Representation Theory of Finite Groups

Example solutions for the practice exam

1. Let $\phi\colon R \to S$ be a homomorphism of rings, and let M be a *simple* S-module. Let ϕ^*M be the Abelian group M viewed as an R-module via $(r, m) \mapsto \phi(r)m$ for $r \in R$ and $m \in M$.

(a) [Assume that ϕ is surjective. Show that ϕ^*M is simple.]

Since M is simple, we have $M \ne 0$. For any non-zero element $t \in M$, the sub-R-module $Rt = \{rt \mid r \in R\}$ equals $St = \{st \mid s \in S\}$ because ϕ is surjective. Since M is simple over S, we have $St = M$. We conclude that every non-zero sub-R-module of ϕ^*M equals ϕ^*M, so ϕ^*M is simple.

(b) [Give an example where ϕ is not surjective and ϕ^*M is not simple.]

Take $\phi\colon \mathbf{R} \to \mathbf{C}$ to be the inclusion and $M = \mathbf{C}$. Then ϕ is not surjective and M is not simple as an $\mathbf{R}$-module since it is isomorphic to $\mathbf{R} \oplus \mathbf{R}$.

(c) [Give an example where ϕ is not surjective, but where ϕ^*M is still simple.]

Take $\phi\colon \mathbf{C} \to \mathbf{C}[x]$ to be the inclusion and let $M = \mathbf{C}$ with the $\mathbf{C}$-action defined by letting x act as 0. Then ϕ is not surjective, but M is simple as a $\mathbf{C}$-module since $M \ne 0$ and any non-zero element of M generates M.

2. Let G be a finite group, let $[G, G]$ be the commutator subgroup of G, and let $G_{\mathrm{ab}} = G/[G, G]$ be the maximal Abelian quotient of G.

(a) [Let g be an element of G with $g \notin [G, G]$. Show that there exists a one-dimensional representation of G on which g acts non-trivially. (*Hint:* one possibility is to use the group ring $\mathbf{C}[G_{\mathrm{ab}}]$.)]

It has been proved during the course that $\mathbf{C}[G_{\mathrm{ab}}]$ is isomorphic to $\prod_{i=1}^{r} \mathrm{Mat}_{n_i}(\mathbf{C})$ where $n_1, \ldots, n_r$ are the dimensions of the irreducible representations of G_{ab}. Since G_{ab} is Abelian, all the n_i are equal to 1, so $\mathbf{C}[G_{\mathrm{ab}}] \cong \mathbf{C}^r$ as $\mathbf{C}$-algebras. Let $\bar{g}$ be the image of g in G_{ab}. Since $g \notin [G, G]$, we have $\bar{g} \ne 1$, so its image in at least one of the factors in the above product decomposition of $\mathbf{C}[G_{\mathrm{ab}}]$ is different from 1. This means that $\bar{g}$ acts non-trivially on the corresponding one-dimensional representation of G_{ab}. Viewing this representation as a representation of G via the canonical map $G \to G_{\mathrm{ab}}$, we obtain a one-dimensional representation of G on which g acts non-trivially.

(b) [Let V be an irreducible representation of G. Show that for every one-dimensional representation W of G, the representation $V \otimes_{\mathbf{C}} W$ is irreducible.]

We note that $V \ne 0$ and $\dim_{\mathbf{C}}(V \otimes_{\mathbf{C}} W) = \dim_{\mathbf{C}} V \cdot \dim_{\mathbf{C}} W = \dim_{\mathbf{C}} V$, hence $V \otimes_{\mathbf{C}} W \ne 0$. Let $W^\vee$ be the dual representation of W; then $W \otimes W^\vee$ is the trivial representation. Let N be a subrepresentation of $V \otimes_{\mathbf{C}} W$. There is a short exact sequence

$$0 \longrightarrow N \longrightarrow V \otimes_{\mathbf{C}} W \longrightarrow Q \longrightarrow 0$$

of $\mathbf{C}[G]$-modules. Tensoring by $W^\vee$ and using the isomorphism $(V \otimes_{\mathbf{C}} W) \otimes_{\mathbf{C}} W^\vee \cong V \otimes_{\mathbf{C}} (W \otimes_{\mathbf{C}} W^\vee) \cong V$, we obtain a short exact sequence

$$0 \longrightarrow N \otimes_{\mathbf{C}} W^\vee \longrightarrow V \longrightarrow Q \otimes_{\mathbf{C}} W^\vee \longrightarrow 0.$$

Since V is irreducible, either $N \otimes_{\mathbf{C}} W^\vee$ or $Q \otimes_{\mathbf{C}} W^\vee$ is the zero module. Hence either N or Q is the zero module. This shows that $V \otimes_{\mathbf{C}} W$ is irreducible.

(Alternative solution: show that the inner product of the character of $V \otimes_{\mathbf{C}} W$ with itself equals the inner product of the character of V with itself.)

