 0\,,$$

with F and γ the canonical operators in (6.2.5). The representation is

$$\widetilde{\pi}_s = \begin{pmatrix} \pi_s & 0 \\ 0 & \pi_0 \end{pmatrix}\,,$$

with the representation π_s and π_0 given in (6.1.2) and (6.1.3).

For each $r \in \{0, \ldots, l\}$ we define the KK-class $[I_r] \in KK(\mathbb{C}, C(\mathbb{WP}_q(m,l)))$ by the Kasparov module

$$I_r := \Big(C(\mathbb{WP}_q(m,l)), i_r, 0\Big) \in \mathbb{E}(\mathbb{C}, C(\mathbb{WP}_q(m,l)))\,,$$

where $i_r : \mathbb{C} \to C(\mathbb{WP}_q)$ is the $*$-homomorphism defined by $i_r : 1 \mapsto \mathbf{p}_r$ with the orthogonal projections $\mathbf{p}_r \in C(\mathbb{WP}_q)$ given in (6.2.3).

Upon collecting these classes as

$$[\mathit{\Pi}] := \oplus_{s=0}^{l}[\mathit{\Pi}_s] \in KK(C(\mathbb{WP}_q), \mathbb{C}^{l+1}) \quad \text{and} \quad [I] := \oplus_{r=0}^{l}[I_r] \in KK(\mathbb{C}^{l+1}, C(\mathbb{WP}_q))\,,$$

it follows that $[I]\widehat{\otimes}_{C(\mathbb{WP}_q)}[\mathit{\Pi}] = [1_{\mathbb{C}^{l+1}}]$ and that $[\mathit{\Pi}]\widehat{\otimes}_{\mathbb{C}^{l+1}}[I] = [1_{C(\mathbb{WP}_q)}]$, from stability of KK-theory (see [8, Corollary 17.8.8]).

We need a final tensoring with the Hilbert C^*-module $\mathcal{E}$. This yields a class

$$[I_r]\widehat{\otimes}_{C(\mathbb{WP}_q)}[\mathcal{E}]\widehat{\otimes}_{C(\mathbb{WP}_q)}[\mathit{\Pi}_s] \in KK(\mathbb{C}, \mathbb{C})\,,$$

for each $s, r \in \{0, \ldots, l\}$. Then, we let $\mathbf{M}_{sr} \in \mathbb{Z}$ denote the corresponding integer in $KK(\mathbb{C}, \mathbb{C}) \simeq \mathbb{Z}$, with $\mathbf{M} := \{\mathbf{M}_{sr}\}_{s,r=0}^{l} \in \mathrm{M}_{l+1}(\mathbb{Z})$ the corresponding matrix.

As a consequence the six term exact sequence in (6.2.1) becomes

$$\begin{array}{ccccc} \oplus_{r=0}^{l} K^0(B) & \xrightarrow{1-\mathbf{M}^d} & \oplus_{s=0}^{l} K^0(B) & \longrightarrow & KK_0\Big(B, C(L_q(d))\Big) \\ \uparrow & & & & \downarrow \\ KK_1(B, C(L_q(d))) & \longleftarrow & \oplus_{s=0}^{l} K^1(B) & \underset{1-\mathbf{M}^d}{\longleftarrow} & \oplus_{r=0}^{l} K^1(B) \end{array} \tag{6.3.1}$$

while, with $\mathbf{M}^t \in \mathrm{M}_{l+1}(\mathbb{Z})$ denoting the matrix transpose of $\mathbf{M} \in \mathrm{M}_{l+1}(\mathbb{Z})$, the six term exact sequence in (6.2.2) becomes

$$\begin{array}{ccccc} \oplus_{s=0}^{l} K_0(B) & \underset{1-(\mathbf{M}^t)^d}{\longleftarrow} & \oplus_{r=0}^{l} K_0(B) & \longleftarrow & KK_0\Big(C(L_q(d)), B\Big) \\ \downarrow & & & & \uparrow \\ KK_1\Big(C(L_q(d)), B\Big) & \longrightarrow & \oplus_{r=0}^{l} K_1(B) & \xrightarrow{1-(\mathbf{M}^t)^d} & \oplus_{s=0}^{l} K_1(B) \end{array}\,. \tag{6.3.2}$$

In order to proceed we therefore need to compute the matrix $\mathbf{M} \in \mathrm{M}_{l+1}(\mathbb{Z})$.

Lemma 6.3.1. *The Kasparov product* $[\mathcal{E}]\widehat{\otimes}_{C(\mathbb{WP}_q)}[\Pi_s] \in KK(C(\mathbb{WP}_q), \mathbb{C})$ *is represented by the Fredholm module* $\mathcal{F}_s$ *in Lemma 6.2.2 for each* $s \in \{0, \dots, l\}$.

Proof. Recall firstly that the class $[\mathcal{E}] \in KK(C(\mathbb{WP}_q), C(\mathbb{WP}_q))$ is represented by the Kasparov module

$$\Big(\mathcal{E}, \phi, 0\Big) \in \mathbb{E}(C(\mathbb{WP}_q), C(\mathbb{WP}_q)),$$

where $\phi : C(\mathbb{WP}_q) \to \mathcal{L}_B(\mathcal{E})$ is induced by the product on the algebra $C(S_q^3)$. It then follows from the observations in the beginning of this section that $(\mathcal{E}, \phi, 0)$ is equivalent to the Kasparov module

$$\Big(C(\mathbb{WP}_q)^2, \Psi\phi\Psi^*, 0\Big) \in \mathbb{E}(C(\mathbb{WP}_q), C(\mathbb{WP}_q)).$$

Suppose next that $s = 0$. The Kasparov product $[\mathcal{E}]\widehat{\otimes}_{C(\mathbb{WP}_q)}[\Pi_0]$ is then represented by the Kasparov module

$$\Big(C(\mathbb{WP}_q)^2\widehat{\otimes}_{\pi_0}\mathbb{C}, \Psi\phi\Psi^* \otimes 1, 0\Big) \in \mathbb{E}(C(\mathbb{WP}_q), \mathbb{C}),$$

which is equivalent to the Kasparov module

$$\left(\mathbb{C}_+ \oplus \mathbb{C}_-\,, \begin{pmatrix} \pi_0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

This proves the claim of the lemma in this case.

