The Weyl Quantization: a brief introduction

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Contents

1 Motivations from classical and quantum mechanics

This section is mainly based on Chapter 2 and Chapter 3 of Hall's book *Quantum theory for mathematicians* [1].

1.1 Classical mechanics

1.1.1 Some conventions

In this section, we make some conventions beforehand:

- In classical mechanics, an observable is assumed to be a real-valued function $f \in C^\infty(\mathbb{R}^n)$, where $\mathbb{R}^n$ is the phase space;
- Newton's law in the case $\mathbb{R}^n$ is formulated as $m\ddot{x}(t) = F(x(t), \dot{x}(t))$, where F is the force.
- The force $F \in C^\infty(\mathbb{R}^n)$ is conserved i.e. the gradient of the potential energy V is the opposite of the force: $-\nabla V(x) = F(x)$.

1.1.2 Hamiltonian mechanics and Poisson bracket

In Hamiltonian mechanics, we think of the energy function as a function of position and momentum, rather than position and velocity. We refer to it as the Hamiltonian of the system.

If a particle in $\mathbb{R}^n$ has the energy defined as the sum of kinetic energy and potential energy $V(x)$, then the Hamiltonian is defined by

$$H(x,p) = \frac{1}{2m}\sum_j p_j + V(x)$$

From the Hamiltonian, we can obtain Newton's law:

$$\begin{cases} \frac{\partial H}{\partial p_j} = \frac{p_j}{m} = v_j = \frac{dx_j}{dt} \\ -\frac{\partial H}{\partial x_j} = -\frac{\partial V(x)}{\partial x_j} = F_j. \end{cases} \tag{1}$$

We call the equations (1) Hamilton's equation.

Definition 1 (Poisson bracket)**.** *Let f and g be two smooth functions on $\mathbb{R}^{2n}$, where an element of $\mathbb{R}^{2n}$ is thought of as a pair (x,p), with $x \in \mathbb{R}^{2n}$ representing the position of a particle and $p \in \mathbb{R}^n$ represents the momentum. Then the Poisson bracket of f and g, denoted $\{f,g\}$, is the function on $\mathbb{R}^{2n}$ given by*

$$\{f,g\} = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j}\right)$$

Remark 1. *Note that for the position function x_j and momentum function p_j we have*

$$\{x_i,x_j\} = 0 = \{p_i,p_j\}, \quad \{x_j,p_i\} = \delta_{i,j}$$

As we have seen in the last talk, the Poisson bracket with the first fixed input can be viewed as an operator of derivation, and the Poisson bracket satisfies the Jacobi identity:

Proposition 1 ([1, Proposition 2.23])**.** *For $f,g,h \in C^\infty(\mathbb{R}^{2n})$, we have*

- $\{f, g+\lambda h\} = \{f,g\} + \lambda\{f,h\}$ *for all* $\lambda \in \mathbb{R}$;
- $\{f,g\} = -\{g,f\}$;
- $\{f,gh\} = \{f,g\}h + g\{f,h\}$;
- $\{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\}$;

Proposition 2 ([1, Proposition 2.25])**.** *If $(x(t),p(t))$ is a solution to Hamilton's equation, then for any smooth function f on $\mathbb{R}^{2n}$ we have*

$$\frac{d}{dt}f(x(t),p(t)) = \{f,H\}(x(t),p(t))$$

Proof. Follows by using the chain rules and Hamilton's equations. ∎

Remark 2. *Observe that Proposition 2 includes Hamiltons equations themselves as special cases, by taking $f(x,p) = x_j$ and by taking $f(x,p) = p_j$.*

Definition 2 (Conserved quantity)**.** *We call a smooth function f on $\mathbb{R}^{2n}$ a conserved quantity if $f(x(t),p(t))$ is independent of t for each solution $(x(t),p(t))$ of Hamilton's equations.*

Then we have f is conserved if and only if $\{f,H\} = 0$. In particular the Hamiltonian H itself is a conserved quantity.

1.2 Quantum mechanics

There are two key ingredients to quantum theory, both of which arose from experiments.

The first ingredient is wave-particle duality which means that the objects are observed to have wavelike and particlelike behavior. From this aspect, the wave functions become the main character in quantum theory.

The second one is the probabilistic behavior which means that it is impossible to predict ahead of time what the precise outcome of an experiment will be. The best one can do is to predict a certain probability of the outcome of an experiment.