(c) [Suppose that G has exactly one irreducible representation of dimension > 1 (up to isomorphism), and let χ be the character of this representation. Show that all $g \in G$ with $g \notin [G, G]$ satisfy $\chi(g) = 0$.]

Let V be the unique irreducible representation of G of dimension > 1. By part (a), there is a one-dimensional representation W of G on which g acts non-trivially. Let $\epsilon\colon G \to \mathbf{C}$ be the character of this representation; then $\epsilon(g) \neq 1$. By part (b), the representation $V \otimes_{\mathbf{C}} W$ is irreducible. By assumption, it is isomorphic to V. Looking at the characters of these two representations, we see that $\chi\epsilon = \chi$. In particular, we get $\chi(g)\epsilon(g) = \chi(g)$. Since $\epsilon(g) \neq 1$, it follows that $\chi(g) = 0$.

3. Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8. (Recall the relations $(-1)^2 = 1$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.)

In this question, you may only use general results about representations, as opposed to results on representations of the particular group Q.

(a) [Show that Q has exactly four irreducible representations of dimension 1 over $\mathbf{C}$ (up to isomorphism), and give these explicitly.]

The one-dimensional representations of Q are homomorphisms $Q \to \mathbf{C}^\times$, so they factor via the largest Abelian quotient $Q_{\mathrm{ab}} = Q/[Q, Q]$ of Q. Note that $iji^{-1}j^{-1} = ij(-i)(-j) = ijij = k^2 = -1$, so $-1 \in [Q, Q]$. The subgroup $\{\pm 1\}$ is normal (even central), and the quotient $Q/\{\pm 1\}$ is isomorphic to the Abelian group V_4, so this is the largest Abelian quotient of Q. Let $a, b, c \in Q/\{\pm 1\} \cong V_4$ be the images of i, j, k. A representation of V_4 is uniquely determined by the images of a and b, which must be in $\{\pm 1\}$. Hence the character table of V_4 is

conj. class	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$
size	1	1	1	1
	1	1	1	1
	1	1	-1	-1
	1	-1	1	-1
	1	-1	-1	1

Viewing these as representations of Q, we obtain exactly four one-dimensional representations of Q:

conj. class	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
size	1	1	2	2	2
	1	1	1	1	1
	1	1	1	-1	-1
	1	1	-1	1	-1
	1	1	-1	-1	1

Let ζ be a fixed square root of -1 in $\mathbf{C}$ (not denoted by i to avoid confusion). There is a representation $\rho\colon Q \to \mathrm{GL}_2(\mathbf{C})$ defined by

$$\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho(j) = \begin{pmatrix} \zeta & 0 \\ 0 & -\zeta \end{pmatrix}.$$

(b) [Compute $\rho(-1)$ and $\rho(k)$.]

We have

$$\rho(-1) = \rho(i^2) = \rho(i)^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\rho(k) = \rho(ij) = \rho(i)\rho(j) = \begin{pmatrix} 0 & -\zeta \\ -\zeta & 0 \end{pmatrix}.$$

(c) [Show that ρ is irreducible.]

From the above matrices, we see that the character χ of ρ is

conj. class	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
size	1	1	2	2	2
χ	2	-2	0	0	0

We obtain

$$\langle \chi, \chi \rangle_Q = \frac{1}{8}(2^2 + (-2)^2 + 0^2 + 0^2 + 0^2) = 1.$$

This shows that ρ is irreducible.

(Alternative solution: a non-trivial subrepresentation of ρ must be a simultaneous one-dimensional eigenspace of $\rho(i)$ and $\rho(j)$; a computation shows that these matrices do not have any eigenspaces in common.)

(d) [Show that every irreducible representation of Q over $\mathbf{C}$ is either one-dimensional or isomorphic to ρ.]

We have constructed five irreducible representations of Q. It has been proved during the course that the number of irreducible representations equals the number of conjugacy classes of Q, which is 5. Therefore every irreducible representation of Q is isomorphic to one of the representations we have constructed.

(e) [Determine the decomposition of $\rho \otimes \rho \otimes \rho \otimes \rho$ as a direct sum of irreducible representations of Q.]

With χ as in part (c), the character of $\rho^{\otimes 4} = \rho \otimes \rho \otimes \rho \otimes \rho$ equals χ^4. Taking the inner product of χ^4 with each of the five irreducible characters of Q shows that χ^4 equals 4 times the sum of the four one-dimensional characters.

$$\rho^{\otimes 4} \cong S_1^{\oplus 4} \oplus S_2^{\oplus 4} \oplus S_3^{\oplus 4} \oplus S_4^{\oplus 4},$$

where the S_i are the four one-dimensional representations of Q up to isomorphism.

4. Let G be a finite group, and let k be a field (possibly of characteristic dividing $\#G$.) Let $V = k[G]$, viewed as a k-linear representation of G via the action

$$\begin{aligned} G \times V &\longrightarrow V \\ (g, v) &\longmapsto gvg^{-1}. \end{aligned}$$

(a) [Show that the kernel of the group homomorphism $\rho\colon G \to \mathrm{Aut}_k(V)$ defined by the above action equals the centre $Z(G)$ of G.]

