Curves on $\mathbf{P}^1 \times \mathbf{P}^1$

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1. Introduction

One of the exercises in last semester's Algebraic Geometry course went as follows:

Exercise. *Let k be a field and $Z = \mathbf{P}^1_k \times_k \mathbf{P}^1_k$. Show that the Picard group* $\operatorname{Pic} Z$ *is the free Abelian group generated by the classes of a horizontal and a vertical line.*

Here $\operatorname{Pic} Z$ is to be interpreted as the divisor class group $\operatorname{Cl} Z$, to which it is naturally isomorphic for Noetherian integral separated locally factorial schemes [Hartshorne, Corollary 6.16]. We view the first $\mathbf{P}^1$ as the result of glueing $\operatorname{Spec}(k[x])$ and $\operatorname{Spec}(k[1/x])$ via $\operatorname{Spec}(k[x, 1/x])$, and similarly for the second $\mathbf{P}^1$ with y instead of x. Then $Z = \mathbf{P}^1_k \times_k \mathbf{P}^1_k$ is the result of glueing the spectra of $k[x, y]$, $k[x, 1/y]$, $k[1/x, y]$ and $k[1/x, 1/y]$ in the obvious way.

To prove the claim (see [Hartshorne, Example II.6.6.1] for a different approach), let L_x and L_y be the vertical and horizontal lines $x = \infty$ and $y = \infty$. More precisely, L_x is determined by the coherent sheaf of ideals $\mathcal{I}_{L_x}$ with

$$\mathcal{I}_{L_x}|_{\operatorname{Spec} A} = \begin{cases} \tilde{A} & \text{for } A = k[x, y] \text{ and } A = k[x, 1/y] \\ 1/x \cdot \tilde{A} & \text{for } A = k[1/x, y] \text{ and } A = k[1/x, 1/y], \end{cases}$$

and similarly for L_y. If Y is a curve on Z different from L_x and L_y (curves are assumed to be integral), the intersection of Y with $\operatorname{Spec}(k[x, y])$ is a plane curve defined by an irreducible polynomial $f \in k[x, y]$. Let a be the degree of f as a polynomial in x and b is its degree as a polynomial in y; then the divisor of f as a rational function on Z equals

$$(f) = Y - a \cdot L_x - b \cdot L_y,$$

so we see that the divisor class of Y is equal to

$$[Y] = a[L_x] + b[L_y].$$

This shows that $\operatorname{Cl} Z$ is generated by $[L_x]$ and $[L_y]$; because there are no rational functions $f \in k(x, y)$ with the property that $(f) = a \cdot L_x + b \cdot L_y$ as a divisor on Z unless $a = b = 0$, the classes $[L_x]$ and $[L_y]$ are linearly independent. If Y is a divisor on Z and a, b are the unique integers with $[Y] = a[L_x] + b[L_y]$, we say that Y is of *type* (a, b).

The isomorphism $\operatorname{Cl} Z \to \operatorname{Pic} Z$ sends the class of a divisor Y of type (a, b) to the invertible sheaf $\mathcal{O}_Z(Y) \cong \mathcal{O}_Z(a \cdot L_x + b \cdot L_y)$. Note that $\mathcal{O}_Z(a \cdot L_x)$ is isomorphic to the pullback $p_1^*\big(\mathcal{O}_{\mathbf{P}^1_k}(a \cdot \infty)\big)$, where the invertible sheaf $\mathcal{O}_{\mathbf{P}^1_k}(a \cdot \infty)$ on $\mathbf{P}^1_k$ is defined by

$$\mathcal{O}_{\mathbf{P}^1_k}(a \cdot \infty)|_{\operatorname{Spec} k[x]} = (k[x])^\sim$$
$$\mathcal{O}_{\mathbf{P}^1_k}(a \cdot \infty)|_{\operatorname{Spec} k[1/x]} = x^a \cdot (k[1/x])^\sim.$$

On the other hand, there is the invertible sheaf $\mathcal{O}_{\mathbf{P}^1_k}(a)$ with

$$\mathcal{O}_{\mathbf{P}^1_k}(a)|_{\operatorname{Spec} k[x/y]} = y^a \cdot (k[x/y])^\sim$$
$$\mathcal{O}_{\mathbf{P}^1_k}(a)|_{\operatorname{Spec} k[y/x]} = x^a \cdot (k[y/x])^\sim,$$

which is clearly isomorphic to $\mathcal{O}_{\mathbf{P}^1_k}(a \cdot \infty)$, so

$$\mathcal{O}_Z(a \cdot L_x) \cong p_1^*(\mathcal{O}_{\mathbf{P}^1_k}(a)).$$

Something similar is true for the second projection. Using

$$\mathcal{O}_Z(a \cdot L_x + b \cdot Ly) \cong \mathcal{O}_Z(a \cdot L_x) \otimes_{\mathcal{O}_Z} (b \cdot L_y)$$

we conclude that $\mathcal{O}_Z(Y)$ is isomorphic to the invertible sheaf $\mathcal{O}(a,b)$ on Z defined by

$$\boxed{\mathcal{O}(a,b) = p_1^*\big(\mathcal{O}_{\mathbf{P}^1_k}(a)\big) \otimes_{\mathcal{O}_Z} p_2^*\big(\mathcal{O}_{\mathbf{P}^1_k}(b)\big).}$$