To combine the two ingredients, we consider a wave function $\psi : \mathbb{R} \to \mathbb{C}$, and let time t fixed. Then the function $|\psi(x)|^2$ is supposed to be the probability density for the position of the particle. That means the probability that the particle belongs to some set $E \subset \mathbb{R}$ is given by the quantity

$$\int_E |\psi(x)|^2 dx.$$

Here we actually require that $\int_{\mathbb{R}} |\psi(x)|^2 dx = 1$ i.e. ψ is a unit vector in $L^2(\mathbb{R})$. Therefore, we then have the expectation value of the position will be

$$E(x) = \int_{\mathbb{R}} x|\psi(x)|^2 dx, \tag{2}$$

provided the integral converges.

The basic idea in quantum theory is to express the expectation value using operators on some Hilbert space H and the inner product of H.

Let us consider the case of the position function. We set the underlying Hilbert space to be $L^2(\mathbb{R})$ and define the position operator $\hat{x} : \mathrm{Dom}(\hat{x}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$(\hat{x}\psi)(x) = x\psi(x),$$

and the expectation value of x is expressed in the inner product

$$E(x) = (\psi, \hat{x}\psi) = \int_{\mathbb{R}} x|\psi(x)|^2 dx, \tag{3}$$

which gives the expected value of the position of the particle as above equation (2).

The quantum version of the observable momentum function is motivated by the de Broglie hypothesis.

Proposition 3 (de Broglie hypothesis). *If the wave function of a particle has spatial frequency k, then the momentum p of the particle is*

$$p = \hbar k,$$

where $\hbar$ is the Planck's constant.

Indeed, the de Broglie hypothesis means that a wave function of the form e^{ikx} represents a particle with momentum $\hbar k$. As the function e^{ikx} is not integrable over $\mathbb{R}$, let us instead consider the domain as a circle. Hence the normalized wave function becomes $e^{ikx}/2\pi$. One can see immediately that $\{e^{ikx}/2\pi : k \in \mathbb{Z}\}$ forms an orthogonal basis of $L^2([0, 2\pi])$ and thus for any unit vector ψ in $L^2([0, 2\pi])$ we have

$$\psi(x) = \sum_k a_k e^{ikx}/2\pi, \text{ such that } \sum_i |a_k|^2 = 1.$$

For the linear combination of $e^{ikx}/2\pi$, the momentum is no longer definite. Instead, one should interpret it as follows: the particle has momentum $\hbar k$ with probability $|a_k|^2$. Therefore we have

$$E(p) = \sum_k \hbar k \cdot |a_k|^2.$$

In an analogy with equation (3), the momentum operator $\hat{p}$ must satisfy the following

$$E(p) = (\psi, \hat{p}\psi).$$

Hence the reasonable definition of the momentum operator is

$$\hat{p} = -i\hbar\frac{d}{dx}.$$

So we can formulate the following definitions.

Definition 3 (Position and momentum operator)**.** *The position operator X and momentum operator P are the operators defined as*

$$\hat{x} : Dom(\hat{x}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}), \psi(x) \mapsto x\psi(x)$$
$$\hat{p} : Dom(\hat{p}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}), \varphi \mapsto -i\hbar\frac{d\psi}{dx},$$

since the operators $\hat{x}$ and $\hat{p}$ are unbounded, we have to specify the domain of each. We call the operator $\hat{x}$ and $\hat{p}$ the quantization of observables x and p.

Remark 3. *A little calculation gives that*

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar I = -i\hbar\{x, p\}, \tag{4}$$

*which is called the **correspondence principle of quantum mechanics**.*

1.2.1 Axioms of quantum mechanics

Physicists stipulate the following axioms for quantum mechanics.

Axiom 1. *The state of the system is represented by a unit vector ψ in an appropriate Hilbert space H. If ψ_1 and ψ_2 are two unit vectors in H with $\psi_1 = c\psi_2$ for some constant $c \in \mathbb{C}$, then ψ_1 and ψ_2 represent the same physical state.*

Axiom 2. *To each real-valued function f on the classical phase space we associate a self-adjoint operator $\hat{f}$ (an operator A is self-adjoint if $A = A^*$, where A^* is the unique operator such that $(Ax, y) = (x, A^*y)$) on a Hilbert space H.*

Axiom 3. *If a quantum system is in a state described by a unit vector $\psi \in H$, the probability distribution for the measurement of some observable f satisfies*

$$E(f^m) = (\psi, (\hat{f})^m\psi).$$

In particular, the expectation value for a measurement of f is given by

$$E(f) = (\psi, \hat{f}\psi),$$

which we abbreviate as $\langle\hat{f}\rangle_\psi$

Axiom 4. *Suppose a quantum system is initially in a state ψ and that a measurement of an observable f is performed. If the result of the measurement is the number $\lambda \in \mathbb{R}$, then immediately after the measurement, the system will be in a state ψ' that satisfies*

$$\hat{f}\psi' = \lambda\psi'.$$

*The passage from ψ to ψ' is called the **collapse of the wave function**. Here $\hat{f}$ is the self-adjoint operator associated to f by Axiom 2.*

Motivated by the relation between the energy and temporal frequency ($E = \hbar\omega$, where ω is the temporal frequency) proposed by Planck, the last Axiom for quantum mechanics is the so-called Schrödinger equation.

