Noncommutative circle bundles, Pimsner Algebras and Gysin Sequences

Francesca Arici

Advances in Noncommutative Geometry
University Paris Diderot
April 21, 2015
The Gysin Sequence for Quantum Lens Spaces
F. Arici, S. Brain, G. Landi
arXiv:1401.6788 [math.QA], to appear in JNCG;

Pimsner Algebras and Gysin Sequences from Principal Circle Actions
F. Arici, J. Kaad, G. Landi
arXiv:1409.5335 [math.QA], to appear in JNCG;

Principal Circle Bundles and Pimsner Algebras
F. Arici, F. D’Andrea, G. Landi
in preparation.
1 Motivation

2 Quantum principal $U(1)$-bundles

3 Pimsner algebras

4 Gysin Sequences

5 Applications

6 Conclusions
Principal circle bundles are a natural framework for many problems in mathematical physics:

- U(1)-gauge theory;
- T-duality;
- Chern Simons field theories.
Principal circle bundles are a natural framework for many problems in mathematical physics:

- U(1)-gauge theory;
- T-duality;
- Chern Simons field theories.

The Gysin sequence: long exact sequence in cohomology for any sphere bundle. In particular, for a principal circle bundle: $\mathbb{U}(1) \hookrightarrow P \overset{\pi}{\longrightarrow} X$.

$$\cdots \longrightarrow H^k(P) \overset{\pi^*}{\longrightarrow} H^{k-1}(X) \overset{\text{eU}}{\longrightarrow} H^{k+1}(X) \overset{\pi^*}{\longrightarrow} H^{k+1}(P) \longrightarrow \cdots$$
In K-theory, the Gysin sequence becomes a cyclic six term exact sequence:

\[
\begin{array}{cccc}
K_0(X) & \xrightarrow{\alpha} & K_0(X) & \xrightarrow{\pi^*} K_0(P) \\
\partial & \uparrow & \partial & \\
K_1(P) & \xleftarrow{\pi^*} & K_1(X) & \xleftarrow{\alpha} K_1(X)
\end{array}
\]
In $K$-theory, the Gysin sequence becomes a cyclic six term exact sequence:

\[
\begin{array}{cccccc}
K^0(X) & \xrightarrow{\alpha} & K^0(X) & \xrightarrow{\pi^*} & K^0(P) \\
\uparrow{[\partial]} & & & \downarrow{[\partial]} & & \\
K^1(P) & \xleftarrow{\pi^*} & K^1(X) & \xleftarrow{\alpha} & K^1(X)
\end{array}
\]
In $K$-theory, the Gysin sequence becomes a cyclic six term exact sequence:

\[
K^0(X) \xrightarrow{\alpha} K^0(X) \xrightarrow{\pi^*} K^0(P) \\
\begin{array}{c}
[\partial] \\
\downarrow \\
K^1(P) \xleftarrow{\pi^*} K^1(X) \xleftarrow{\alpha} K^1(X)
\end{array}
\]

(1)

where $\alpha$ is the multiplication by the Euler class

\[
\chi(L) = 1 - [L]
\]

(2)

of the line bundle $L \to X$ with associated circle bundle $\pi : P \to X$.  

5/38 Noncommutative circle bundles, Pimsner Algebras and Gysin Sequences Francesca Arici
In $K$-theory, the Gysin sequence becomes a cyclic six term exact sequence:

$$
\begin{align*}
K^0(X) & \xrightarrow{\alpha} K^0(X) \xrightarrow{\pi^*} K^0(P) \\
\uparrow{[\partial]} & \quad & \downarrow{[\partial]} \\
K^1(P) & \xleftarrow{\pi^*} K^1(X) & K^1(X) \xleftarrow{\alpha}
\end{align*}
$$

(1)

where $\alpha$ is the multiplication by the Euler class

$$
\chi(\mathcal{L}) = 1 - [\mathcal{L}]
$$

(2)

of the line bundle $\mathcal{L} \to X$ with associated circle bundle $\pi : P \to X$. 