Suppose thus that $s \in \{1, \dots, l\}$. The Kasparov product $[\mathcal{E}]\widehat{\otimes}_{C(\mathbb{WP}_q)}[\Pi_s]$ is then represented by the Kasparov module given by the $\mathbb{Z}_2$-graded Hilbert space

$$\Big(C(\mathbb{WP}_q))^2\widehat{\otimes}_{\pi_s}\ell^2(\mathbb{N}_0)\Big)_+ \oplus \Big(C(\mathbb{WP}_q)^2\widehat{\otimes}_{\pi_0}\ell^2(\mathbb{N}_0)\Big)_- \simeq \mathcal{H}_+ \oplus \mathcal{H}_-$$

with associated $*$-homomorphism

$$\rho_s = \begin{pmatrix} \pi_s(\Psi\phi\Psi^*) & 0 \\ 0 & \pi_0(\Psi\phi\Psi^*) \end{pmatrix} : C(\mathbb{WP}_q) \to \mathcal{B}\Big(\mathcal{H}_+ \oplus \mathcal{H}_-\Big),$$

and with Fredholm operator F and grading γ the canonical ones in (6.2.5). This proves the claim of the lemma in these cases as well. □

Combining the results of Lemma 6.3.1 and Proposition 6.2.4 one obtains the following:

Proposition 6.3.2. *The matrix* $\mathbf{M} = \{\mathbf{M}_{sr}\} \in \mathrm{M}_{l+1}(\mathbb{Z})$ *has entries*

$$\mathbf{M}_{sr} = \langle [\mathcal{F}_s], [I_r] \rangle = \begin{cases} 1 \textit{ for } & s = r \\ 1 \textit{ for } & r = 0 \\ 0 & \textit{otherwise} \end{cases}.$$

A combination of Proposition 6.3.2 and the six term exact sequences in (6.3.1) and (6.3.2) then allows us to compute the K-theory and the K-homology of the quantum lens space $L_q(dlm; m, l)$ for all $d \in \mathbb{N}$.

When $C = \mathbb{C}$, the sequence in (6.3.1) reduces to

$$0 \longrightarrow K_1(C(L_q(d)) \longrightarrow \mathbb{Z}^{l+1} \xrightarrow{1-\mathbf{M}^d} \mathbb{Z}^{l+1} \longrightarrow K_0(C(L_q(d)) \longrightarrow 0$$

while the one in (6.3.2) becomes

$$0 \longleftarrow K^1(C(L_q(d)) \longleftarrow \mathbb{Z}^{l+1} \xleftarrow{1-(\mathbf{M}^t)^d} \mathbb{Z}^{l+1} \longleftarrow K^0(C(L_q(d)) \longleftarrow 0\,.$$

Let us use the notation $\iota : \mathbb{Z} \to \mathbb{Z}^l$, $1 \mapsto (1, \ldots, 1)$ for the diagonal inclusion and let $\iota^t : \mathbb{Z}^l \to \mathbb{Z}$ denote the transpose, $\iota^t : (m_1, \ldots, m_l) \mapsto m_1 + \ldots + m_l$.

Theorem 6.3.3. *Let $m, l \in \mathbb{N}$ be coprime and let $d \in \mathbb{N}$. Then*

$$\begin{aligned} K_0\Big(C(L_q(dlm; m, l))\Big) &\simeq \mathrm{Coker}(1 - \mathbf{M}^d) \simeq \mathbb{Z} \oplus \Big(\mathbb{Z}^l/\mathrm{Im}(d \cdot \iota)\Big) \\ K_1\Big(C(L_q(dlm; m, l))\Big) &\simeq \mathrm{Ker}(1 - \mathbf{M}^d) \simeq \mathbb{Z}^l \end{aligned}$$

and

$$\begin{aligned} K^0\Big(C(L_q(dlm; m, l))\Big) &\simeq \mathrm{Ker}(1 - (\mathbf{M}^t)^d) \simeq \mathbb{Z} \oplus \Big(\mathrm{Ker}(\iota^t)\Big) \\ K^1\Big(C(L_q(dlm; m, l))\Big) &\simeq \mathrm{Coker}(1 - (\mathbf{M}^t)^d) \simeq \mathbb{Z}/(d\mathbb{Z}) \oplus \mathbb{Z}^l\,. \end{aligned}$$

We finish by stressing that the results on the K-theory and K-homology of the lens spaces $L_q(dlm; m, l)$ are different from the ones obtained for instance in [46]. In fact our lens spaces are not included in the class of lens spaces considered there. Thus, for the moment, there seems to be no alternative method which results in a computation of the KK-groups of these spaces.

Conclusion

We studied the noncommutative topology of principal circle bundles in the context of Pimsner algebras. Chapter 3 was devoted to the theory of Pimsner algebras and generalized crossed products. Their connection with principal circle bundles was analyzed in Chapter 4, where we also provided a handful of examples.

In Part III we focused on two special classes of examples: quantum lens spaces of any odd-dimension, seen as principal circle bundles over quantum projective spaces, and weighted lens spaces in dimension three, seen as principal circle bundles over weighted projective lines.