Axiom 5. *The time-evolution of the wave function ψ in a quantum system is given by the Schrödinger equation,*

$$\frac{d\psi}{dt} = \frac{1}{i\hbar}\hat{H}\psi(t).$$

Here $\hat{H}$ is the operator corresponding to the classical Hamiltonian H by means of Axiom 2.

Finally, recall the Proposition 2 indicating the derivative of any observable f is equal to the Poisson bracket $\{f, H\}$. The corresponding result in quantum version is the following.

Proposition 4 ([1, Proposition 3.14])**.** *Suppose ψ is a solution of Schrödinger equation and A is a self-adjoint operator on H. Assuming certain domain conditions hold, we have*

$$\frac{d}{dt}\langle A\rangle_{\psi(t)} = \frac{d}{dt}(\psi(t), A\psi(t)) = (\psi(t), \frac{1}{i\hbar}[A, \hat{H}]\psi(t)) = \langle\frac{1}{i\hbar}[A, \hat{H}]\rangle_{\psi(t)}$$

2 The Weyl Quantization

This section is mainly based on [2, Chapter 4].

We have seen the "quantization" of position and momentum in the previous section. For an arbitrary observable, we shall use the Weyl quantization scheme.

2.1 Preliminaries

Definition 4 (Fourier transform)**.** *Let $f \in L^2(\mathbb{R}^n)$, the Fourier transform of f is*

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i\xi x} f(x)dx,$$

and the Fourier inverse transform is

$$\mathcal{F}^{-1}(f)(x) = \int_{\mathbb{R}^n} e^{2\pi i x\xi} f(\xi)d\xi.$$

Definition 5 (Schwartz class)**.** *The Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is the set of all smooth complex valued functions f on $\mathbb{R}^n$ such that for all $\alpha, \beta \in \mathbb{N}^n$ there are constants $C_{\alpha,\beta}$ such that*

$$|x^\alpha D_x^\beta f| \leq C_{\alpha,\beta}$$

for all $x \in \mathbb{R}^n$. This is equivalent to assuming there exist estimates of the form:

$$|D_x^\beta f(x)| \leq C_{m,\beta}(1 + |x|)^m \text{ for all } (m, \beta),$$

where $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ and $D_x^\beta f(x) = (\frac{\partial}{\partial x_1})^{\beta_1} \dots (\frac{\partial}{\partial x_n})^{\beta_n} f(x)$.

If we set the seminorm $p_{\alpha,\beta}(f) := \sup_{x\in\mathbb{R}^n} |x^\alpha D_x^\beta f(x)|$, the Schwartz class becomes a complete locally convex topological vector space. Then the Fourier transform becomes a homeomorphism from $\mathcal{S}(\mathbb{R}^n)$ to itself.

2.1.1 Gaussian integrals

In this section, we shall list the integrals that we will frequently use in Weyl quantization.

Lemma 1 (Gaussian integral, [2, Lemma 3.1.1]). *Let $\varphi(x) = (x, Ax) + (a, x) + a_0 = x^t Ax + a^t x + a_0$ be a quadratic function on $\mathbb{R}^n$, where A is a complex symmetric matrix with a positive definite real part, a as a complex vector and a_0 a complex constant. Then*

$$I := \int e^{-\varphi(x)} dx = \pi^{n/2} (\det A)^{-1/2} e^{-\varphi(z_0)},$$

where $z_0 \in \mathbb{C}^n$ is a stationary point for φ i.e. a solution of $\partial\varphi/\partial x_i = 0$, and

$$(\det A)^{-1/2} = (\det(B + iC))^{-1/2}$$

is understood as a continuous branch of the function $(\det(B + iC))^{-1/2}$ on the interval $t \in [0, 1]$, with a positive initial value at $t = 0$.

Proof. Since the proof is standard, we only sketch the proof: since z_0 is a stationary point, we can write

$$\varphi(x) = \varphi(z_0) + (x - z_0)^t A(x - z_0).$$

Then do the diagonalization for the term $(x - z_0)^t A(x - z_0)$ and integral it. $\square$

The next corollaries are based on the Lemma 1, so we shall omit their proofs.