\hspace{1em}
1 Motivation

2 Quantum principal $U(1)$-bundles

3 Pimsner algebras

4 Gysin Sequences

5 Applications

6 Conclusions
As structure group we consider the Hopf algebra

\[ O(U(1)) := \mathbb{C}[z, z^{-1}]/\langle 1 - zz^{-1} \rangle. \]
As structure group we consider the Hopf algebra

\[ \mathcal{O}(U(1)) := \mathbb{C}[z, z^{-1}] / \langle 1 - zz^{-1} \rangle. \]

Let \( \mathcal{A} \) be a complex unital algebra that it is a right comodule algebra over \( \mathcal{O}(U(1)) \), i.e we have a homomorphism of unital algebras

\[ \Delta_R : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{O}(U(1)). \]
As structure group we consider the Hopf algebra

\[ \mathcal{O}(U(1)) := \mathbb{C}[z, z^{-1}]/\langle 1 - zz^{-1} \rangle. \]

Let \( \mathcal{A} \) be a complex unital algebra that it is a right comodule algebra over \( \mathcal{O}(U(1)) \), i.e. we have a homomorphism of unital algebras

\[ \Delta_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}(U(1)). \]

We will denote by

\[ \mathcal{B} := \{ x \in \mathcal{A} \mid \Delta_R(x) = x \otimes 1 \} \]

the unital subalgebra of coinvariant elements for the coaction.
Definition

One says that the datum \((\mathcal{A}, \mathcal{O}(U(1)), \mathcal{B})\) is a quantum principal \(U(1)\)-bundle when the canonical map

\[
\chi : \mathcal{A} \otimes \mathcal{B} \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}(U(1)), \quad x \otimes y \mapsto x \cdot \Delta_R(y),
\]

is an isomorphism.
Definition

One says that the datum \((\mathcal{A}, \mathcal{O}(U(1)), \mathcal{B})\) is a *quantum principal U(1)-bundle* when the *canonical map*

\[
\chi : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}(U(1)) , \quad x \otimes y \mapsto x \cdot \Delta_R(y),
\]

is an isomorphism.

Examples of quantum principal U(1)-bundles: quantum spheres and lens spaces over quantum projective spaces (both \(\theta\) and \(q\)-deformations).

Graded algebra structure: the coordinate algebra decomposes as a direct sum of line bundles over \(\mathcal{B}\).
Let $\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n$ be a $\mathbb{Z}$-graded unital algebra and let $\mathcal{O}(U(1))$ as before. The unital algebra homomorphism

$$\Delta_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}(U(1)) \quad x \mapsto x \otimes z^{-n}, \text{ for } x \in \mathcal{A}_n.$$ 

turns $\mathcal{A}$ into a right comodule algebra over $\mathcal{O}(U(1))$. The subalgebra of coinvariant elements coincides with $\mathcal{A}_0$. 
Let \( \mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n \) be a \( \mathbb{Z} \)-graded unital algebra and let \( \mathcal{O}(U(1)) \) as before. The unital algebra homomorphism

\[
\Delta_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}(U(1)) \quad x \mapsto x \otimes z^{-n}, \quad \text{for } x \in \mathcal{A}_n.
\]

turns \( \mathcal{A} \) into a right comodule algebra over \( \mathcal{O}(U(1)) \).

The subalgebra of coinvariant elements coincides with \( \mathcal{A}_0 \).

Question: when is a graded algebra a principal circle bundle?
Definition

Let $\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n$ a $\mathbb{Z}$-graded algebra. $\mathcal{A}$ is *strongly graded* if and only if any of the following equivalent conditions is satisfied.

1. For all $n, m \in \mathbb{Z}$ we have $\mathcal{A}_n \mathcal{A}_m = \mathcal{A}_{n+m}$.

2. For all $n \in \mathbb{Z}$ we have $\mathcal{A}_n \mathcal{A}_{-n} = \mathcal{A}_0$.

3. $\mathcal{A}_1 \mathcal{A}_{-1} = \mathcal{A}_0 = \mathcal{A}_{-1} \mathcal{A}_1$. 