The aim of this talk is to study the cohomology of the sheaves $\mathcal{O}(a,b)$ and to derive some consequences for the kind of curves that exist on Z. We will do the following:

1. Prove the Künneth formula: if X and Y are Noetherian separated schemes over a field k, there is a natural isomorphism
$$H(X \times_k Y, p_1^*\mathcal{F} \otimes_{\mathcal{O}_{X\times_k Y}} p_2^*\mathcal{G}) \cong H(X,\mathcal{F}) \otimes_k H(Y,\mathcal{G})$$
for all quasi-coherent sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y.
2. Deduce a connectedness result for closed subschemes and a genus formula for curves on Z.
3. Prove Bertini's theorem: if X is a non-singular subvariety of $\mathbf{P}^n_k$ with k an algebraically closed field, there exists a hyperplane $H \subset \mathbf{P}^n_k$ not containing X such that $H \cap X$ is a regular scheme.
4. Deduce that if k is algebraically closed field, there exist non-singular curves of type (a,b) on Z for all $a, b > 0$.

2. Tensor products of complexes

Let A be a ring, (C,d) a complex of right A-modules and (C',d') a complex of left A-modules, i.e. C and C' are graded A-modules

$$C = \bigoplus_{n\in\mathbf{Z}} C^n \quad \text{and} \quad C' = \bigoplus_{n\in\mathbf{Z}} C'^n$$

and d, d' are A-module endomorphisms such that $dd = 0$ and $d(C^n) \subseteq C^{n+1}$ (similarly for d'). Let $C \otimes_A C'$ be the usual tensor product, graded in such a way that

$$(C \otimes_A C')^n = \bigoplus_{p+q=n} C^p \otimes_A C'^q.$$

There is a group endomorphism D of $C \otimes_A C'$ defined by

$$D(x \otimes y) = dx \otimes y + (-1)^p x \otimes d'y \quad \text{for } x \in C^p;$$

it fulfills $D((C \otimes_A C')^n) \subset (C \otimes_A C')^{n+1}$ and $DD = 0$, so $((C \otimes_A C'), D)$ is a complex of Abelian groups.

For any complex (C,d) of Abelian groups, we write $Z(C)$ for the subgroup of cocycles, $B(C)$ for the subgroup of coboundaries and $H(C)$ for the cohomology of C:

$$Z(C) = \ker d, \quad B(C) = \operatorname{im} d, \quad H(C) = Z(C)/B(C).$$

If x and y are cocycles in C and C', respectively, then $x \otimes y$ is a cocycle in $C \otimes_A C'$, because

$$D(x \otimes y) = dx \otimes y + (-1)^p x \otimes d'y = 0 \quad \text{for } x \in C^p.$$

This means that there is a natural A-bilinear map

$$\begin{aligned} Z(C) \times Z(C') &\to Z(C \otimes_A C') \\ (x,y) &\mapsto x \otimes y. \end{aligned}$$

If either $x \in B(C)$ or $y \in B(C')$, then the image of (x,y) under this map is in $B(C \otimes_A C')$, because for example

$$dx \otimes y = D(x \otimes y) \quad \text{for all } x \in C, y \in Z(C')$$

This means that we can divide out by the coboundaries in each of the groups and get a natural A-bilinear map

$$H(C) \times H(C') \to H(C \otimes_A C')$$

and therefore (by the universal property of the tensor product) a natural group homomorphism

$$\boxed{\gamma_{C,C'}\colon H(C) \otimes_A H(C') \to H(C \otimes_A C').}$$

In the next section we will need the following result:

Lemma. *Let A be a ring, (C,d) a complex of right A-modules and (C',d') a complex of left A-modules. Assume $d=0$. Then $H(C)\cong C$ and $\gamma_{C,C'}$ induces a natural group homomorphism*

$$\begin{aligned} C\otimes_A H(C') &\longrightarrow H(C\otimes_A C') \\ x\otimes\bar{y} &\longmapsto \overline{x\otimes y}. \end{aligned} \tag{1}$$

If C is flat over A, then this map is an isomorphism.

Proof. We only need to prove the last claim. Because C is flat, $\ker(D)=\ker(1\otimes d')=C\otimes_A\ker(d')$, so the natural map $C\otimes_A Z(C')\to Z(C\otimes_A C')$ is an isomorphism. Furthermore, the image of $C\otimes_A B(C')$ in $C\otimes_A Z(C')$ corresponds to the subgroup $B(C\otimes_A C')$ under this isomorphism, since both are generated by elements of the form $x\otimes d'y$ with $x\in C$ and $y\in C'$. This implies the map defined above is an isomorphism.

3. The Künneth formula

From now on we restrict our attention to the case where A is a field k. Then all complexes have the structure of k-vector spaces, and all modules are flat. For a treatment without this restriction, see [Bourbaki]. We will prove the following theorem (note that the previous lemma is a special case of this):

Theorem (Künneth formula). *Let (C,d) and (C',d') be complexes over k. Then the natural k-linear map*

$$\gamma_{C,C'}\colon H(C)\otimes_k H(C')\to H(C\otimes_k C')$$

is an isomorphism.