Corollary 1 ([2, Corollary 3.1.2]). *Let $A = \epsilon B^2 + iC$, where B is a positive definite matrix and C is non-degenerate matrix. Then we have*

$$I = \lim_{\epsilon\to 0} \int (-x^t(\epsilon B_i^2 C)x - a^t x - a_0) dx = \pi^{n/2} |\det C|^{-1/2} e^{-\varphi(z_0)} \exp\left(-\frac{i\pi}{4} sgn(C)\right),$$

where z_0 is a stationary point of the function $\varphi(x) = ix^t Cx + a^t x + a_0$ and $sgn(C)$ denotes the difference of the number of positive eigenvalues and the number of negative eigenvalues.

Corollary 2 ([2, Corollary 3.1.3]). *Let $f(x)$ be a polynomial. Under the assumptions of 1,*

$$I = \int f(x) e^{-\varphi(x)} dx = \pi^{n/2} (\det A)^{-1/2} e^{-\varphi(z_0)} \exp\left(\frac{1}{4}(\partial_\tau, A^{-1}\partial_\tau)\right) f(z_0 + \tau)|_{\tau=0},$$

where the exponential function is understood as a formal expansion.

2.2 Symbol classes and the composition formula

In quantum mechanics, as introduced in the first section, we would like to transfer the classical observable i.e. real smooth function to the operator on some Hilbert space. We call this process quantization. We have seen the quantization of position and momentum. In this section, we shall introduce Weyl quantization.

Let $\mathbb{R}^{2n}_x = \mathbb{R}^n_q \times \mathbb{R}^n_p$ be the phase space of classical mechanics. There is a standard symplectic form

$$\omega = dq^1 \wedge dp^1 + \ldots dq^n \wedge dp^n,$$

where $x = (x_1, \ldots, x^{2n}) = (q^1, p_1, \ldots, q^n, p_n)$ turning $\mathbb{R}^{2n}$ into a symplectic space.

Definition 6. *A function $a(x,h) \in C^\infty(\mathbb{R}^{2n} \times (0,1])$ belongs to the symbol class Σ_h^m if for any integer $N \geq 0$ there is a decomposition*

$$a(x,h) = a_0(x) + ha_1(x) + \cdots + h^{N-1}a_{N-1}(x) + h^N r_N(x,h),$$

where

$$|a_k^{(\alpha)}(x)| \leq C_{\alpha,k}\sqrt{1+|x|^2}^{m-2k-|\alpha|}$$

and

$$|r_N^{(\alpha)}(x,h)| \leq C_{N,\alpha}\sqrt{1+|x|^2}^{m-2N-|\alpha|}$$

uniformly with respect to $h \in (0,1]$.

We omit the subscript h whenever h is fixed. The union of all Σ_h^m over m is denoted by Σ_h, and the intersection of all Σ_h^m is denoted by $\Sigma_h^{-\infty}$.

Now, we define the Weyl quantization.

Definition 7. *Let $h \in (0,1)$ and $a(q,p)$ be a reasonable smooth function(sometimes in Σ^m) on $\mathbb{R}^{2n}$, the Weyl quantization A of $a(q,p)$, denoted by*

$$A = \hat{a} = Op(a(q,p)) = a(q, -ih\frac{\partial}{\partial q}),$$

is the operator on $\mathcal{S}(\mathbb{R}^n)$, defined by the following

$$(Au)(q) = (2\pi h)^{-n}\int_{\mathbb{R}^{2n}} \exp\left(\frac{i}{h}p(q-q')\right) a\left(\frac{q+q'}{2}, p\right) u(q')dq'dp \tag{5}$$

where $u(q) \in \mathcal{S}(\mathbb{R}^n)$ and $p(q-q')$ is the abbreviation of the standard inner product of p and $q-q'$ on $\mathbb{R}^n$. The function $a(x)$ is called the (Weyl) symbol of the operator A.