Noncommutative circle bundles, Pimsner Algebras and Gysin Sequences
Definition

Let $\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} A_n$ a $\mathbb{Z}$-graded algebra. $\mathcal{A}$ is strongly graded if and only if any of the following equivalent conditions is satisfied.

1. For all $n, m \in \mathbb{Z}$ we have $A_n A_m = A_{n+m}$.
2. For all $n \in \mathbb{Z}$ we have $A_n A_{-n} = A_0$.
3. $A_1 A_{-1} = A_0 = A_{-1} A_1$.

**strong grading $\iff$ principal action**
To prove bijectivity of $\chi$, one has to construct sequences

$$\{\xi_j\}_{j=1}^N, \{\beta_i\}_{i=1}^M \text{ in } A_1 \quad \text{and} \quad \{\eta_j\}_{j=1}^N, \{\alpha_i\}_{i=1}^M \text{ in } A_{-1}$$

with the property that

$$\sum_{j=1}^N \xi_j \eta_j = 1_A = \sum_{i=1}^M \alpha_i \beta_i.$$
To prove bijectivity of $\chi$, one has to construct sequences

$$\{\xi_j\}_{j=1}^N, \{\beta_i\}_{i=1}^M \text{ in } A_1 \quad \text{and} \quad \{\eta_j\}_{j=1}^N, \{\alpha_i\}_{i=1}^M \text{ in } A_{-1}$$

with the property that

$$\sum_{j=1}^N \xi_j\eta_j = 1_A = \sum_{i=1}^M \alpha_i\beta_i.$$

This means that the modules $A_1$ and $A_{-1}$ are finetely generated projective.

Indeed, we construct idempotents

$$\Phi_1 : A_1 \to (A_0)^N \quad \Psi_1 : (A_0)^N \to A_1$$

$$\Phi_{-1} : A_{-1} \to (A_0)^N \quad \Psi_{-1} : (A_0)^N \to A_1$$

with $\Psi_1\Phi_1 = \text{Id}_{A_1}$ and $\Psi_{-1}\Phi_{-1} = \text{Id}_{A_{-1}}$. 
The module $\mathcal{A}_1$ and its inverse $\mathcal{A}_{-1}$ play a crucial role. They can be thought of as modules of sections of line bundles. This phenomenon is related to a natural construction: Pimsner algebras.
<table>
<thead>
<tr>
<th></th>
<th>Motivation</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Quantum principal U(1)-bundles</td>
</tr>
<tr>
<td>3</td>
<td>Pimsner algebras</td>
</tr>
<tr>
<td>4</td>
<td>Gysin Sequences</td>
</tr>
<tr>
<td>5</td>
<td>Applications</td>
</tr>
<tr>
<td>6</td>
<td>Conclusions</td>
</tr>
</tbody>
</table>
Noncommutative line bundles

**Definition**

A *self Morita equivalence bimodule (SMEB)* over $B$ is a pair $(E, \phi)$ where $E$ is a full right Hilbert $C^*$-module over $B$ and

$$\phi : B \to \mathcal{K}(E)$$

is an isomorphism.

Example: $A = C(X)$ and $E = \Gamma(\mathcal{L})$ the module of sections of a Hermitian line bundle $\mathcal{L} \to X$. 
The $C^*$-algebraic dual

$$E^* := \{ \lambda_{\xi}, \xi \in E \mid \lambda_{\xi}(\eta) = \langle \xi, \eta \rangle \} \subseteq \text{Hom}^*_B(E, B)$$

can be given the structure of a (right) Hilbert $C^*$-module over $B$ using $\phi$, with right action

$$\lambda_{\xi} b := \lambda_{\xi} \phi(b),$$

and inner product on $E^*$ is given by

$$\langle \lambda_{\xi}, \lambda_{\eta} \rangle := \phi^{-1}(|\xi\rangle\langle\eta|).$$