In Chapter 5 we constructed an exact sequence in K-theory for quantum lens spaces $C(L_q^{2n+1}(d))$ of any dimension. This sequence allowed us to compute the K-theory groups of the algebra $C(L_q^{2n+1}(d))$. While a computation of these groups was already present in [46], it is worth stressing that our approach is substantially different, more geometrical, and it provides explicit representatives of the generators of the group $K_0(C(L_q^{2n+1}(d)))$, in the form of combinations of pulled-back line bundles. We described these generators for some particular cases in Section 5.3.

Similarly, in Chapter 6 we constructed an exact sequence in KK-theory for weighted quantum lens spaces in dimension three. There a central character is played by an integer matrix of index pairings. The novelty of our approach lies in the construction of this matrix, and to the best of our knowledge, there seems to be no alternative method for computing the KK-groups of the quantum lens spaces $C(L_q(dkl; k, l))$ for $d \neq 1$. It is worth mentioning that the K-groups of these spaces are different from their classical counterparts.

The relevance of the mapping cone extension (2.4.4) in constructing the Gysin exact sequence was already noticed in [3, Section 5], where we employed techniques borrowed from [17]. An interesting problem would be understanding the connection between the exact sequence (3.3.5) and the mapping cone exact sequence for the inclusion $B \hookrightarrow \mathcal{O}_{\mathcal{E}}$, in the case of a self-Morita equivalence bimodule, and even in the most general case of a general full correspondence. More on this problem can be found in Appendix B. This problem is currently under investigation and results in this direction will be reported elsewhere.

Appendices

Appendix A

Principal circle bundles and Gysin sequences

A.1 Vector bundles and principal bundles

A.1.1 Complex vector bundles

Let X be a Hausdorff topological space. A *complex vector bundle* of rank n over the *base* X consists of a topological space E, named the *total space*, together with a continuous surjection $\pi : E \to X$ such that each fiber $E_x := \pi^{-1}(x)$ has the structure of complex n-dimensional vector space, and such that for every point $x \in X$ there exists an open neighborhood U of x in X and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{C}^n$, mapping each fiber E_x complex linearly onto $\{x\} \times \mathbb{C}^n$. The latter condition is referred to as *local triviality.*

Example A.1. The *trivial rank n bundle over X* consists of the product manifold $E = X \times \mathbb{C}^n$, with projection $\pi : X \times \mathbb{C}^n \to X$ onto the first factor. In this case one has a global trivialization with $U = X$ and Φ the identity map on $\pi^{-1}(X) = X \times \mathbb{C}^n$.

Given two vector bundles $\pi' : E' \to X'$ and $p' : E' \to X'$, a vector bundle map is a continuous map $\tau : E \to E'$ such that there exists a map $u : X \to X'$ satisfying

$$\pi' \circ \tau = f \circ \pi, \tag{A.1.1}$$

i.e. such that

$$\begin{array}{ccc} E & \xrightarrow{\ \tau\ } & E' \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'}, \\ X & \xrightarrow{\ f\ } & X' \end{array}$$

and such that the induced fiber-wise maps are linear maps. Condition (A.1.1) is equivalent to $\tau(E_x) \subseteq E'_{f(x)}$, so the bundle map is fiber preserving.

If both $\pi : E \to X$ and $\pi' : E' \to X$ are vector bundles over the same base, then a vector bundle map is a continuous map $\tau : E_1 \to E_2$ satisfying $\pi_2 \circ \tau = \pi_1$, defining fiber-wise linear maps. In that case a bundle map $\tau : E \to E'$ is named an *equivalence* if the induced fiber-wise maps $\tau_x : E_x \to E'_x$ are linear isomorphisms for every $x \in X$. A vector bundle is *trivial* if it is equivalent to the product bundle $\pi : X \times \mathbb{C}^n \to X$.

Given two vector bundles $E \to X$ and $E' \to X$, of rank n and m, respectively, one defines their tensor product as the rank nm vector bundle $E \otimes E' \to X$ constructed using the fiber-wise tensor product of vector spaces.

Given a vector bundle $E \to X$, the *dual bundle* $E^* \to X$ is the vector bundle with fibers the dual spaces to the fibers of E. It is a complex vector bundle of the same rank.

Line bundles and the Picard group

A complex line bundle is a vector bundle $E \to X$ with fiber isomorphic to the complex line $\mathbb{C}$.

Equivalence classes of line bundles over a topological space X form a group, under the operation of tensor product over $\mathbb{C}$, with inverse the dual line bundle and unit element the trivial line bundle. This groups is named the *topological Picard group of* X, denoted by $\mathrm{Pic}(X)$. One has a map

$$c_1 : \mathrm{Pic}(X) \simeq H^2(X, \mathbb{Z}), \tag{A.1.2}$$

which is an isomorphism of groups, satisfying

$$c_1(L \otimes L') = c_1(L) + c_1(L').$$

There are several ways to define this isomorphism: in Chern-Weyl theory (see for instance [60, Section 14]) one defines this map by picking a connection on the line bundle and by considering an appropriate multiple of the trace of the curvature of the connection. Equivalently, this map can be defined in terms of sheaf cohomology (see [42, Exercise III.4.5 and Section B.5]).