Example 1. *Consider $a = q^i$ and $a = p_i$ i.e. take a as position function and momentum function. Then we have*

$$Op(q^i)u = q^i u \quad \text{and} \quad Op(p^i)u = -ih\frac{\partial u}{\partial q^i}$$

which are exactly the quantization of position and momentum functions. More precisely, for position function q^i, we have

$$\begin{aligned}
&(2\pi h)^{-n}\int e^{\frac{i}{h}p(q-q')} a\left(\frac{q+q'}{2}, p\right) u(q')dpdq' \\
&= (2\pi h)^{-n}\int e^{\frac{i}{h}p(q-q')} \cdot \frac{q^i + q^{i'}}{2} \cdot u(q')dpdq' \\
&= (2\pi h)^{-n}\int e^{\frac{i}{h}pq}(\int e^{-\frac{i}{h}pq'}\frac{q^{i'}}{2} \cdot u(q')dq')dp + (2\pi h)^{-n}\int e^{\frac{i}{h}pq}(\int e^{-\frac{i}{h}pq'}\frac{q^i}{2} \cdot u(q')dq') \\
&= (2\pi h)^{-n}\int e^{\frac{i}{h}pq}\mathcal{F}'(\frac{p^i}{2} \cdot u(p))dp + q^i \cdot (2\pi h)^{-n}\int e^{\frac{i}{h}pq}\mathcal{F}'(u(p))dq') \\
&= q^i u(q),
\end{aligned}$$

where $\mathcal{F}'$ denotes the Fourier transform with coefficient $\frac{i}{h}$ in the power of exponential and the last equality is due to the inverse Fourier transform. For the momentum function, one can use the same method and integrating by parts to show that the quantization of the momentum function p_i gives $-ih\frac{\partial u}{\partial q^i}$.

Proposition 5 ([2, Proposition 3.1.7])**.** *If $a(x) \in \Sigma^m$, then the operator $A = Op(a)$ maps $\mathcal{S}(\mathbb{R}^n)$ continuously to itself.*

Let us consider the adjoint of the operator A. Let $u, v \in \mathcal{S}(\mathbb{R}^n)$, we have

$$(v, Au) = (2\pi h)^{-n} \int \exp\left(\frac{i}{h} p(q-q')\right) a\left(\frac{q+q'}{2}, p\right) u(q')\bar{v}(q) dq' dp dq,$$

which gives the adjoint A^* has symbol $\bar{a}(x)$. Then any operator having real symbol is self-adjoint, which agrees with the axiom of quantum mechanics.

Definition 8 (Integral operator)**.** *An integral operator is an operator of the form:*

$$(Tf)(x) = \int_{t_1}^{t_2} K(t,x) f(t) dt.$$

We call the function $K(t,x)$ the kernel of the operator.

Then, back to the Weyl quantization, by performing change of variables, we have

$$(v, Au) = (2\pi h)^{-n} \int a(x) K_{u,v}(x) dx,$$

where $q + q' = 2s, q - q' = t$ and $K_{u,v}(x) = K_{u,v}(s,p) = \int \exp\left(\frac{i}{h} pt\right) u(s - t/2)\bar{v}(s + t/2) dt$, which belongs to $\mathcal{S}(\mathbb{R}^{2n})$ and it is the symbol of the one-dimensional operator $\widehat{K}_{u,v} := u(v, \cdot)$.

For the symbol $a(x)$ in $\Sigma^{-\infty}$, the Weyl quantization may be represented as an integral operator with kernel function

$$K(q,q') = (2\pi h)^{-n} \int \exp\left(\frac{i}{h} p(q-q')\right) a\left(\frac{q+q'}{2}, p\right) dp. \tag{6}$$

If we perform a change of variables again, we get

$$K\left(s + \frac{t}{2}, s - \frac{t}{2}\right) = (2\pi h)^{-n} \int \exp\left(\frac{i}{h} pt\right) a(s,p) dp.$$

Then by the Fourier inversion formula, we get

$$a(s,p) = \int \exp\left(-\frac{i}{h} pt\right) K\left(s + \frac{t}{2}, s - \frac{t}{2}\right) dt.$$

Definition 9. *We define the trace of the operator $A = Op(a)$ for $a \in \Sigma^{-\infty}$ as*

$$\operatorname{tr} A = \int K(q,q) dq = (2\pi h)^{-n} \int a(x) dx,$$

where K is the kernel of Weyl quantization and the last equality follows from (6)*.*

Proposition 6 ([2, Proposition 3.1.8])**.** *Let A_1, A_2 be Weyl quantization with symbols $a_1(x), a_2(x) \in \Sigma^{-\infty}$. Then*

$$\operatorname{tr} A_1 A_2 = (2\pi h)^{-n} \int a_1(x) a_2(x) dx.$$

In particular, the trace does not depend on the order of composition of operators.