If we define $\phi^*$ as

$$\phi^*(a)(\lambda_{\xi}) := \lambda_{\xi} a^*,$$

the pair $(\phi^*, E^*)$ gives an isomorphism $\phi^* : B \to \mathcal{K}(E^*)$. 
We can take interior tensor product modules, that we will denote using
with
\[
E^{(n)} := \begin{cases} 
E \hat{\otimes} \phi^n & n > 0 \\
B & n = 0 \\
(E^*) \hat{\otimes} \phi_*^{-n} & n < 0
\end{cases} .
\]

Out of these we construct the Hilbert module

\[
\mathcal{E}_\infty := \bigoplus_{n \in \mathbb{Z}} E^{(n)}
\]

on which we will represent the Pimsner algebra.
We have natural creation and annihilation operators $S_\xi, S_\xi^* : \mathcal{E}_\infty \to \mathcal{E}_\infty$, defined at levels $1, 0, -1$ by

$$S_\xi(\eta) = \xi \otimes \eta$$
$$S_\xi(b) = \xi b$$
$$S_\xi(\lambda_\eta) = \phi^{-1}(\theta_{\xi, \eta})$$

$$S_\xi^*(\eta) = \langle \xi, \eta \rangle$$
$$S_\xi^*(b) = \lambda_\xi b$$
$$S_\xi^*(\lambda_\eta) = \lambda_\xi \otimes \lambda_\eta,$$

and extended on higher tensor powers.
**Definition**

The *Pimsner algebra* of the pair \((\phi, E)\), denoted \(\mathcal{O}_E\), is the smallest \(C^*\)-subalgebra of \(\text{End}^*_B(\mathcal{E}_\infty)\) which contains the operators \(S_\xi : \mathcal{E}_\infty \to \mathcal{E}_\infty\) for all \(\xi \in E\).

We have an inclusion \(\widehat{\phi} : \mathcal{O}_E \to \text{End}^*_B(\mathcal{E}_\infty)\)
Pimsner's Construction

**Definition**

The *Pimsner algebra* of the pair $(\phi, E)$, denoted $\mathcal{O}_E$, is the smallest $C^*$-subalgebra of $\text{End}_B^*(\mathcal{E}_\infty)$ which contains the operators $S_\xi : \mathcal{E}_\infty \to \mathcal{E}_\infty$ for all $\xi \in E$.

We have an inclusion $\hat{\phi} : \mathcal{O}_E \to \text{End}_B^*(\mathcal{E}_\infty)$.

The representation of $U(1)$ on $\mathcal{E}_\infty$ given by

$$t \circ x = t^n x \quad \forall t \in S^1, \ x \in E^{(n)}$$

induces an circle action on $\mathcal{O}_E$. 
Let $A$ be a $C^*$-algebra with an action $\{\sigma_z\}_{z \in S^1}$. When can we recover $A$ as a Pimsner algebra?

For each $n \in \mathbb{Z}$, one can define the spectral subspaces

$$A(n) := \{ \xi \in A \mid \sigma_z(\xi) = z^{-n} \xi \quad \text{for all} \ z \in S^1 \}.$$ 

It is easy to check that $A^*_n = A(-n)$ and that $A(n)A(m) \subseteq A(n+m)$. 
Definition

The action \( \sigma \) has *large spectral subspaces* if \( A^*_n A_n = A_0 \) for all \( n \in \mathbb{Z} \).

Note that \( \sigma \) has large spectral subspaces if and only if

\[
A^*_1 A_1 = A_0 = A_1 A^*_1. \tag{3}
\]
**Definition**

The action $\sigma$ has *large spectral subspaces* if $A^*_n A_n = A_0$ for all $n \in \mathbb{Z}$.

Note that $\sigma$ has large spectral subspaces if and only if

$$A^*_1 A_1 = A_0 = A_1 A^*_1.$$ (3)

**Theorem**

Let $\phi : A_0 \to \text{End}^*_{A_0}(A_1)$ simply defined by $\phi(a)(\xi) := a\xi$. Suppose that $A_1$ and $A_{-1}$ are full and countably generated over $A_0$.