A.1.2 Principal bundles

Let X be a Hausdorff topological space and G a topological group. A *continuous principal bundle over* X *with structure group* G is a triple (P, π, G) where P is a topological space, $\pi : P \to X$ is a continuous surjection, and there is a right action of G on P such that the following are satisfied:

1. The action preserves the fibers of π, i.e.

$$\pi(p \cdot g) = \pi(p); \tag{A.1.3}$$

2. For every point $x \in X$ there exists an open neighborhood U of x in X and a homeomorphism $\Psi : \pi^{-1}(U) \to U \times G$ of the form

$$\Psi(p) = (\pi(p), \psi(p)),$$

where $\psi : \pi^{-1}(U) \to G$ satisfies

$$\psi(p \cdot g) = \psi(p)g \tag{A.1.4}$$

for all $p \in \pi^{-1}(U)$ and $g \in G$. This property is named *local triviality.*

While (A.1.3) implies that the group G acts on the bundle fiber-wise, condition (A.1.4) has the important consequence (*cf.* [62, Lemma 4.1.1]) that the fiber of P at p coincides with the orbit of the point p, i.e.

$$P_p := \pi^{-1}(\pi(p)) = \{p \cdot g \mid g \in G\} = p \cdot G.$$

We wil sometimes denote the bundle (P, π, G) with the compact notation $\pi : P \to X$. If the projection map is understood from the context, we will write $G \hookrightarrow P \to X$.

Example A.2. The *trivial principal G-bundle over X* consists of the product manifold $P = X \times G$, the projection $\pi : X \times G \to X$ onto the first factor and the action $(x, h) \cdot g = (x, hg)$. In this case one has a global trivialization with $U = X$ and Ψ the identity map on $\pi^{-1}(X) = X \times G$.

Let us fix a topological group G and let us consider two principal G-bundles $\pi : P \to X$ and $\pi' : P' \to X'$. For convenience we denote the two-actions of G on P and P' with the same dot. Then a *principal bundle map* is a continuous map $\tau : P \to P'$ such that

$$\tau(p \cdot g) = \tau(p) \cdot g,$$

for all $p \in P_1$ and $g \in G$. The map τ preserves the fibers of the bundles, and in particular, it determines a map $f : X \to X'$ defined by

$$\pi' \circ \tau = f \circ \pi,$$

i.e. such that

$$\begin{array}{ccc} P & \xrightarrow{\tau} & P' \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi'} . \\ X & \xrightarrow{f} & X' \end{array}$$

If both $\pi : P \to X$ and $\pi' : P' \to X$ are principal G-bundles over the same base space, then a bundle map $\tau : P \to P'$ is named an *equivalence* if the induced map $f : X \to X$ is the identity on X. It is easy to check (see for instance [62, Exercise 4.3.3]) that τ is necessarily a homeomorphism and its inverse $\tau^{-1} : P' \to P$ is also an equivalence. A principal G-bundle is said to be trivial if it is equivalent to the trivial bundle $\pi : X \times G \to G$.

Associated bundles

Let V be a finite dimensional vector space and $\rho : G \to \mathrm{GL}(V)$ a representation of G on V. Then ρ gives rise to an action of G on V by

$$(g, v) \mapsto g \cdot v = \rho(g)(v).$$

In particular, this allows one to define a right action of G on $P \times V$ by

$$(p, \xi) \cdot g = (p \cdot g, g^{-1} \cdot v).$$

We denote by $P \times_\rho V$ the orbit space of $P \times V$ by this action. More precisely, one defines an equivalence relation $\sim$ on $P \times V$ as follows: $(p_1, v_1) \sim (p_2, v_2)$ if and only if there exists $g \in G$ such that $(p_2, v_2) = (p_1, v_1) \cdot g$. The equivalence class of (p, v) is denoted by $[p, v] := \{(p \cdot g, g^{-1} \cdot v) \mid g \in G\}$. As a set, $P \times_\rho V = \{[p, v] \ : \ (p, v) \in P \times V\}$, and one endows $P \times_\rho V$ with the quotient topology. The map $\pi_\rho([p, v]) = \pi(p)$ is continuous and $\pi_\rho : P \times_\rho V \to X$ is a locally trivial vector bundle with fiber V. It is called the *vector bundle associated to* $\pi : P \to X$ via the representation ρ.

Example A.3. Let $\pi : P \to X$ be any principal circle bundle, and $V = \mathbb{C}$. If $\rho : \mathbb{S}^1 \to \mathrm{GL}(\mathbb{C}) = \mathbb{C}^\times$ is any one-dimensional representation of the circle group on $\mathbb{C}$, then the associated vector bundle $P_\rho := P \times_\rho \mathbb{C}$ has fibers that are copies of $\mathbb{C}$ and it is a complex line bundle. A natural choice of representation is given by $\rho : \mathbb{S}^1 \to \mathbb{C}^\times$, obtained by taking $\rho(g)z = gz$ (note that if $g = \mathrm{e}^{i\theta}$ for $0 \leq \theta < 2\pi$, then $\rho(g)$ is rotation by θ). We will denote the corresponding complex line bundle by $P \times_{\mathbb{S}^1} \mathbb{C}$.

Classification of principal circle bundles

The correspondence between principal circle bundles and associated line bundles is one-to-one. This in particular allows one to classify principal circle bundles using the classification result of (A.1.2).

Theorem A.1.1 ([20], see also [15, Exercise 4.4.3])**.** *Circle bundles over a topological space X are classified by the group $H^2(X, \mathbb{Z})$, via their Euler class $\chi(P)$. This class agrees with the first Chern class of the associated line bundle.*

A.2 The Gysin sequence

The Gysin exact sequence, defined in [39], is a long exact sequence in cohomology, naturally associated to any sphere bundle ([9, Proposition 14.33]) and to any oriented vector bundle ([60, Chapter 12]). In this work we focus on the topology of circle bundles, hence we will concentrate on that particular case.

