Perturbations of conditionally periodic motion (Kolmogorov–Arnold–Moser theory)

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1 Introduction

In classical mechanics, we are sometimes lucky enough to encounter an *integrable* system, one in which there as many conserved quantities as there are degrees of freedom. Such systems are in a sense the easiest systems to study: they can be described in terms of *action-angle variables*, for which the Hamiltonian depends only on the conserved quantities (the action variables). For these systems, the canonical equations of motion and their solutions are very simple.

However, in many 'real' situations, such as the solar system, there are no longer enough constants of motion. When the system is nearly integrable, i.e. the Hamiltonian can be written as a small perturbation of an integrable one, it turns out that much of the nice behaviour of integrable systems can be rescued. This is the content of a famous theorem from 1954 by A.N. Kolmogorov [4], which was extended by V.I. Arnold and J. Moser. The general theory describing small perturbations of integrable systems is called *Kolmogorov–Arnold–Moser theory*, or KAM theory. An overview of KAM theory is given in [2].

In section 2, the situation is explained in the case of an integrable system (the unperturbed system). The simple behaviour of such a system, which is called *conditionally periodic motion*, is described here. Section 3 deals with small perturbations of integrable systems and introduces Kolmogorov's theorem. There will be no attempt at a proof of Kolmogorov's theorem, because it is rather long and technical; only a couple of ideas for the proof mentioned by Arnold [1] are given.

A surprising relation to number theory appears in connection with the question under what conditions the system will still exhibit conditionally periodic motion. This relation is stated in section 4, along with possibly complicated behaviour in regions of the phase space where this condition is not satisfied. In section 5 some practical considerations are given which appear when applying KAM theory to real systems.

2 Conditionally periodic motion

Suppose we have a mechanical system with n degrees of freedom, described by a Hamiltonian H_0 on a $2n$-dimensional manifold M (the phase space) with a symplectic 2-form ω. Furthermore, we assume that the system is integrable (this means that there are n independent first integrals, or constants of motion, $I_1, I_2, \ldots, I_n$), and that the integral curves of the system lie on compact sets within the phase space.

In this situation, these compact sets are n-dimensional tori in the phase space, called *invariant tori.* Furthermore, it is possible to introduce *action-angle variables* as coordinates on the phase space. The action variables are a set of n independent first integrals $I = (I_1, I_2, \ldots, I_n)$,, any particular value of which fixes an invariant torus with coordinates $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$. The symplectic 2-form can be written in these coordinates as

$$\omega = \sum_{i=1}^{n} dI_i \wedge d\phi_i,$$

and the Hamiltonian H_0 depends only on I. From this it is not hard to see that the canonical equations of motion are

$$\frac{dI_i}{dt} = 0 \quad \text{and} \quad \frac{d\phi_i}{dt} = \omega_i, \text{ where } \omega_i := \frac{\partial H_0}{\partial I_i}.$$

From this, we see that the conserved quantities I of a given trajectory determine a set of freqencies $(\omega_1, \omega_2, \ldots, \omega_n)$. If there is a non-trivial $\mathbf{Z}$-linear relation between the ω_i, i.e.

$$\sum_{i=1}^{n} a_i\omega_i = 0 \quad \text{with } a_i \in \mathbf{Z} \text{ not all } 0,$$

the torus on which the orbit lies is called *resonant*; otherwise it is called *non-resonant*. For $n = 2$, the set of frequencies (ω_1, ω_2) is resonant precisely when ω_1/ω_2 is a rational number.

3 Perturbations of conditionally periodic motion

We now look at what happens if we introduce a perturbation of an integrable Hamiltonian system. We write the Hamiltonian as

$$H(I, \phi) = H_0(I) + \epsilon H_1(I, \phi),$$

where ϵH_1 is small compared to H_0. An important theorem of A.N. Kolmogorov from 1954 says that, under suitable conditions, most of the invariant tori of the original system are still present in the perturbed system, only somewhat deformed. There are several versions of the theorem; the original formulation uses the condition of *non-degeneracy* of the frequencies. This condition requires the map

$$I \mapsto (\omega_1, \omega_2, \ldots, \omega_n)(I) = \left(\frac{\partial H}{\partial I_1}, \frac{\partial H}{\partial I_2}, \ldots, \frac{\partial H}{\partial I_n}\right)$$

to have full rank on the invariant torus we are considering. Formulated more explicitly, the non-degeneracy condition is

$$\det\left(\frac{\partial \omega_i}{\partial I_j}\right)_{i,j=1}^{n} = \det\left(\frac{\partial^2 H}{\partial I_i \partial I_j}\right)_{i,j=1}^{n} \neq 0.$$

Kolmogorov's theorem says the following:

Theorem (Kolmogorov, 1954). *If the non-degeneracy condition is fulfilled and ϵ is sufficiently small, the phase space of the perturbed system is 'mostly' filled by deformations of the non-resonant invariant tori of the unperturbed system, in the sense that the complement of these deformed tori has small measure if ϵ is small.*

Kolmorogov's theorem originally only indicated the existence of a single invariant torus in the perturbed system. The theorem was improved upon by V.I. Arnold and J. Moser in the 1960s, and the result is nowadays known as KAM theory.

The idea of the proof is as follows. At a particular initial point in the phase space, the frequencies ω_i will generally change as a result of the perturbation. Because of the non-degeneracy, however, we can shift the initial point a little bit so that we land on a point with exactly the original frequencies. It turns out that it is possible to apply an iterative procedure to the original torus; this procedure can be shown to converge to an invariant torus in the perturbed system under a certain condition on the frequencies, which will be described in the next section.