Proof. Using formula (6) for the kernels K_1 and K_2, we obtain

$$\begin{aligned}\operatorname{tr} A_1A_2 &= \int K_1(q,q')K_2(q',q)dqdq' \\ &= (2\pi h)^{-2n}\int \exp\left(-\frac{i}{h}(p_1-p_2)(q-q')\right)a_1\left(\frac{q+q'}{2},p_1\right)a_2\left(\frac{q'+q}{2},p_2\right)dqdq'dp_1dp_2\end{aligned}$$

Performing the change of variables $p_1 - p_2 = u$, we have

$$\operatorname{tr} A_1A_2 = (2\pi h)^{-2n}\int a_2(s,p_2)dsdp_2\int \exp\left(\frac{i}{h}tu\right)a_1(s,p_2+u)dudt$$

Finally, the Fourier integral formula

$$(2\pi h)^{-n}\int \exp\left(\frac{i}{h}tu\right)a_1(s,p+u)dudt = a_1(s,p)$$

completes the proof. ∎

Now, we introduce the composition formula for the Weyl quantizations.

Theorem 1 ([2, Theorem 3.2.1])**.** *Let $A = Op(a)$ and $B = Op(b)$ be Weyl quantizations with symbols $a(x), b(x) \in \Sigma^{-\infty}$. Then AB is also a Weyl quantization with Weyl symbol*

$$c(x) = (\pi h)^{-2n}\int_{\mathbb{R}^{4n}} \exp\left(\frac{2i}{h}\omega(t,\tau)\right)a(x+t)b(x+\tau)dtd\tau,$$

where $\omega(u,r;v,s) = rv - su$.

Proof. We sketch the proof below.

$$\begin{aligned}(ABu)(q) &= (A(Bu))(q) \\ &= (2\pi h)^{-n}\int \exp\left(\frac{i}{h}p_1(q-q_1)\right)a\left(\frac{q+q_1}{2},p_1\right)(Bu)(q_1)dq_1dp_1 \\ &= (2\pi h)^{-2n}\int \exp\left(\frac{i}{h}(p_1(q-q_1)+p_2(q_1-q_2))\right)a\left(\frac{q+q_1}{2},p_1\right)b\left(\frac{q_1+q_2}{2},p_2\right)u(q_2)dq_2dp_1dp_2\end{aligned}$$

Thus the kernel of AB is given by

$$K(q_1,q_2) = (2\pi h)^{-2n}\int \exp\left(\frac{i}{h}(p_1(q-q_1)+p_2(q_1-q_2))\right)a\left(\frac{q+q_1}{2},p_1\right)b\left(\frac{q_1+q_2}{2},p_2\right)dq_1dp_1dp_2$$

By using Fourier inversion formula, we can obtain

$$c(x) = (\pi h)^{-2n}\int_{\mathbb{R}^{4n}} \exp\left(\frac{2i}{h}\omega(t,\tau)\right)a(x+t)b(x+\tau)dtd\tau$$

is the symbol of AB. ∎

Similarly, for the symbols a, b in $\Sigma_h^{m_a}$ and $\Sigma_h^{m_b}$ respectively, we have the following theorem.

Theorem 2 ([2, Theorem 3.2.2])**.** *The operator $Op(a)Op(b)$ has symbol c in $\Sigma_h^{m_a+m_b}$ given by*

$$c(x,h) = \sum_{k<N}\frac{1}{k!}\left(\frac{-ih}{2}\right)^k(\omega^{-1}(\partial_z,\partial_\tau))^k a_1(x+t)a_2(x+\tau)|_{t=\tau=0} + h^N R_N(x,h), \tag{7}$$

where $R_N \in \Sigma_h^{m_a+m_b-2N}$.

In particular, for the linear functions, we have the corollary below.

Corollary 3 ([2, Proposition 3.2.4])**.** *Let $a = \sum_{i=1}^{2n} \alpha_i x^i$ be a linear function, then $Op(a) \circ Op(a) \cdots \circ Op(a) = Op(a^k)$.*

Proof. First notice that $\{a, a^k\} = \{a, a^{k-1}\} = \ldots \{a, a\} = 0$ by the following calculation:

$$\begin{aligned}
\{a, a^k\} &= \sum_{i,j} \frac{\partial a}{\partial x_i}\frac{\partial a^k}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial a^k}{\partial x_i} \\
&= \sum_{i,j} \frac{\partial a}{\partial x_i}\frac{\partial a \cdot a^{k-1}}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial a \cdot a^{k-1}}{\partial x_i} \\
&= \sum_{i,j} \frac{\partial a}{\partial x_i}(\alpha_j + \frac{\partial a^{k-1}}{\partial x_j}) - \frac{\partial a}{\partial x_j}(\alpha_i + \frac{\partial a^{k-1}}{\partial x_i}) \\
&= \sum_{i,j} \frac{\partial a}{\partial x_i}\frac{\partial a^{k-1}}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial a^{k-1}}{\partial x_i} \\
&= \{a, a^{k-1}\}
\end{aligned}$$

Then by the equation (7) we obtain that $Op(a) \circ Op(a^{k-1}) = a^k - \frac{ih}{2}\{a, a^{k-1}\} = a^k$. Applying an induction argument completes the proof. ∎

2.3 Gaussian symbols

In this section, we consider a special class of symbols.