A.2.1 The Gysin sequence in cohomology

Let $\pi : P \to X$ be a principal circle bundle. The pull-back map $\pi^* : H^k(X,\mathbb{Z}) \to H^k(P,\mathbb{Z})$ and the push-forward map $\pi_* : H^k(P,\mathbb{Z}) \to H^{k-1}(P,\mathbb{Z})$ fit into a long exact sequence:

$$\cdots \longrightarrow H^k(P,\mathbb{Z}) \xrightarrow{\pi_*} H^{k-1}(X,\mathbb{Z}) \xrightarrow{\alpha} H^{k+1}(X,\mathbb{Z}) \xrightarrow{\pi^*} H^{k+1}(P,\mathbb{Z}) \longrightarrow \cdots, \tag{A.2.1}$$

where $\alpha : H^{k-1}(X,\mathbb{Z}) \to H^{k+1}(X,\mathbb{Z})$ is defined on forms $\omega \in H^{k-1}(X,\mathbb{Z})$ as the cup product $\alpha(\omega) = \chi(P) \cup \omega$ with the Chern class of the associated line bundle $P \times_{\mathbb{S}^1} \mathbb{C}$.

The above sequence is called the *Gysin sequence* for the principal circle bundle $\pi : P \to X$. It has the important feature of relating the cohomology groups of the total space and of the base space of the bundle $\pi : P \to X$. Furthermore, it has some notable applications in mathematical physics, as we described in the Introduction.

A.2.2 The Gysin sequence in K- theory

The Gysin exact sequence (A.2.1) admits a version in topological K-theory. In this section we simply follow Karoubi's book [49], simplifying the construction by focusing on the case of line bundles and circle bundles. Let $\pi : L \to X$ be a complex line bundle; we assume the base space X to be compact.

Let us choose a metric on L, which is always possible by the existence of partitions of unity.

We write $B(L)$ for the *ball bundle* of L, the bundle over X whose fibre $B(L)_x$ at the point $x \in X$ is the closed unit ball of the fibre L_x of L. Similarly we write $S(L)$ for the *sphere bundle* of L, whose fibre $S(L)_x$ at $x \in \mathbb{CP}^n$ is the unit circle in the fibre L_x. Then $B(L) - S(L)$ denotes the *open ball bundle*.

Let now $K^*(B(L), S(L))$ denote the relative K-theory groups. One has a six term exact sequence in topological K-theory [49, IV.1.13]:

$$\begin{array}{ccccc} K^0(B(L),S(L)) & \longrightarrow & K^0(B(L)) & \longrightarrow & K^0(S(L)) \\ \partial\uparrow & & & & \downarrow\partial \\ K^1(S(L)) & \longleftarrow & K^1(B(L)) & \longleftarrow & K^1(B(L),S(L)). \end{array} \tag{A.2.2}$$

Here π'^* is the map induced by the projection $\pi' : S(L) \to X$ and the vertical arrows are *connecting homomorphisms*. The map α is simply the product with the Euler class $\chi(L)$ of the line bundle L, defined in [49, Section V.3].

Since $S(L)$ is closed in $B(L)$, by [49, Proposition II.5.19] we have isomorphisms

$$K^*(L) \simeq K^*(B(L) - S(L)) \simeq K^*(B(L), S(L)).$$

Moreover, $B(L)$ has the same homotopy type of X, via the inclusion of the latter into $B(L)$ determined by the zero section of L.

If $P \to X$ is a principal circle bundle with associated line bundle $L \to X$, upon identification of $S(L)$ with the total space P, the exact sequence becomes

$$\begin{array}{ccccc} K^0(X) & \longrightarrow & K^0(X) & \longrightarrow & K^0(P) \\ {\scriptstyle\partial}\uparrow & & & & \downarrow{\scriptstyle\partial} \\ K^1(P) & \longleftarrow & K^1(X) & \longleftarrow & K^1(X). \end{array} \tag{A.2.3}$$

Example A.4 (The Gysin sequence for the complex projective space). Let $\mathbb{CP}^n$ denote the complex projective space of $\mathbb{C}^{n+1}$ and let L be a complex line bundle over $\mathbb{CP}^n$ equipped with a Hermitian fibre metric. We write again $B(L)$ for the ball bundle and $S(L) \simeq P$ for the sphere bundle of L, with $B(L) - S(L)$ denoting the open ball bundle.

Using the vanishing $K^1(\mathbb{CP}^n) = 0$ (*cf.* [49, Cor. IV.2.8]), the sequence A.2.2 transforms into the K-theoretic *Gysin sequence* for the bundle $S(L)$:

$$0 \longrightarrow K^1(S(L)) \xrightarrow{\partial} K^0(\mathbb{CP}^n) \xrightarrow{\alpha} K^0(\mathbb{CP}^n) \xrightarrow{\pi^*} K^0(S(L)) \longrightarrow 0\,. \tag{A.2.4}$$

Now let L be the tautological line bundle over $\mathbb{CP}^n$, whose total space is $\mathbb{C}^{n+1}$ and whose fibre L_x at $x \in \mathbb{CP}^n$ is the one-dimensional complex vector subspace of $\mathbb{C}^{n+1}$ which defines that point. Via the associated bundle construction, the bundle L may be identified with the quotient of $\mathbb{S}^{2n+1} \times \mathbb{C}$ by the equivalence relation

$$(x, t) \sim (\lambda x, \lambda^{-1} t), \qquad \lambda \in \mathbb{S}^1 \subseteq \mathbb{C}.$$

Similarly, its d-th tensor power $L^{\otimes d}$ may be identified with the quotient of $\mathbb{S}^{2n+1} \times \mathbb{C}$ by the equivalence relation $(x, t) \sim (\lambda x, \lambda^{-d} t)$. Moreover, $L^{\otimes d}$ can be given the fibre metric defined by $\varphi\left((x, t'), (x, t)\right) = t'\bar{t}$. It follows that the sphere bundle $S(L^{\otimes d})$ can be identified with the lens space $L^{2n+1}(d) := \mathbb{S}^{2n+1}/\mathbb{Z}_d$ (where the cyclic group $\mathbb{Z}_d$ of order d acts upon the sphere $\mathbb{S}^{2n+1}$ via the d-th roots of unity) by the map $(x, t) \mapsto \sqrt[d]{t} \cdot x$.

