PRACTICE EXAM
GALOIS REPRESENTATIONS AND AUTOMORPHIC FORMS

You only have to do *two* of the problems of your choice.
You are allowed to refer to results from the notes, but not to the exercises.

Problem 1. Let K be a field of characteristic 0.

(a) Consider the polynomial ring $K[X_1, X_2]$. For all $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$, we define $g \cdot X_1 = aX_1 + cX_2$, $g \cdot X_2 = bX_1 + dX_2$. Show that these formulas can be extended to define a representation of $\mathrm{GL}_2(K)$ on the vector space of polynomials $K[X_1, X_2]$.

(b) Consider the vector space V of polynomials f in two variables X_1, X_2 with coefficients in K which are homogeneous of degree 2. Show that $V \subset K[X_1, X_2]$ is an irreducible subrepresentation of dimension 3.
Hint: Show that the only subspaces of V that are stable under the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(K)$ are $\{0\}$, $\mathbb{C}X_1^2$, $\mathbb{C}X_1^2 + \mathbb{C}X_1X_2$ and V; similarly, determine the subspaces of V that are stable under the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

From the K-basis (X_1^2, X_1X_2, X_2^2) of V, we obtain an isomorphism $V \cong K^3$ and a representation

$$\mathrm{Sym}^2\colon \mathrm{GL}_2(K) \to \mathrm{GL}_3(K).$$

This representation is called the *symmetric square*. From now on, we will take $K = \overline{\mathbb{Q}}_l$. Let F be a number field, and let ℓ be a prime number. Consider a semi-simple Galois representation

$$\rho\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell).$$

We write

$$r = \mathrm{Sym}^2(\rho)\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$$

for the composition of ρ with the representation $\mathrm{Sym}^2\colon \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell) \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$.

(c) Show that at every F-place v where the representation ρ is unramified, the representation r is unramified as well, and we have

$$\mathrm{charpol}(r(\mathrm{Frob}_v)) = X^3 - (t_v^2 - d_v)X^2 + d_v(t_v^2 - d_v)X - d_v^3 \in \overline{\mathbb{Q}}_\ell[X], \tag{1}$$

where $t_v = \mathrm{Tr}\,\rho(\mathrm{Frob}_v)$ and $d_v = \det(\rho(\mathrm{Frob}_v))$ in $\overline{\mathbb{Q}}_\ell$.

(d) Consider another semi-simple Galois representation

$$r'\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell),$$

such that for almost all F-places v where r' is unramified, the characteristic polynomial of $r'(\mathrm{Frob}_v) \in \mathrm{GL}_3(\overline{\mathbb{Q}}_\ell)$ is given by equation (1). Show that r' is isomorphic to r.

Date: 23 December 2016.

Problem 2. Let F be a number field, and let $\chi\colon \mathbb{A}_F^\times \to \mathbb{C}^\times$ be a Hecke character, *i.e.* a continuous morphism which is trivial on $F^\times$ embedded diagonally in the idèles $\mathbb{A}_F^\times = \prod'_v(F_v^\times : \mathcal{O}_{F_v}^\times)$. Assume that F is *totally real*, *i.e.* all Archimedean places are real. Let $S = \{v_1, \ldots, v_r\}$ be the set of Archimedean places of F, all of which are real by assumption; here $r = [F : \mathbb{Q}]$. By a version of Dirichlet's unit theorem from algebraic number theory, the abelian group $\mathcal{O}_F^\times$ is isomorphic to $\mathbb{Z}^{r-1}$ times a finite group, and the image of the group homomorphism

$$\begin{aligned} \mathcal{O}_F^\times &\to \mathbb{R}^r \\ x &\mapsto (\log|x|_{v_i})_{i=1}^r \end{aligned}$$

is a discrete subgroup of rank $r-1$ in $\mathbb{R}^r$. In particular, the $\mathbb{R}$-vector space spanned by this subgroup has dimension $r-1$.

In this exercise we will show that there exists a real number $w \in \mathbb{R}$ such that the character $\chi \cdot |\cdot|_{\mathbb{A}_F^\times}^{-w}\colon \mathbb{A}_F^\times \to \mathbb{C}^\times$ has finite image.

(a) Let $\chi_\infty\colon F_\infty^\times \to \mathbb{C}^\times$ be the restriction of χ to

$$F_\infty^\times := (F \otimes_{\mathbb{Q}} \mathbb{R})^\times \cong \prod_{v|\infty} F_v^\times$$

via the inclusion of $F_\infty^\times$ into the infinite part of the idèles $\mathbb{A}_F^\times$. Show that χ_∞ is trivial on a subgroup of $\mathcal{O}_F^\times$ which is of finite index.

(b) Let H be a subgroup of finite index in $\mathcal{O}_F^\times$. Show that the additive group $\mathrm{Hom}(F_\infty^\times/H, \mathbb{R})$ of continuous group homomorphisms $F_\infty^\times/H \to \mathbb{R}$ has a natural structure of a real vector space of dimension 1.