Proof. Write $Z=Z(C)$, $B=B(C)$, $H=H(C)$ and $H'=H(C')$. Consider the short exact sequence of complexes defining $Z(C)$ and $B(C)$:

$$0\longrightarrow Z\xrightarrow{\;j\;}C\xrightarrow{\;d\;}B(1)\longrightarrow 0.$$

Here $B(1)$ denotes the complex B shifted one place to the left, i.e. $B(1)^n=B^{n+1}$. Taking the tensor product with C' gives a short exact sequence of complexes

$$0\longrightarrow Z\otimes_k C'\xrightarrow{\;j\otimes 1\;}C\otimes_k C'\xrightarrow{\;d\otimes 1\;}(B\otimes_k C')(1)\longrightarrow 0.$$

We take the cohomology sequence of this short exact sequence. The coboundary map will go from $H(B\otimes_k C')$ to $H(Z\otimes_k C')$. To find out what it does, we write down the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (Z\otimes_k C')^{n-1} & \xrightarrow{j\otimes 1} & (C\otimes_k C')^{n-1} & \xrightarrow{d\otimes 1} & (B\otimes_k C')^{n} & \longrightarrow & 0 \\ & & \downarrow D & & \downarrow D & & \downarrow D & & \\ 0 & \longrightarrow & (Z\otimes_k C')^{n} & \xrightarrow{j\otimes 1} & (C\otimes_k C')^{n} & \xrightarrow{d\otimes 1} & (B\otimes_k C')^{n+1} & \longrightarrow & 0. \end{array}$$

Because $d=0$ on B and because B is flat over k, the kernel of $D\colon(B\otimes_k C')^n\to(B\otimes_k C')^{n+1}$ equals

$$\ker(D)=\ker(1\otimes d')\cong B\otimes_k\ker(d'),$$

so $\ker D$ is generated by elements of the form $dx\otimes y$ with $x\otimes y\in(C\otimes_k C')^{n-1}$ such that $y\in Z(C')$. The image of $x\otimes y\in(C\otimes_k C')^{n-1}$ in $(C\otimes_k C')^n$ is now $D(x\otimes y)=dx\otimes y$, which is in $(Z\otimes_k C')^n$. We see therefore that the coboundary map sends the class of $dx\otimes y$ to that of $(i\otimes 1)(dx\otimes y)$, where $i\colon B\to Z$ is the inclusion. In other words, the coboundary map equals $H(i\otimes 1)$. The long exact sequence is now

$$H^n(B\otimes_k C')\xrightarrow{H(i\otimes 1)}H^n(Z\otimes_k C')\xrightarrow{H(j\otimes 1)}H^n(C\otimes_k C')\xrightarrow{H(d\otimes 1)}H^{n+1}(B\otimes_k C')\xrightarrow{H(i\otimes 1)}H^{n+1}(Z\otimes_k C').$$

We can also take the tensor product with H' of the short exact sequence defining H to obtain an exact sequence

$$0 \longrightarrow B \otimes_k H' \xrightarrow{i\otimes 1} Z \otimes_k H' \xrightarrow{p\otimes 1} H \otimes_k H' \longrightarrow 0.$$

We connect this sequence with the long exact sequence above via the natural maps

$$\begin{aligned}
\gamma_{B,C'}\colon B \otimes_k H' &\to H(C \otimes_k C')\\
\gamma_{Z,C'}\colon Z \otimes_k H' &\to H(C \otimes_k C')\\
\gamma_{C,C'}\colon H \otimes_k H' &\to H(C \otimes_k C'),
\end{aligned}$$

the first two of which are the isomorphisms occurring in the lemma from Section 2. This gives a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
(B\otimes_k H')^n & \xrightarrow{i\otimes 1} & (Z\otimes_k H')^n & \xrightarrow{p\otimes 1} & (H\otimes_k H')^n & \longrightarrow & 0 & & \\
\downarrow \gamma_{B,C'} & & \downarrow\gamma_{Z,C'} & & \downarrow \gamma_{C,C'} & & & & \\
H^n(B\otimes_k C') & \xrightarrow{H(i\otimes 1)} & H^n(Z\otimes_k C') & \xrightarrow{H(j\otimes1)} & H^n(C\otimes_k C') & \xrightarrow{H(d\otimes 1)} & H^{n+1}(B\otimes_k C') & \xrightarrow{H(i\otimes1)} & H^{n+1}(Z\otimes_k C')\\
&&&&&& \gamma_{B,C'}\uparrow && \gamma_{Z,C'}\uparrow\\
&&&& 0 & \longrightarrow & (B\otimes_k H')^{n+1} & \xrightarrow{i\otimes 1} & (Z\otimes_k H')^{n+1}
\end{array}$$

The lower right part shows that $H(i \otimes 1)$ is injective, so $H(d \otimes 1) = 0$ by exactness. From the rest of the diagram we now see that $\gamma_{C,C'}$ is an isomorphism.