Definition 10. *A function $a \in \Sigma^{-\infty}$ is called a Gaussian symbol if it has the form*

$$a = c \cdot e^{-\frac{1}{h}g(x,x)},$$

where $g(x, x) = x^t G x$ is a complex quadratic form with positive definite real part and c is a constant.

The composition of Gaussian symbols is also a Gaussian symbol:

Proposition 7 ([2, Proposition 3.3.1])**.** *The class of Gaussian symbols is closed under composition:*

$$Op\left(\exp\left(-\frac{1}{h}g_1(x,x)\right)\right) \circ Op\left(\exp\left(-\frac{1}{h}g_2(x,x)\right)\right) = Op\left(c \cdot \exp\left(-\frac{1}{h}g_3(x,x)\right)\right),$$

where

$$G_3 = G_1 - (G_1 - i\Omega)(G_1 - \Omega G_2^{-1}\Omega)^{-1}(G_1 + i\Omega), \tag{8}$$

and

$$c = \sqrt{\det G_2 \det\big(G_1 - \Omega_2^{-1}\Omega\big)},$$

where $g_i(x, x) = x^t G_i x$.

Corollary 4 ([2, Corollary 3.3.3])**.** *Any Gaussian symbol may be represented as a composition of two other Gaussian symbols.*

Proof. The idea of this proof is to assume Gaussian symbol with the matrix G can be written as the composition of two other Gaussian symbols, one of which has the matrix $\lambda \cdot G$ with $\lambda > 0$. Then substitute $G_3 = G$ and $G_1 = \lambda \cdot G$ in equation (8). One can obtain G_2 by solving the equation. ∎

Example 2. *Let us consider the Gaussian symbol of the form $p_\epsilon(x) = e^{-\frac{\epsilon}{h}|x|^2}$. We have that $p_\epsilon \to 1$ as $\epsilon \to 0$. By the proposition above, we have that $Op(p_{\epsilon_1})Op(\epsilon_2) = (1 + \epsilon_1\epsilon_2)^{-n}Op(p_{\epsilon_3})$ where $\epsilon_3 = \frac{\epsilon_1+\epsilon_2}{1+\epsilon_1\epsilon_2}$.*

Lemma 2 ([2, Lemma 3.3.4])**.** *The set $\{Op(p_\epsilon)(u) : u \in \mathcal{S}(\mathbb{R}^n), \epsilon > 0\}$ is dense in $\mathcal{S}(\mathbb{R}^n)$.*

2.4 Operators in $L^2(\mathbb{R}^n)$

Theorem 3 ([2, Theorem 3.4.1])**.** *The operator $A = Op(a)$ with $a \in \Sigma_h^0$ is bounded in $L^2(\mathbb{R}^n)$ uniformly with respect to $h \in (0,1]$.*

Definition 11 (Hilbert–Schmidt operator)**.** *An operator A is called a Hilbert–Schmidt operator if there is an orthonormal basis $\{e_i\}$ in H, such that*

$$\|A\|_2^2 := \sum_i \|Ae_i\|^2 < \infty. \tag{9}$$

The subspace consisting of Hilbert–Schmidt operators is denoted by $\mathcal{L}_2(H)$.

Definition 12 (Trace class operator)**.** *An operator A is called a trace class operator if there exists two Hilbert–Schmidt operators B, C such that $A = BC$. The norm*

$$\|A\|_1 := \inf_{B,C \in \mathcal{L}_2(H): A = BC} \|B\|_2 \|C\|_2,$$

is called a trace norm. The subspace consisting of trace class operators is denoted by $\mathcal{L}_1(H)$.

We list the following properties of $\mathcal{L}_1(H)$ and $\mathcal{L}_2(H)$ without proofs. The proof can be found in [3, Section 2.4] .