Taking $L^{\otimes d}$ in the above sequence (A.2.4) one finds, just as in [49, IV.1.14], the K-theoretic *Gysin sequence* for the lens space $L^{2n+1}(d)$:

$$0 \longrightarrow K^1(L^{2n+1}(d)) \xrightarrow{\delta_{10}} K^0(\mathbb{CP}^n) \xrightarrow{\alpha} K^0(\mathbb{CP}^n) \xrightarrow{\pi^*} K^0(L^{2n+1}(d)) \longrightarrow 0\,. \tag{A.2.5}$$

Here, since $L^{\otimes d}$ is a line bundle, its Euler class (giving the map α) is given simply by

$$\chi(L^{\otimes d}) := 1 - [L^{\otimes d}]\,.$$

Appendix B

The mapping cone exact sequence and Pimsner algebras

This appendix is devoted to an alternative proof of the exactness of the Gysin sequence for the quantum lens spaces $C(L_q^{2n+1}(d))$. In [3] the Gysin sequence was constructed explicitly by providing the maps appearing in it. Our strategy to prove exactness was to relate the sequence to a six-term exact sequence in K-theory coming from the mapping cone of the inclusion $B \hookrightarrow A$, with $A = C(L_q^{2n+1}(d))$ being the C^*-algebra of continuous functions on the quantum lens space and $B = C(\mathbb{CP}_q^n)$.

In this appendix we show that the Gysin exact sequence described in 5.2.3 and the mapping cone exact sequence associated to the inclusion of $i : C(\mathbb{CP}_q^n) \hookrightarrow C(L_q^{2n+1}(d))$ fit into a nice commutative diagram.

B.1 K-theory of the mapping cone

Let $i : B \hookrightarrow A$ an inclusion of C^*-algebras. We define that the *mapping cone* $M(B, A)$ of the pair (B, A) to be the mapping cone C_i of the inclusion, as in Definition 2.4.13. It is easy to see that

$$M(B, A) \simeq \{f \in C([0, 1], A) \mid f(0) = 0,\ f(1) \in B\}.$$

The group $K_0(M(B, A))$ has a particularly elegant description in terms of partial isometries. Indeed, let us write $V_m(B, A)$ for the set of partial isometries $v \in \mathrm{M}_m(A)$ such that the associated projections v^*v and vv^* belong to $\mathrm{M}_m(B)$. Using the inclusion $V_m(B, A) \hookrightarrow V_{m+1}(B, A)$ given by setting $v \mapsto v \oplus 0$, one defines

$$V(B, A) := \bigcup_m V_m(B, A)$$

and then generates an equivalence relation $\sim$ on $V(B, A)$ by declaring that:

1. $v \sim v \oplus p$ for all $v \in V(B, A)$ and $p \in \mathrm{M}_m(B)$;

2. if $v(t)$, $t \in [0, 1]$, is a continuous path in $V(B, A)$ then $v(0) \sim v(1)$.

Following [69, Lemma 2.5], there is a well-defined bijection between $V(B,A)/\!\sim$ and the K-theory $K_0(M(B,A))$. It is also shown there that if v and w are partial isometries with the same image in $K_0(M(B,A))$ one can arrange them to have the same initial projection, i.e. $v^*v = w^*w$, without changing their class in $V(B,A)/\!\sim$. Having done so, with the same reasoning as in [17, Lemma 3.3], one defines an addition in $V(B,A)/\!\sim$ by $[v \oplus w^*] = [v] + [w^*] = [v] - [w] = [vw^*]$ so that $V(B,A)/\!\sim$ and $K_0(M(B,A))$ are isomorphic as Abelian groups.

The class of the unbounded Kasparov module $(\mathcal{E}_\infty, \mathcal{N})$ in $KK_1(A,B)$ described in Remark 5.2.1, has a canonical lift to the group $KK_0(M(B,A),B)$. Let P be as before, the spectral projection for $\mathcal{N}$ corresponding to the non-negative real axis. Following the same reasoning as in Section 4 of [17], we write

$$T_\pm := \pm\partial_t \otimes 1 + 1 \otimes \mathcal{N}$$

for the unbounded operators with domains

$$\begin{aligned} \mathfrak{Dom}(T_\pm) : = \Big\{ f \in C_c^\infty([0,\infty)) \otimes \mathfrak{Dom}(\mathcal{N}) \mid f = \sum_{i=1}^n f_i \otimes x_i,\ x_i \in \mathfrak{Dom}(\mathcal{N}), \Big\} \\ \text{and} \qquad P(f(0)) = 0 \ (+ \text{ case}), \quad (1-P)(f(0)) = 0 \ (- \text{ case}), \end{aligned}$$

where smoothness at the boundary of $[0,\infty)$ is defined by taking one-sided limits. With $Y := L^2([0,\infty)) \otimes \mathcal{E}_\infty$, one finds that

$$\widehat{\mathcal{N}} : \mathfrak{Dom}(T_+) \oplus \mathfrak{Dom}(T_-) \to Y \oplus Y, \qquad \widehat{\mathcal{N}} := \begin{pmatrix} 0 & T_- \\ T_+ & 0 \end{pmatrix},$$

is a densely defined unbounded symmetric linear operator. By modifying the domains slightly, one obtains a $\mathbb{Z}_2$-graded Hilbert $M(B,A)$-B-bimodule $\widehat{\mathcal{E}}$ endowed with an odd unbounded linear operator $\widehat{\mathcal{N}} : \mathfrak{Dom}(\widehat{\mathcal{N}}) \to \widehat{\mathcal{E}}$ which in addition is self-adjoint and regular by [17, Proposition 4.13]. It then follows from [17, Proposition 4.14] that the pair $(\widehat{\mathcal{E}}, \widehat{\mathcal{N}})$ determines a class in the bivariant K-theory $KK_0(M(B,A),B)$.