4. The cohomology of sheaves of the form $\mathcal{F} \otimes_k \mathcal{G}$

Let X and Y be two compact separated schemes over a field k. Consider the scheme $Z = X \times_k Y$ together with its projection morphisms $p_1\colon Z \to X$ and $p_2\colon Z \to Y$. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent sheaves on X and Y, respectively. Recall that the pullbacks $p_1^*\mathcal{F}$ and $p_2^*\mathcal{G}$ of $\mathcal{F}$ and $\mathcal{G}$ to Z are defined by

$$\begin{aligned}
p_1^*\mathcal{F} &= \mathcal{O}_Z \otimes_{p_1^{-1}\mathcal{O}_X} p_1^{-1}\mathcal{F}\\
p_2^*\mathcal{G} &= \mathcal{O}_Z \otimes_{p_2^{-1}\mathcal{O}_Y} p_2^{-1}\mathcal{G}.
\end{aligned}$$

It is a general fact that the pullback of a quasi-coherent sheaf is quasi-coherent. We use this for $p_1^*\mathcal{F}$ and $p_2^*\mathcal{G}$. Suppose $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ are affine opens of X and Y, respectively, M is an A-module such that $\mathcal{F}|_U \cong M^\sim$ and N is a B-module such that $\mathcal{F}|_V \cong N^\sim$. Then the restrictions of $p_1^*\mathcal{F}$ and $p_2^*\mathcal{G}$ to the affine open subscheme $W = U \times_k V = \operatorname{Spec}(A \otimes_k B)$ of Z are

$$\begin{aligned}
p_1^*\mathcal{F}|_W &\cong (p_1^*\mathcal{F}(W))^\sim & \qquad p_2^*\mathcal{G}|_W &\cong (p_2^*\mathcal{G}(W))^\sim\\
&\cong ((A \otimes_k B) \otimes_A \mathcal{F}(U))^\sim & &\cong ((A \otimes_k B) \otimes_B \mathcal{G}(V))^\sim\\
&\cong (B \otimes_k M)^\sim, & &\cong (A \otimes_k N)^\sim.
\end{aligned}$$

From this we get the following expression for the sheaf $p_1^*\mathcal{F} \otimes_{\mathcal{O}_Z} p_2^*\mathcal{G}$:

$$\begin{aligned}
p_1^*\mathcal{F} \otimes_{\mathcal{O}_Z} p_2^*\mathcal{G}|_W &\cong ((B \otimes_k M) \otimes_{A\otimes_k B} (A \otimes_k N))^\sim\\
&\cong (M \otimes_k N)^\sim.
\end{aligned}$$

In particular, we see that

$$p_1^*\mathcal{F} \otimes_{\mathcal{O}_Z} p_2^*\mathcal{G}(U \times_k V) \cong \mathcal{F}(U) \otimes_k \mathcal{G}(V)$$

for all open affine subschemes U of X and V of Y. It seems therefore useful to introduce the abbreviated notation

$$\boxed{\mathcal{F} \otimes_k \mathcal{G} = p_1^*\mathcal{F} \otimes_{\mathcal{O}_Z} p_2^*\mathcal{G}}$$

for quasi-coherent sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y. (To prevent confusion, this notation should only be used if the sheaves are quasi-coherent.)

We are now going to compare the cohomology of the sheaf $\mathcal{F} \otimes_k \mathcal{G}$ on Z to the cohomology of $\mathcal{F}$ on X and $\mathcal{G}$ on Y. This we will do using a variant of Čech cohomology with respect to finite affine coverings of X, Y and Z.

Definition. The *unordered Čech complex* of a sheaf $\mathcal{F}$ of Abelian groups on a topological space X with respect to an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ is the complex defined by

$$C^n(\mathcal{U},\mathcal{F}) = \prod_{i_0,\ldots,i_n\in I} \mathcal{F}(U_{i_0,\ldots,i_n})$$

where, as usual,

$$U_{i_0,\ldots,i_n} = U_{i_0}\cap\ldots\cap U_{i_n}.$$

The maps $d\colon C^n \to C^{n+1}$ are defined using the same formula as for the usual (*alternating*) Čech complex:

$$d\bigl(\{s_{i_0,\ldots,i_n}\}_{i_0,\ldots,i_n\in I}\bigr) = \left\{\sum_{j=0}^{n+1}(-1)^j s_{i_0,\ldots,\hat{\imath}_j,\ldots,i_{n+1}}|_{U_{i_0,\ldots,i_{n+1}}}\right\}_{i_0,\ldots,i_{n+1}\in I}.$$

Notice that, in contrast to the alternating Čech cohomology, all the $C^n(\mathcal{U},\mathcal{F})$ are non-zero (unless $X=\emptyset$), but that the product occuring in the definition of $C^n(\mathcal{U},\mathcal{F})$ is finite if I is finite.