Then the circle action $\{\sigma_z\}$ has large spectral subspaces.

Moreover, the Pimsner algebra $\mathcal{O}_{A_1}$ is isomorphic to $A$. 

---

**Noncommutative circle bundles, Pimsner Algebras and Gysin Sequences**

Francesca Arici
Proposition (Gabriel-Grensing)

Let $A$ be a unital, commutative $C^*$-algebra. Suppose that the first spectral subspace $E = A_{(1)}$ generates $A$ as a $C^*$-algebra, and that it is finitely generated projective over $B = A_{(0)}$.
Then the following facts hold

1. $B = C(X)$ for some compact space $X$;
2. $E = \Gamma(\mathcal{L})$ for some line bundle $\mathcal{L} \to X$;
3. $A = C(P)$, where $P \to X$ is the principal $S^1$ bundle over $X$ associated to the line bundle $\mathcal{L}$. 
1 Motivation

2 Quantum principal $U(1)$-bundles

3 Pimsner algebras

4 Gysin Sequences

5 Applications

6 Conclusions
Since $\phi : B \to \mathcal{K}(E)$, we have a well defined class

$$[E] := [(E, \phi, 0)] \in KK_0(B, B)$$ (4)
Since $\phi : B \to \mathcal{K}(E)$, we have a well defined class

$$[E] := [(E, \phi, 0)] \in KK_0(B, B)$$  \hfill (4)

Since $\widehat{\phi} : \mathcal{O}_E \to \text{End}^*_B(\mathcal{E}_\infty)$ is the inclusion, we have a class

$$[\partial] := [(\mathcal{E}_\infty, \widehat{\phi}, F)] \in KK_1(\mathcal{O}_E, B)$$  \hfill (5)
Since $\phi : B \rightarrow \mathcal{K}(E)$, we have a well defined class

$$[E] := [(E, \phi, 0)] \in KK_0(B, B)$$  \hspace{1cm} (4)

Since $\hat{\phi} : \mathcal{O}_E \rightarrow \text{End}_B^*(\mathcal{E}_\infty)$ is the inclusion, we have a class

$$[\partial] := [(\mathcal{E}_\infty, \hat{\phi}, F)] \in KK_1(\mathcal{O}_E, B)$$  \hspace{1cm} (5)

To define the operator $F$, let $P : \mathcal{E}_\infty \rightarrow \mathcal{E}_\infty$ denotes the orthogonal projection with

$$\text{Im}(P) = \bigoplus_{n=1}^{\infty} E^{(n)} \oplus B \subseteq \mathcal{E}_\infty,$$

and set $F := 2P - 1 \in \text{End}_B^*(\mathcal{E}_\infty)$. 
For any separable $C^*$-algebra $C$ the Kasparov product induces the group homomorphisms

$$[E] : KK_*(B, C) \to KK_*(B, C), \quad [E] : KK_*(C, B) \to KK_*(C, B)$$

and

$$[\partial] : KK_*(B, C) \to KK_{*+1}(\mathcal{O}_E, C), \quad [\partial] : KK_*(C, \mathcal{O}_E) \to KK_{*+1}(C, B),$$
For any separable $C^*$-algebra $C$ the Kasparov product induces the group homomorphisms

$$[E] : KK_*(B, C) \to KK_*(B, C), \quad [E] : KK_*(C, B) \to KK_*(C, B)$$

and

$$[\partial] : KK_*(B, C) \to KK_{*-1}(O_E, C), \quad [\partial] : KK_*(C, O_E) \to KK_{*-1}(C, B),$$

The inclusion $j : B \hookrightarrow O_E$ also induces maps in $KK$-theory.