The internal Kasparov product of $K_0(M(B,A))$ with the class of $(\widehat{\mathcal{E}}, \widehat{\mathcal{N}})$ yields a map

$$\mathrm{Ind}_{\widehat{\mathcal{N}}} : K_0(M(B,A)) \to K_0(B). \tag{B.1.1}$$

Following [17, Theorem 5.1] , the internal Kasparov product of the K-theory $K_0(M(B,A))$ with the class of $(\widehat{X}, \widehat{\mathfrak{D}})$ in the bivariant K-theory $KK_0(M(B,A),B)$ is represented by the index

$$\mathrm{Ind}_{\widehat{\mathcal{N}}}([v]) := \mathrm{Ker}(PvP)|_{v^*vP\mathcal{E}_\infty^m} - \mathrm{Ker}(Pv^*P)|_{vv^*P\mathcal{E}_\infty^m},$$

the result being an element of $KK_0(\mathbb{C},B) = K_0(B)$. Here $v \in \mathrm{M}_m(A)$ is a partial isometry representing a class in $K_0(M(B,A))$ and considered as a map

$$v : v^*vP\mathcal{E}_\infty^m \to vv^*P\mathcal{E}_\infty^m.$$

B.1.1 The mapping cone exact sequence

The extension

$$0 \to SA \xrightarrow{\iota} M(B,A) \xrightarrow{\mathrm{ev}} B \to 0, \tag{B.1.2}$$

where $\iota(f \otimes a)(t) := f(t)a$ and $\mathrm{ev}(f) := f(1)$, admits a completely positive cross section, yielding six term exact sequence in KK-theory. We will concentrate on the one in K-theory, which has the form

$$\begin{array}{ccccc} K_0(SA) & \xrightarrow{\iota_*} & K_0(M(B,A)) & \xrightarrow{\mathrm{ev}_*} & K_0(B) \\ \uparrow{\scriptstyle \partial'} & & & & \downarrow{\scriptstyle \partial'} \\ K_1(B) & \xleftarrow[\mathrm{ev}_*]{} & K_1(M(B,A)) & \xleftarrow[\iota_*]{} & K_1(SA). \end{array} \tag{B.1.3}$$

Using the vanishing of $K_1(B)$, the sequence (B.1.3) degenerates to

$$0 \longrightarrow K_0(SA) \xrightarrow{\iota_*} K_0(M(B,A)) \xrightarrow{\mathrm{ev}_*} K_0(B) \to \tag{B.1.4}$$

$$\xrightarrow{\partial'} K_1(SA) \longrightarrow K_1(M(B,A)) \longrightarrow 0\,.$$

Where $\iota_* : K_0(S)) \to K_0(M(B,A))$. is the map in K-theory induced by the inclusion $\iota : SA \to M(B,A)$, and ev_* in (B.1.3) can be given by

$$\mathrm{ev}_* : K_0(M(B,A)) \to K_0(B), \qquad \mathrm{ev}_*([v]) := [v^*v] - [vv^*],$$

for $v \in \mathrm{M}_m(A)$ a partial isometry representing a class in $K_0(M(B,A))$ (*cf.* [69, Lemma 2.3]).

The boundary map ∂' is defined as in [44, Remark 4.9.3]: for $[p]-[q] \in K_0(B)$ one chooses representatives p,q over B and, from these, self-adjoint lifts x,y over $M(B,A)$. Then the exponentials $\mathrm{e}^{2\pi i x}$ and $\mathrm{e}^{2\pi i y}$ are unitaries over $C(S^1) \otimes A$ which are equal to the identity modulo $C_0((0,1)) \otimes A$, so one defines

$$\partial'([p]-[q]) := [\mathrm{e}^{2\pi i x}] - [\mathrm{e}^{2\pi i y}] \in K_1(SA). \tag{B.1.5}$$

B.1.2 Exactness of the Gysin sequence

Recall that A is a Cuntz-Krieger algebra associated to a graph which is connected, row-finite and has neither sources nor sinks [46]. It follows [17, Lemma 6.7] that $K_1(M(B,A)) = 0$ and that the index map (B.1.1) is an isomorphism [17, Proposition 6.8]. Thus

$$K_0(M(B,A)) \simeq K_0(B) \simeq \mathbb{Z}^{n+1}, \tag{B.1.6}$$

where the second isomorphism is the result of Proposition 5.2.2, since $B = C(\mathbb{CP}^n_q)$.