Let $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in J}$ be finite coverings by open affine subschemes of X and Y, respectively. Because X and Y are separated over k, the intersection of any positive number of such affines is again affine [Hartshorne, Exercise II.4.3]. We look at the unordered Čech complex of the sheaf $\mathcal{F}\otimes_k\mathcal{G}$ on Z with respect to the affine open covering $\mathcal{U}\times\mathcal{V}$. By the property (1) of $\mathcal{F}\otimes_k\mathcal{G}$ and because I and J are finite,

$$\begin{aligned} C^n(\mathcal{U}\times_k\mathcal{V},\mathcal{F}\otimes_k\mathcal{G}) &= \prod_{(i_0,j_0),\ldots,(i_n,j_n)\in I\times J} \mathcal{F}\otimes_k\mathcal{G}\bigl(U_{i_0}\times_k V_{j_0}\cap\ldots\cap U_{i_n}\times_k V_{j_n}\bigr) \\ &\cong \bigoplus_{i_0,\ldots,i_n\in I}\ \bigoplus_{j_0,\ldots,j_n\in J} \mathcal{F}(U_{i_0,\ldots,i_n})\otimes_k\mathcal{G}(V_{j_0,\ldots,j_n}).\end{aligned}$$

Since the tensor product is distributive over direct sums, we see that

$$\begin{aligned} C^n(\mathcal{U}\times_k\mathcal{V},\mathcal{F}\otimes_k\mathcal{G}) &\cong \left(\bigoplus_{i_0,\ldots,i_n\in I}\mathcal{F}(U_{i_0,\ldots,i_n})\right)\otimes_k\left(\bigoplus_{j_0,\ldots,j_n\in J}\mathcal{G}(V_{j_0,\ldots,j_n})\right) \\ &\cong C^n(\mathcal{U},\mathcal{F})\otimes_k C^n(\mathcal{V},\mathcal{G}).\end{aligned}$$

We take the direct sum over all n and conclude that

$$C(\mathcal{U}\times_k\mathcal{V},\mathcal{F}\otimes_k\mathcal{G}) \cong \bigoplus_{n=0}^{\infty} C^n(\mathcal{U},\mathcal{F})\otimes_k C^n(\mathcal{V},\mathcal{G}). \tag{2}$$

Fact. *There exists a natural homotopy equivalence of complexes*

$$\bigoplus_{n=0}^{\infty} C^n(\mathcal{U},\mathcal{F})\otimes_k C^n(\mathcal{V},\mathcal{G}) \sim C(\mathcal{U},\mathcal{F})\otimes_k C(\mathcal{V},\mathcal{G}).$$

After applying this fact, which follows from the *Eilenberg–Zilber theorem* [Godement, Théorème 3.9.1], to the right-hand side of (2) and taking cohomology, we obtain a natural isomorphism

$$H(C(\mathcal{U}\times_k\mathcal{V},\mathcal{F}\otimes_k\mathcal{G})) \cong H(C(\mathcal{U},\mathcal{F})\otimes_k C(\mathcal{V},\mathcal{G})).$$

Now the Künneth formula implies that

$$\boxed{\check{H}(\mathcal{U}\times_k\mathcal{V},\mathcal{F}\otimes_k\mathcal{G}) \cong \check{H}(\mathcal{U},\mathcal{F})\otimes_k\check{H}(\mathcal{V},\mathcal{G}).}$$

If X and Y are Noetherian, then from the fact that the Čech cohomology is isomorphic to the derived functor cohomology for open affine coverings (the proof of [Hartshorne, Theorem III.4.5] also works for the unordered Čech cohomology) we get the following theorem:

Theorem. *Let X and Y be Noetherian separated schemes over a field k. For all quasi-coherent sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, there is a natural isomorphism of k-vector spaces*

$$H(X,\mathcal{F})\otimes_k H(Y,\mathcal{G}) \cong H(X\times_k Y,\mathcal{F}\otimes_k\mathcal{G}).$$

5. Application to the sheaves $\mathcal{O}(a,b)$ and curves on $\mathbf{P}^1_k \times_k \mathbf{P}^1_k$

We have seen in Dirard's talk (see also [Hartshorne, Section III.5]) that for any ring A the cohomology of the sheaves $\mathcal{O}_X(n)$ on $X = \mathbf{P}^r_A$ is given by

$$\begin{aligned} H^0(X, \mathcal{O}_X(n)) &\cong S_n \\ H^i(X, \mathcal{O}_X(n)) &= 0 \quad \text{for } 0 < i < r \\ H^r(X, \mathcal{O}_X(n)) &\cong \operatorname{Hom}_A(S_{-n-r-1}, A) \end{aligned}$$

for all $n \in \mathbf{Z}$, where S_n is the component of degree n in $S = A[x_0, \ldots, x_r]$. In particular, for A equal to the field k and for $r = 1$,

$$\begin{aligned} H^0(\mathbf{P}^1_k, \mathcal{O}_{\mathbf{P}^1_k}(n)) &\cong k[x_0, x_1]_n \\ H^1(\mathbf{P}^1_k, \mathcal{O}_{\mathbf{P}^1_k}(n)) &\cong k[x_0, x_1]^\vee_{-n-2} \end{aligned}$$

The dimensions are therefore equal to

$$\begin{aligned} \dim_k H^0(\mathbf{P}^1_k, \mathcal{O}_{\mathbf{P}^1_k}(n)) &= \max\{n+1, 0\} \\ \dim_k H^1(\mathbf{P}^1_k, \mathcal{O}_{\mathbf{P}^1_k}(n)) &= \max\{-n-1, 0\}. \end{aligned}$$