$$j^* : KK_*(O_E, C) \to KK_*(B, C), \quad j_* : KK_*(C, B) \to KK_*(C, O_E),$$

We get two six term exact sequences.
In particular, for $\mathbb{C} = \mathbb{C}$ we get exact sequences in $K$-theory

$$
\begin{align*}
K_0(B) & \xrightarrow{1-[E]} K_0(B) \xrightarrow{j_*} K_0(O_E) \\
\uparrow{[\partial]} & \quad \quad \downarrow{[\partial]}, \\
K_1(O_E) & \leftarrow{j_*} K_1(B) \leftarrow 1-[E] K_1(B)
\end{align*}
$$

and in $K$-homology

$$
\begin{align*}
K^0(B) & \xleftarrow{1-[E]} K^0(B) \xleftarrow{j^*} K^0(O_E, C) \\
\downarrow{[\partial]} & \quad \quad \quad \uparrow{[\partial]}, \\
K^1(O_E) & \xrightarrow{j^*} K^1(B) \xrightarrow{1-[E]} K^1(B)
\end{align*}
$$
The previous sequences be interpreted as a *Gysin sequence* in K-theory and K-homology for the ‘line bundle’ \( E \) over the ‘noncommutative base space’ \( B \).

- Multiplication by the Euler class is replaced with the Kasparov product with \( 1 - [E] \).
1 Motivation

2 Quantum principal $U(1)$-bundles

3 Pimsner algebras

4 Gysin Sequences

5 Applications

6 Conclusions
The coordinate algebra $\mathcal{A}(S^2_{q}^{2n+1})$ of the quantum $S^2_{q}^{2n+1}$:

$\ast$-algebra generated by $2n + 2$ elements $\{z_i, z_i^*\}_{i=0, \ldots, n}$ s.t.:

\[
\begin{align*}
    z_i z_j &= q^{-1} z_j z_i & 0 \leq i < j \leq n, \\
    z_i^* z_j &= q z_j z_i^* & i \neq j, \\
    [z_n^*, z_n] &= 0, & [z_i^*, z_i] &= (1 - q^2) \sum_{j=i+1}^{n} z_j z_j^* & i = 0, \ldots, n-1, \\
    1 &= z_0 z_0^* + z_1 z_1^* + \ldots + z_n z_n^*.
\end{align*}
\]

(L. Vaksman, Ya. Soibelman)
U(1)-action on the algebra $\mathcal{A}(S_q^{2n+1})$:

$$(z_0, z_1, \ldots, z_n) \mapsto (\lambda z_0, \lambda z_1, \ldots, \lambda z_n), \quad \lambda \in U(1).$$

The coordinate algebra $\mathcal{A}(\mathbb{C}P_q^n)$ of the quantum projective space $\mathbb{C}P_q^n$ is the subalgebra of invariant elements. We have a decomposition

$$\mathcal{A}(S_q^{2n+1}) = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_k.$$

The $U(1)$-action restricts to an action of the finite cyclic group $\mathbb{Z}_r$.

$$\mathcal{A}(L_q^{(n,r)}) := \mathcal{A}(S_q^{2n+1})^{\mathbb{Z}_r}$$
We have a decomposition

$$\mathcal{A}(L_{q}^{(n,r)}) = \bigoplus_{k \in \mathbb{Z}} A_{rk}.$$  

The $C^*$-algebras $C(S^{2n+1}_q)$, $C(L_{q}^{(n,r)})$ and $C(\mathbb{CP}^n_q)$ of continuous functions: completions of $\mathcal{A}(S^{2n+1}_q)$, $\mathcal{A}(L_{q}^{(n,r)})$ and $\mathcal{A}(\mathbb{CP}^n_q)$ in the universal $C^*$-norms

Let $r \geq 1$, then

$$C(L_{q}^{(n,r)}) = \mathcal{O}_{E_{(r)}}$$

with $E_{(r)}$ the $r$-th spectral subspaces for the circle action on $C(S^{2n+1}_q)$. 
Since $K_1(\mathbb{CP}_q^n) = 0$, we can compute $K_0(L_q^{(n,r)})$ as the kernel of a matrix representing the multiplication map $1 - [E] : K_0(\mathbb{CP}_q^n) \rightarrow K_0(\mathbb{CP}_q^n)$.