As a consequence, there is a very easy description of the partial isometries which generate $K_0(M(B,A))$. Recall from the discussion at the end of Subsection 5.2.1 that, upon pulling-back to A, one finds that for all $k \in \mathbb{Z}$ the projections $\mathbf{p}_{dk}$ become equivalent to the identity. This is equivalent to saying that for any $m \in \mathbb{Z}$ the projections $\mathbf{p}_{dk}$ and $\mathbf{p}_{d(k+m)}$ are equivalent for all $k \in \mathbb{Z}$. Indeed, one can explicitly exhibit partial isometries relating these projectors. Taking the particular case $m = 1$, these partial isometries are the elements $v_N \in \mathrm{M}_{(\mathrm{d}_{d(k+1)}, \mathrm{d}_{dk})}(A)$, with the integers $\mathrm{d}_{(\cdot)}$ as in (5.1.5), given by

$$v_k = \Psi_{d(k+1)} \Psi_{dk}^{\dagger}, \qquad k = 0, -1, \dots, -n; \tag{B.1.7}$$

clearly $v_k^* v_k = \mathbf{p}_{dk}$ and $v_k v_k^* = \mathbf{p}_{d(k+1)}$ for $k = 0, -1, \dots, -n$. With our conventions, the entries of v_k are elements of A homogeneous of degree $-d$ for the action of $\widetilde{\mathrm{U}}(1)$.

Proposition B.1.1 ([3, Proposition 5.2]). *The partial isometries* (B.1.7) *form a basis of* $K_0(M(B,A))$.

Proof. From (B.1.6) we just need $n+1$ independent generators. Now, since the map (B.1.1) is an isomorphism, the partial isometries v_N are independent (and thus a basis for $K_0(M(B,A))$ if and only if the classes $\mathrm{Ind}_{\widehat{N}}([v_k])$ are so. Since $\mathbf{p} v_k \mathbf{p}$ is essentially a ‘left degree shift’ operator on the elements of non-negative homogeneous degree in $\mathbf{p}\mathcal{E}_\infty^{\mathrm{d}_{dk}}$ it has no cokernel. Its kernel thus determines the index:

$$\mathrm{Ind}_{\widehat{N}}([v_k]) = [\mathbf{p}_{dk}].$$

Now, it follows from Proposition 5.2.2 that the matrix of pairings $\{\langle [\mu_m], [\mathbf{p}_{-dk}] \rangle = \binom{dk}{m}\}$ is invertible, thus proving that the elements $\mathbf{p}_{dk}$ for $k = 0, -1, \dots, -n$ are independent. We note that these projections do not form a basis for $K_0(B)$: the matrix of pairings (while invertible over $\mathbb{Q}$) is not invertible over $\mathbb{Z}$, that is it does not belong to $\mathrm{GL}_{n+1}(\mathbb{Z})$. □

Finally we introduce a pair of maps

$$B_B : K_0(B) \to K_0(B), \qquad B_A : K_1(SA) \to K_0(A).$$

The former is defined simply by the multiplication

$$B_B([p] - [q]) := -[\mathbf{p}_{-d}]([p] - [q])$$

in $K_0(B)$. The latter map B_A is the inverse of the Bott isomorphism

$$\beta_A : K_0(A) \to K_1(SA)$$

of Theorem 1.3.5.

We are ready to state and prove the central result of this appendix:

Theorem B.1.2 ([3, Theorem 5.3])**.** *There is a diagram*

$$\begin{CD} 0 @>>> K_1(A) @>{\iota_*}>> K_0(M(B,A)) @>{\mathrm{ev}_*}>> K_0(B) @>{\partial'}>> K_1(SA) @>>> 0 \\ @. @VV{\mathrm{id}}V @VV{\mathrm{Ind}_{\widehat{\mathcal{N}}}}V @VV{B_B}V @VV{B_A}V \\ 0 @>>> K_1(A) @>{[\partial]}>> K_0(B) @>{1-[\text{Ł}^{(d)}]}>> K_0(B) @>{j_*}>> K_0(A) @>>> 0 \end{CD} \tag{B.1.8}$$

in which every square commutes and each vertical arrow is an isomorphism of groups.

Proof. Upon using the isomorphism $K_1(A) \simeq K_0(SA)$, that the first square commutes is precisely [17, Theorem 5.1]. For the second square we explicitly compute that for each $k = 0, -1, \ldots, -n$, one has

$$\alpha\left(\mathrm{Ind}_{\widehat{\mathcal{N}}}([v_k])\right) = (1 - [\mathbf{p}_{-d}])[\mathbf{p}_{dk}] = -[\mathbf{p}_{-d}]([\mathbf{p}_{dk}] - [\mathbf{p}_{d(k+1)}]) = B_B(\mathrm{ev}_*([v_k])).$$

For the third square, we argue as in [17, Lemma 3.1]. Recall that in defining the map (B.1.5) we chose self-adjoint lifts x, y over $M(B, A)$. We choose here in particular the lifts $x := t \otimes j(p)$ and $y := t \otimes j(q)$. These are both self-adjoint and vanish at $t = 0$; at $t = 1$ they are matrices over B. It follows that

$$[\mathrm{e}^{2\pi i x}] - [\mathrm{e}^{2\pi i y}] = [\mathrm{e}^{2\pi i (t\otimes p)}] - [\mathrm{e}^{2\pi i (t\otimes q)}] = -\mathrm{Bott}([p] - [q]) \in K_1(SA).$$

Thus it follows that, modulo the isomorphism $\mathrm{Bott} : K_0(A) \to K_1(SA)$, we have

$$\partial'([p] - [q]) = -([j(p)] - [j(q)]), \tag{B.1.9}$$

i.e. that ∂' is induced up to Bott peridocity by minus the algebra inclusion $j : B \to A$. Now using the fact that the image of the class of $\mathbf{p}_{-d}$ in $K_0(A)$ along $j : B \to A$ is trivial, the above (B.1.9) may in fact be written

$$\partial'([p] - [q]) = -[j(\mathbf{p}_{-d})]([j(p)] - [j(q)]),$$

up to Bott periodicity, from which the result follows. ☐

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