It is now a matter of simple calculations and applying the Künneth formula to find the following table for the cohomology of the sheaves $\mathcal{O}(a,b)$ on $Z = \mathbf{P}^1_k \times_k \mathbf{P}^1_k$:

	$\dim_k H^0(Z, \mathcal{O}(a,b))$	$\dim_k H^1(Z, \mathcal{O}(a,b))$	$\dim_k H^2(Z, \mathcal{O}(a,b))$
$a \geq -1, \quad b \geq -1$	$(a+1)(b+1)$	0	0
$a \geq -1, \quad b \leq -1$	0	$(a+1)(-b-1)$	0
$a \leq -1, \quad b \geq -1$	0	$(-a-1)(b+1)$	0
$a \leq -1, \quad b \leq -1$	0	0	$(a+1)(b+1)$

We can now look at a few applications of this. Let Y be a locally principal closed subscheme of Z, and let $i \colon Y \to Z$ be the inclusion map, which is a closed immersion. Viewing Y as a divisor on Z, we have an exact sequence of coherent sheaves:

$$0 \longrightarrow \mathcal{O}_Z(-Y) \longrightarrow \mathcal{O}_Z \longrightarrow i_*\mathcal{O}_Y \longrightarrow 0.$$

The corresponding long exact cohomology sequence is

$$\begin{aligned} 0 \longrightarrow H^0(Z, \mathcal{O}_Z(-Y)) &\longrightarrow H^0(Z, \mathcal{O}_Z) \longrightarrow H^0(Z, i_*\mathcal{O}_Y) \\ \longrightarrow H^1(Z, \mathcal{O}_Z(-Y)) &\longrightarrow H^1(Z, \mathcal{O}_Z) \longrightarrow H^1(Z, i_*\mathcal{O}_Y) \\ \longrightarrow H^2(Z, \mathcal{O}_Z(-Y)) &\longrightarrow H^2(Z, \mathcal{O}_Z) \longrightarrow H^2(Z, i_*\mathcal{O}_Y) \longrightarrow 0. \end{aligned}$$

Because i is a closed immersion, we know that

$$H(Z, i_*\mathcal{O}_Y) \cong H(Y, \mathcal{O}_Y).$$

Furthermore, the case $a = b = 0$ gives us that $H^0(Z, \mathcal{O}_Z) \cong k$, $H^1(Z, \mathcal{O}_Z) = 0$ and $H^2(Z, \mathcal{O}_Z) = 0$, so the long exact sequence breaks down into two exact sequences

$$0 \longrightarrow H^0(Z, \mathcal{O}_Z(-Y)) \longrightarrow k \longrightarrow H^0(Y, \mathcal{O}_Y) \longrightarrow H^1(Z, \mathcal{O}_Z(-Y)) \longrightarrow 0$$

and

$$0 \longrightarrow H^1(Y, \mathcal{O}_Y) \longrightarrow H^2(Z, \mathcal{O}_Z(-Y)) \longrightarrow 0.$$

If Y is of type (a,b) with $a,b>0$, then $\mathcal{O}_Z(-Y)\cong\mathcal{O}(-a,-b)$; for these sheaves we have by the bottom row of the table above

$$H^0(Z,\mathcal{O}_Z(-Y))=0,\quad H^1(Z,\mathcal{O}_Z(-Y))=0,$$
$$\dim_k H^2(Z,\mathcal{O}_Z(-Y))=(-a+1)(-b+1)=(a-1)(b-1).$$

Therefore,

$$\boxed{H^0(Y,\mathcal{O}_Y)\cong k \quad\text{and}\quad \dim_k H^1(Y,\mathcal{O}_Y)=(a-1)(b-1) \qquad \text{if } a,b>0.}$$

The interpretation of this is that Y is connected, and if Y is a non-singular curve it has genus $(a-1)(b-1)$.

6. Bertini's theorem

In this section we study intersections of projective varieties with hyperplanes. A hyperplane $H\subset\mathbf{P}^n$ is by definition the zero set of a single homogeneous polynomial $f\in k[x_0,\ldots,x_n]$ of degree 1. Let V be the subspace of homogeneous elements of degree 1 in $k[x_0,\ldots,x_n]$. Form the projective space

$$\begin{aligned}\mathfrak{H}&=(V\setminus\{0\})/k^\times\\&=(k[x_0,\ldots,x_n]_1\setminus\{0\})/k^\times\end{aligned}$$

and view it as a projective variety over k; it is isomorphic to $\mathbf{P}^n_k$. Because two non-zero sections of $\mathcal{O}_{\mathbf{P}^n}$ determine the same hyperplane if and only if one is a multiple of the other by an element of $k^\times$, there is a canonical bijection between $\mathfrak{H}$ and the set of hyperplanes in $\mathbf{P}^n_k$.