Proposition 8 ([2, Proposition 3.4.4])**.** *Hilbert–Schmidt operators have the following properties:*

- *the sum* (9) *does not depend on the choice of basis;*
- *if $A \in \mathcal{L}_2(H)$ then $A^* \in \mathcal{L}_2(H)$ and $\|A\|_2 = \|A^*\|_2$;*
- *if $A \in \mathcal{L}_2(H)$ then A is compact;*
- *the space $\mathcal{L}_2(H)$ is a Hilbert space with the inner product $(A, B)_2 = \sum_i (Ae_i, Be_i)$;*
- *the space $\mathcal{L}_2(H)$ is a two-sided ideal in $\mathcal{L}(H)$ and $\|BAC\|_2 \le \|B\| \|A\|_2 \|C\|$ for any $A \in \mathcal{L}_2(H)$, and $B, C \in \mathcal{L}(H)$;*
- *Let $s_i^2 > 0$ be the nonzero eigenvalues of A^*A. Then $\|A\|_2^2 = \sum_i s_i^2$ where the sum counts for multiplicities.*

Proposition 9 ([2, Proposition 3.4.5])**.** *Trace class operators have the following properties:*

- *The space $\mathcal{L}_1(H)$ is a Banach space with respect to the norm $\| \cdot \|_1$;*
- *If $A \in \mathcal{L}_1(H)$, then $A^* \in \mathcal{L}_1(H)$ with the same trace norm;*
- *The space $\mathcal{L}_1(H)$ is a two-sided ideal in $\mathcal{L}(H)$ and $\|BAC\|_1 \le \|B\| \|A\|_1 \|C\|$ for any $A \in \mathcal{L}_1(H)$, and $B, C \in \mathcal{L}(H)$;*
- *Define a trace of an operator $A \in \mathcal{L}_1(H)$ by $\operatorname{tr} A = (B, C^*)_2$, where $B, C \in \mathcal{L}_2(H), A = BC$. Then tr is a bounded linear functional on $\mathcal{L}_1(H)$;*
- *For any orthonormal basis $\{e_i\}$, we have $\operatorname{tr} A = \sum_i (Ae_i, e_i)$;*
- *Let $s_i^2 > 0$ be the nonzero eigenvalues of A^*A. Then $\|A\|_2 = \sum_i s_i$ where the sum is over multiplicities;*
- *If λ_i are nonzero eigenvalues of A, then $\operatorname{tr} A = \sum_i \lambda_i$ where the sum is over multiplicities.*

The last property of trace class operators is known as Lidskij's theorem [4, Proposition A.3.3], it has an important corollary:

Corollary 5 ([2, Corollary 3.4.6]). *If A, B are two operators such that $AB \in \mathcal{L}_1(H)$ and $BA \in \mathcal{L}_1(H)$, then* $trAB = \operatorname{tr} BA$.

Proof. If $\lambda \neq 0$ is an eigenvalue of AB with eigenvector e_λ, then $(AB - \lambda)e_\lambda = 0$ and $Be_\lambda \neq 0$. Ap[plying B to both sides gives that

$$B(AB - \lambda)e_\lambda = (BA - \lambda)Be_\lambda,$$

thus we have λ is also an eigenvalue for BA with the eigenvector Be_λ.

If $(AB - \lambda)^k e_\lambda = 0$ for some $k \in \mathbb{N}_{>1}$, then by the same argument, one can show that $(BA - \lambda)^k Be_\lambda = 0$ as well. Therefore, all nonzero eigenvalues (taking into account their multiplicities) of AB are eigenvalues of BA and vice versa, which completes the proof. ∎

Note that in the finite dimensional case, the trace of an operator coincides with the usual definition of operator trace. In particular, if $\dim(H) = n$, then the trace of the identity I is n.

This has a interesting consequence:

Corollary 6. *The canonical relation $[A, B] = -i\hbar I$ does not hold in $\mathcal{L}_1(H)$.*

Proof. We have on the one hand,

$$\operatorname{tr}(AB - BA) = \operatorname{tr} AB - \operatorname{tr} BA = 0,$$

which means that the commutator has zero trace. On the other hand, the trace of $-i\hbar I$ is clearly nonzero. ∎

Finally, let us consider a natural question: how to determine whether an operator obtained through Weyl quantization is trace class (or Hilbert–Schmidt). If so, how can we estimate the corresponding norms and does there exists any relation between the trace of the symbol and the trace of the corresponding operator? The following theorem gives the answer.

Theorem 4 ([2, Proposition 3.4.7]). *If $a \in \Sigma^m$, and $A = Op(a)$, then*

1. $A \in \mathcal{L}_2(L^2(\mathbb{R}^n))$ *for* $m < -n$ *and*

$$\|A\|_2^2 = (2\pi h)^{-n} \int |a(x)|^2 dx,$$

2. $A \in \mathcal{L}_1(L^2(\mathbb{R}^n))$ *for* $m < -2n$ *and*

$$\operatorname{tr} A = (2\pi h)^{-n} \int a(x) dx.$$

Finally, from this theorem, we know that two traces coincide and we need not to distinguish them anymore.

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