(c) Deduce that there exists a real number w satisfying

$$\log|\chi_\infty(x)| = w \log\left(\prod_{v|\infty} |x_v|_{F_v}\right) \quad \text{for all } x = (x_v)_{v|\infty} \in F_\infty^\times.$$

(d) Identify $\mathbb{A}_F^{\infty,\times}/F^\times\widehat{\mathcal{O}}_F^\times$ with the class group of F, and deduce that this quotient is finite. Then show that for any compact open subgroup $U \subset \mathbb{A}_F^{\infty,\times}$ the quotient $\mathbb{A}_F^{\infty,\times}/F^\times U$ is finite.

(e) Show that the character $\mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$, $x \mapsto |x|_{\mathbb{A}_F^\times}^{-w} \cdot \chi(x)$ has finite image.

Problem 3. In this problem we assume that the global Langlands conjecture is true and investigate some of its consequences. Let F be a number field, and let F' be a quadratic extension of F.

(a) Let V be a two-dimensional $\mathbb{C}$-vector space, and let ϕ be an endomorphism of V. Write the characteristic polynomial of ϕ as $X^2 - tX + d$. Show that the characteristic polynomial of $\phi \circ \phi$ equals $X^2 - (t^2 - 2d)X + d^2$.

(b) Let π be a cuspidal algebraic automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$, and let S be the set of all finite places v of F such that both the smooth representation π_v of $\mathrm{GL}_2(F_v)$ is unramified at v and the extension F'/F is unramified at v. For each $v \in S$, recall that the Satake parameter of π_v is a semi-simple conjugacy class in $\mathrm{GL}_2(\mathbb{C})$; we write its characteristic polynomial as $X^2 - t_v X + d_v \in \mathbb{C}[X]$.

Assuming the global Langlands conjecture, prove that there exists a unique automorphic representation Π of $\mathrm{GL}_2(\mathbb{A}_{F'})$ with the following properties: Π is unramified at all places w of F' lying above a place $v \in S$, and for every such place w, the Satake parameter of Π_w is the unique semi-simple conjugacy class in $\mathrm{GL}_2(\mathbb{C})$ whose characteristic polynomial is given by

$$\begin{cases} X^2 - t_v X + d_v & \text{if } v \text{ is split in } F', \\ X^2 - (t_v^2 - 2d_v)X + d_v^2 & \text{if } v \text{ is inert in } F'. \end{cases}$$

(c) Let E be an elliptic curve over F. Let S be the set of finite places of F such that E has good reduction at v and the extension F'/F is unramified at v. For all $v \in S$, let $\kappa(v)$ be the residue field of F at v, let $q_v = \#\kappa(v)$, and let $a_v(E) = 1 - \#E(\kappa(v)) + q_v$. Assuming the global Langlands conjecture, prove that the Euler product

$$\prod_{v \in S \text{ split in } F'} \frac{1}{(q_v^{-2s} - a_v(E) q_v^{-s} + q_v)^2} \cdot \prod_{v \in S \text{ inert in } F'} \frac{1}{q_v^{-4s} - (a_v(E)^2 - 2q_v) q_v^{-2s} + q_v^2}$$

converges for $\Re s$ sufficiently large and (after multiplying by suitable Euler factors at the places outside S) has an analytic continuation to the whole complex plane that satisfies a functional equation (which you do not need to specify).

Problem 4. Let p and ℓ be distinct prime numbers. Let $\langle p \rangle$ be the subgroup of $\mathbb{Q}_p^\times$ generated by p, and let $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. For all integers $r \geq 0$, let A_r be the Abelian group defined by

$$A_r = (\overline{\mathbb{Q}}_p^\times / \langle p \rangle)[\ell^r] = \{x \in \overline{\mathbb{Q}}_p^\times \mid x^{\ell^r} \in \langle p \rangle\} / \langle p \rangle$$

with the natural action of $G_{\mathbb{Q}_p}$.

(a) Show that A_r is (non-canonically) isomorphic to $\mathbb{Z}/\ell^r\mathbb{Z} \times \mathbb{Z}/\ell^r\mathbb{Z}$.

(b) Show that there exists a Galois-equivariant short exact sequence

$$1 \longrightarrow \mu_{\ell^r}(\overline{\mathbb{Q}}_p) \longrightarrow A_r \longrightarrow B_r \longrightarrow 1$$

where B_r is a cyclic group of order ℓ^r with trivial action of $G_{\mathbb{Q}_p}$.

(c) Define $\mathbb{Q}_\ell(1) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_r \mu_{\ell^r}(\overline{\mathbb{Q}}_p)$ and

$$V = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \varprojlim_r A_r.$$

Let $I_{\mathbb{Q}_p} \subset G_{\mathbb{Q}_p}$ be the inertia subgroup, and let $V^{I_{\mathbb{Q}_p}} \subseteq V$ be the subspace of inertia invariants. Show that there is an isomorphism $\mathbb{Q}_\ell(1) \xrightarrow{\sim} V^{I_{\mathbb{Q}_p}}$.

(d) Show that the L-function of the representation V of $G_{\mathbb{Q}_p}$ equals $(1 - p \cdot p^{-s})^{-1}$.