Theorem (Bertini). *Let X be a non-singular closed subvariety of $\mathbf{P}^n_k$, where k is an algebraically closed field. Then there exists a hyperplane $H\subset\mathbf{P}^n_k$, not containing X, such that the scheme $H\cap X$ is regular. Moreover, the set of all hyperplanes with this property is an open dense subset of $\mathfrak{H}$.*

Proof. Consider a closed point x of X. There is an $i\in\{0,2,\ldots,n\}$ such that x is not in the hyperplane defined by x_i; after renaming the coordinates we may assume $i=0$. Then f/x_0 is a regular function in a neighbourhood of x for all $f\in V$, so there is a k-linear map

$$\begin{aligned}\phi_x\colon V&\to\mathcal{O}_{X,x}\\ f&\mapsto f/x_0,\end{aligned}$$

where $\mathcal{O}_{X,x}$ is the local ring of X at x. If X is contained in the hyperplane H defined by f, then $\phi_x(f)=0$; conversely, $\phi_x(f)=0$ means that f vanishes on some open neighbourhood of x in X, hence on all of X since X is irreducible. We conclude that $\phi_x(f)=0 \iff X\subseteq H$. Furthermore, $\phi_x(f)\in\mathfrak{m}_x \iff x\in H$.

Assume $X\not\subseteq H$ but $x\in X\cap H$, so that $\phi_x(f)\in\mathfrak{m}_x\setminus\{0\}$. Then $\mathfrak{f}=\phi_x(f)\mathcal{O}_{X,x}$ is a non-zero ideal of $\mathcal{O}_{X,x}$ contained in $\mathfrak{m}_x$. Now the local ring of $H\cap X$ at x is $\mathcal{O}_{X,x}/\mathfrak{f}$, and its maximal ideal is $\mathfrak{n}=\mathfrak{m}_x/\mathfrak{f}$. The fact that $\mathcal{O}_{X,x}$ is an integral domain and $\mathfrak{f}$ is a non-zero principal ideal implies that

$$\dim(\mathcal{O}_{X,x}/\mathfrak{f})=\dim(\mathcal{O}_{X,x})-1.$$

Furthermore, $\mathfrak{n}^2=(\mathfrak{m}_x^2+\mathfrak{f})/\mathfrak{f}$ and $\mathfrak{n}/\mathfrak{n}^2\cong\mathfrak{m}_x/(\mathfrak{m}_x^2+\mathfrak{f})$. In particular,

$$\dim_k\mathfrak{n}/\mathfrak{n}^2\le\dim_k\mathfrak{m}_x/\mathfrak{m}_x^2$$

with equality if and only if $\mathfrak{f}\subseteq\mathfrak{m}^2$. Recall that $\dim_k\mathfrak{m}_x/\mathfrak{m}_x^2\ge\dim\mathcal{O}_{X,x}$ with equality if and only if $\mathcal{O}_{X,x}$ is a regular local ring. Applying this also to $\mathcal{O}_{X,x}/\mathfrak{f}$ we see that $\mathcal{O}_{X,x}/\mathfrak{f}$ is regular if $\mathfrak{f}\not\subseteq\mathfrak{m}^2$ (in which case $\dim_k\mathfrak{n}/\mathfrak{n}^2=\dim\mathcal{O}_{X,x}/\mathfrak{f}$), and not regular if $\mathfrak{f}\subseteq\mathfrak{m}$. Hence $\mathcal{O}_{X,x}/\mathfrak{f}$ is a regular local ring if and only if $\phi_x(f)\in\mathfrak{m}_x\setminus\mathfrak{m}_x^2$.

Let $B_x\subset\mathfrak{H}$ be the set of hyperplanes that are defined by an element $f\in V$ for which $\phi_x(f)\in\mathfrak{m}_x^2$. In other words, if we put

$$\begin{aligned}\bar\phi_x\colon V&\to\mathcal{O}_{X,x}/\mathfrak{m}_x^2\\ f&\mapsto f/x_0 \bmod \mathfrak{m}_x^2\end{aligned}$$

then

$$B_x = (\ker \bar{\phi}_x \setminus \{0\})/k^\times \subseteq \mathfrak{H}.$$

This is a subvariety of $\mathfrak{H}$, the interpretation of which is as follows: a hyperplane H is in B_x if and only if either $H \supseteq X$ or $x \in H \cap X$ and x is a singular point of $H \cap X$. Let us take a closer look at B_x. We put $y_i = x_i/x_0$ for $1 \le i \le n$, so that $\operatorname{Spec} k[y_1, \ldots, y_n]$ is an affine open neighbourhood of x. Let $g_1, \ldots, g_m \in k[y_1, \ldots, y_n]$ be local equations for X, and let $(a_1, \ldots, a_n)$ be the coordinates of the point x. Then $\mathcal{O}_{X,x}$ is isomorphic to $A_\mathfrak{p}$, where

$$\begin{aligned} A &= (k[y_1, \ldots, y_n]/(g_1, \ldots, g_m)), \\ \mathfrak{p} &= (y_1 - a_1, \ldots, y_n - a_n), \end{aligned}$$

and $\mathfrak{m}_x$ corresponds to $\mathfrak{p}A_\mathfrak{p}$ under this isomorphism. Furthermore, the k-vector space $\mathcal{O}_{X,x}/\mathfrak{m}_x^2$ has dimension