An element $g \in G$ is in the kernel of ρ if and only if it commutes with every element of V. Since V consists of the $\mathbf{C}$-linear combinations of elements of G, this is equivalent to commuting with every element of G, i.e. with g being in $Z(G)$.

Let c be the number of conjugacy classes of G, and let l be the length of V as a $k[G]$-module.

(b) [Prove the inequality $l \geq c$. (*Hint*: find non-trivial submodules of V.)]

Let $G/\sim$ be the set of conjugacy classes of G. For each $C \in G/\sim$, let $k\langle C\rangle$ be the k-linear subspace of V spanned by the elements of C. Since G acts on V by conjugation, each $k\langle C\rangle$ is a non-trivial sub-$k[G]$-module of V, and V is the direct sum of the $k\langle C\rangle$. We obtain

$$\mathrm{length}_{k[G]}(V) = \sum_{C\in G/\sim} \mathrm{length}_{k[G]}\, k\langle C\rangle \geq \#(G/\sim).$$

(c) [*Bonus question:* Show that if G is not Abelian, then l is strictly larger than c.]

If G is not Abelian, then there is a conjugacy class C with $\#C > 1$. Let $x = \sum_{g\in C} g$; then x is stable under conjugation, and hence $kx \subseteq k\langle C\rangle$ is a sub-$k[G]$-module that is neither zero nor equal to $k\langle C\rangle$. This implies $\mathrm{length}\, k\langle C\rangle \geq 1$, so the inequality obtained in part (b) is strict.

5. [Let A_5 be the alternating group of order 60, and let $g = (1\,2\,3\,4\,5) \in A_5$. We view the cyclic group C_5 of order 5 as a subgroup of A_5 by $C_5 = \langle g\rangle \subset A_5$. Let $\zeta = \exp(2\pi i/5) \in \mathbf{C}$, and let V be the one-dimensional representation of C_5 on which g acts as ζ. Determine the decomposition of $\mathrm{Ind}_{C_5}^{A_5} V$ as a direct sum of irreducible representations of A_5.]

Let ξ be the character of V, and let $\xi' = \mathrm{ind}_{C_5}^{A_5}$ be the induced character (i.e. the character of $\mathrm{Ind}_{C_5}^{A_5} V$). By Frobenius reciprocity, for each irreducible character χ of A_5, we have

$$\langle \chi, \xi'\rangle_{A_5} = \langle \chi|_{C_5}, \xi\rangle_{C_5}.$$

Let $\chi_1, \ldots, \chi_5$ be the irreducible characters of A_5. Here is a table of the characters $\chi_i|_{C_5}$ (read off from the character table of A_5 using the hint) and ξ of C_5:

conj. class	$\{1\}$	$\{g\}$	$\{g^2\}$	$\{g^3\}$	$\{g^4\}$
size	1	1	1	1	1
$\chi_1\|_{C_5}$	1	1	1	1	1
$\chi_2\|_{C_5}$	3	$-\zeta^2-\zeta^3$	$-\zeta-\zeta^4$	$-\zeta-\zeta^4$	$-\zeta^2-\zeta^3$
$\chi_3\|_{C_5}$	3	$-\zeta-\zeta^4$	$-\zeta^2-\zeta^3$	$-\zeta^2-\zeta^3$	$-\zeta-\zeta^4$
$\chi_4\|_{C_5}$	4	-1	-1	-1	-1
$\chi_5\|_{C_5}$	5	0	0	0	0
ξ	1	ζ	ζ^2	ζ^3	ζ^4

We compute (using the identity $1+\zeta+\zeta^2+\zeta^3+\zeta^4 = 0$)

$$\begin{aligned}
\langle \chi_1|_{C_5}, \xi\rangle &= \frac{1}{5}(1+\zeta+\zeta^2+\zeta^3+\zeta^4) = 0,\\
\langle \chi_2|_{C_5}, \xi\rangle &= \frac{1}{5}(3-(\zeta^3+\zeta^4)-(\zeta^3+\zeta)-(\zeta^4+\zeta^2)-(\zeta+\zeta^2)) = 1,\\
\langle \chi_3|_{C_5}, \xi\rangle &= \frac{1}{5}(3-(\zeta^2+1)-(\zeta^4+1)-(1+\zeta)-(1+\zeta^3)) = 0,\\
\langle \chi_4|_{C_5}, \xi\rangle &= \frac{1}{5}(4-\zeta-\zeta^2-\zeta^3-\zeta^4) = 1,\\
\langle \chi_5|_{C_5}, \xi\rangle &= \frac{1}{5}(5+0+0+0+0) = 1.
\end{aligned}$$

This shows that $\mathrm{Ind}_{C_5}^{A_5} V$ is the direct sum of the irreducible representations with characters χ_2, χ_4 and χ_5.