This leads to

$$K_0(L_q^{(n,r)}) = \mathbb{Z} \oplus \mathbb{Z}/\alpha_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/\alpha_n \mathbb{Z} \quad K_1(L_q^{(n,r)}) = \mathbb{Z},$$

where the $\alpha_i$'s depend on the divisibility properties of the integer $r$.

Explicit algebraic generators.

Joint work with S. Brain and G. Landi.
$U(1)$-action on the algebra $\mathcal{A}(S^{2n+1}_q)$: for a weight vector $\ell = (\ell_0, \ldots, \ell_n)$

$$(z_0, z_1, \ldots, z_n) \mapsto (\lambda^{\ell_0} z_0, \lambda^{\ell_1} z_1, \ldots, \lambda^{\ell_n} z_n), \quad \lambda \in U(1).$$

The coordinate algebra $\mathcal{A}(\mathbb{W}^n_q(\ell))$ of the quantum projective space $\mathbb{W}^n_q(\ell)$ is the subalgebra of invariant elements.

The $C^*$-algebras $C(\mathbb{W}^n_q(\ell))$ of continuous functions: completion in the universal $C^*$-norm.
We focus on $n=1$: weighted projective line. $C(W_q(k,l))$ is the universal $\mathbb{C}^*$-algebra generated by the elements $z_0^l(z_1^*)^k$ and $z_1z_1^*$. Notice that it does not depend on $k$ and

$$K_0(C(W_q(k,l))) = \mathbb{Z}^{l+1}, \quad K_1(C(W_q(k,l))) = 0.$$ 

Define the $\mathbb{C}^*$-algebra of the weighted quantum lens spaces $L_q(dkl, k, l)$ as a Pimsner algebra

$$C(L_q(dkl, k, l)) := \mathcal{O}_{E_d}$$

for the $d$-th spectral subspace $E_{(d)}$ for the weighted $U(1)$-action on $S^3_q$. 

Quantum weighted lens spaces and projective spaces
We have a Gysin sequence in $K$-theory

$$0 \longrightarrow K_1(C(L_q(dkl, k, l))) \longrightarrow \mathbb{Z}^{l+1} \xrightarrow{1-M^d} \mathbb{Z}^{l+1} \longrightarrow K_0(C(L_q(dkl, k, l))) \longrightarrow 0$$

Where $M = \{M_{sr}\} \in M_{l+1}(\mathbb{Z})$ is a matrix of pairings between the $K$-theory and $K$-homology of $C(W_q(k, l))$.

We compute the $K$-theory groups as

$$K_1(C(L_q(dkl, k, l))) = \text{Ker}(1 - M^d) \quad K_0(C(L_q(dkl, k, l))) = \text{Coker}(1 - M^d)$$

Joint work with J. Kaad and G.Landi.
1 Motivation

2 Quantum principal $\text{U}(1)$-bundles

3 Pimsner algebras

4 Gysin Sequences

5 Applications

6 Conclusions
Summing up

- Quantum principal bundles are strongly graded algebras.
- Self Morita Equivalence are the $C^*$-algebraic version of line bundles.
- The corresponding Pimsner algebra $O_E$ is then the total space algebra of a principal circle bundle over $B$.
- Gysin-like sequences relates the KK-theories of $O_E$ and of $B$.
- Explicit computations and representatives.
- Rich class of examples.
- Still open: understand the structure of other principal bundles.
Thank you very much for your attention!
The Gysin Sequence for Quantum Lens Spaces
F. Arici, S. Brain, G. Landi
arXiv:1401.6788 [math.QA], to appear in JNCG;

Pimsner Algebras and Gysin Sequences from Principal Circle Actions
F. Arici, J. Kaad, G. Landi
arXiv:1409.5335 [math.QA], to appear in JNCG;

Principal Circle Bundles and Pimsner Algebras
F. Arici, F. D’Andrea, G. Landi
in preparation.