$$\dim_k(\mathcal{O}_{X,x}/\mathfrak{m}_x^2) = \dim_k(\mathcal{O}_{X,x}/\mathfrak{m}_x) + \dim_k(\mathfrak{m}_x/\mathfrak{m}_x^2) = 1 + \dim X$$

and is spanned over k by the elements 1, $y_1 - a_1$, $\ldots$, $y_n - a_n$ (easy check). This shows that $\bar{\phi}_x$ is surjective, and

$$\begin{aligned} \dim \ker \bar{\phi}_x &= \dim_k V - \dim_k(\mathcal{O}_{X,x}/\mathfrak{m}_x^2) \\ &= (n+1) - (1 + \dim X) \\ &= n - \dim X, \end{aligned}$$

from which we conclude that $\dim B_x = n - \dim X - 1$.

The polynomials g_1, $\ldots$, g_m which locally define X are modulo $\mathfrak{m}_x^2$ congruent to the polynomials

$$\bar{g}_i = \sum_{j=1}^n (y_j - a_j) \frac{\partial g_i}{\partial y_j}(a_1, \ldots, a_n) \quad (1 \le i \le m).$$

Because $\phi_x(f)$ is of the form $b_0 + \sum_{j=1}^n b_j y_j$, we see that

$$\phi_x(f) \in \mathfrak{m}_x^2 \iff f/x_0 \in \sum_{i=1}^m k\bar{g}_i,$$

or, equivalently,

$$\ker \bar{\phi}_x = \sum_{i=1}^m k x_0 \bar{g}_i \quad \text{and} \quad B_x = \left(\sum_{i=1}^m k x_0 \bar{g}_i \setminus \{0\} \right) /k^\times.$$

Consider the fibred product $X \times_k \mathfrak{H}$. Because of the above characterisation of $\ker \bar{\phi}_x$, there is a closed subscheme B of $X \times_k \mathfrak{H}$ such that the closed points of B are precisely the points of $X \times_k \mathfrak{H}$ corresponding to the pairs (x, H) with x a closed point of X and $H \in B_x$.

We have seen that the fibre of B above each point of X has dimension $n - \dim X - 1$, so B itself has dimension $(n - \dim X - 1) + \dim X = n - 1$. Because X is proper over k and proper morphisms are preserved under base extension, the projection $p_2 \colon X \times_k \mathfrak{H} \to \mathfrak{H}$ is proper too. This implies that $p_2(B)$ is a closed subset of $\mathfrak{H}$ of dimension at most $n - 1$, and from this we conclude that $\mathfrak{H} - p_2(B)$ is an open dense subset of $\mathfrak{H}$. For each $H \in \mathfrak{H} \setminus p_2(B)$, the scheme $H \cap X$ is regular at every point by the construction of B.

7. Application to the existence of non-singular curves of type (a, b)

Let k be an algebraically closed field, and let a, b be positive integers. We want to show that there are non-singular curves of type (a, b) on $\mathbf{P}^1_k \times_k \mathbf{P}^1_k$. First we embed $\mathbf{P}^1_k \times_k \mathbf{P}^1_k$ into $\mathbf{P}^n_k$, where $n = ab + a + b$, using the a-uple, b-uple and Segre embeddings:

$$\mathbf{P}^1_k \times_k \mathbf{P}^1_k \longrightarrow \mathbf{P}^a_k \times_k \mathbf{P}^a_k \longrightarrow \mathbf{P}^n_k.$$

Recall that the a-uple embedding is defined by

$$(x_0 : x_1) \mapsto (x_0^a : x_0^{a-1} x_1 : \ldots : x_1^a)$$

and similarly for the b-uple embedding; the Segre embedding is defined by

$$((s_0 : \ldots : s_a), (t_0 : \ldots : t_b)) \mapsto (\ldots : s_i t_j : \ldots)$$

in lexicographic order. Let j denote the composed embedding $\mathbf{P}^1_k \times_k \mathbf{P}^1_k \to \mathbf{P}^n_k$. The image of j is a non-singular surface X in $\mathbf{P}^n_k$ that is isomorphic to $\mathbf{P}^1_k \times_k \mathbf{P}^1_k$. We apply Bertini's theorem to find a hyperplane H in $\mathbf{P}^N_k$ such that $H \cap X$ is a one-dimensional regular closed subscheme of X. This hyperplane is given by a homogeneous linear polynomial in the coordinates $\{z_{i,j} : 0 \le i \le a, 0 \le j \le b\}$ of $\mathbf{P}^n_k$. Now

$$z_{i,j} = j(x_0^{a-i} x_1^i y_0^{b-j} y_1^j),$$

so $Y = j^{-1}(H \cap X)$, viewed as a divisor on $\mathbf{P}^1_k \times_k \mathbf{P}^1_k$, is of type (a, b). We have seen earlier that this implies that Y is connected. The local rings of Y are regular local rings, so in particular they are integral domains [Hartshorne, Remark II.6.11.1A]. This means that there cannot be two irreducible components of Y intersecting each other; therefore Y is irreducible, and hence a non-singular curve.

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