There are many variations on the above theorem. One important counterpart is the *isoenergetic* KAM *theorem*, which states that the same result holds if the system is *isoenergetically non-degenerate*. This means that on each energy level set, the map that associates to a point x the ratios of the frequencies ω_i at that point is non-degenerate, i.e. the map

$$x \mapsto (\omega_1 : \omega_2 : \cdots : \omega_n) = \left(\frac{\partial H_0}{\partial I_1} : \frac{\partial H_0}{\partial I_2} : \cdots : \frac{\partial H_0}{\partial I_n}\right)$$

from the $(n-1)$-dimensional energy level manifold containing x to $(n-1)$-dimensional projective space has full rank. An equivalent condition is (note that $\frac{\partial^2 H_0}{\partial I^2}$ is a $(n \times n)$-matrix and $\frac{\partial H_0}{\partial I}$ is a column vector)

$$\det \begin{pmatrix} \frac{\partial^2 H_0}{\partial I^2} & \frac{\partial H_0}{\partial I} \\ \left(\frac{\partial H_0}{\partial I}\right)^T & 0 \end{pmatrix} \neq 0.$$

The conditions of non-degeneracy and isoenergetic non-degeneracy do not imply one another; there are non-degenerate systems that are isoenergetically degenerate, and also isoenergetically non-degenerate systems that are degenerate.

4 Resonance, chaos and number theory

In the proof of Kolmogorov's theorem, the condition guaranteeing that everything converges, so that new invariant tori can be found, is given by a so-called *Diophantine condition*, which is well-known in number theory. It selects from the set of frequency vectors $(\omega_1, \omega_2, \ldots, \omega_n)$ a certain subset where the ratios between the frequencies are in some sense sufficiently irrational. More specifically, the condition on $(\omega_1, \omega_2, \ldots, \omega_n)$ is that there exist $C > 0, \nu > 0$ such that

$$\Big| \sum_{i=1}^{n} a_i \omega_i \Big| \geq C \|\mathbf{a}\|^{-\nu} \quad \text{for all } \mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{Z}^n.$$

For $n = 2$, the condition on $\alpha = \omega_1/\omega_2$ is similar[1] to the condition that for some $\gamma > 0, \tau > 2$ we have

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{\gamma}{q^{-\tau}} \quad \text{for all } \frac{p}{q} \in \mathbf{Q}.$$

This last condition is well-known in the theory of algebraic and transcendent numbers; it says that α must be 'sufficiently irrational' and is satisfied for example by numbers $\alpha \notin \mathbf{Q}$ which are algebraic, i.e. which are zeroes of polynomials with rational coefficients.

The cases $n = 2$ and $n > 2$ are fundamentally different in the behaviour of trajectories lying outside the deformed non-resonant tori in the phase space. In the case of two degrees of freedom, we have to do with a four-dimensional phase space. The regions where the value of the Hamiltonian equals a fixed value E are three-dimensional, and each torus separates such a hypersurface into an inner and an outer region. This means that any trajectory in the phase space will generally lie between two deformed tori, and will be confined to a bounded region of phase space.

For more than two degrees of freedom, this property of tori no longer holds, so the trajectories of the system are no longer bounded. In regions of the phase space which are not covered by deformations of the non-resonant tori, the behaviour of the perturbed system can be influenced strongly by *resonance*. This may be viewed as the kind of situation in which the effect of the perturbation is amplified because the perturbation is in some way in phase with the motion of the system. A famous example of resonance (although not a Hamiltonian system) is the driven pendulum: if the frequency of the driving force equals the natural frequency of the pendulum, the oscillations are amplified, while the contribution of the force is on average zero if the frequencies are unrelated.

The appearance of resonance often leads to *chaos*, i.e. strong sensitivity to initial conditions. An example is the three-body problem: in almost all initial configurations of three spherical masses attracting each other by gravity, the orbits are very complicated, and usually one of the masses is 'kicked away' to infinity.

5 Practical applications

A number of applications of KAM theory are given in Appendix 8 of [1]. One important application of KAM-like ideas is the solar system (and more generally the n-body problem). The simplest approach is to consider the Hamiltonian for two planets of mass m_1 and m_2

[1] Maybe the conditions are equivalent; I haven't checked, but it is not important here.

revolving around a fixed star with mass $M_\star$. In appropriate units where $G = 1, M_\star = 1$ and $m_1, m_2 \ll 1$, the Hamiltonian for this system is

$$H(r_i, \phi_i, p_i, L_i) = \sum_{i=1}^{2} \left[\frac{p_i^2}{2m_i} + \frac{L_i^2}{2m_i r^2} - \frac{m_i}{r_i} \right] - \frac{m_1 m_2}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\phi_1 - \phi_2)}}.$$

The first term on the right represents the kinetic energy and the gravitational attraction of the planets by the star, while the second term represents the attraction between the planets. The 'unperturbed' system, where this second term is neglected, is integrable; if m_1, m_2 are small enough, we can regard the second term as a perturbation of this integrable system.

A number of problems arise when applying KAM theory to our own solar system. The first difficulty is that the resonances in our own solar system appear to be too large for the KAM theorem to apply. Furthermore, it is impossible in practice to determine the frequencies of system exactly, which is important to decide whether they are irrational enough.

This last problem is a general 'feature' in actual applications of the KAM theorem. In these cases, it is possible to make use of *Nekhoroshev's theorem* [5], which gives quantitative estimates for the stability of general trajectories in perturbed Hamiltonian systems. A combined approach to KAM theory and Nekhoroshev's theorem, which in a way complement each other, is described in [3].

References

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