**Representation Theory of Finite Groups**

**Practice exam** (3 hours)

*Note:*

- You may consult books and lecture notes. The use of electronic devices is not allowed.
- You may use results proved in the lecture or in the exercises, unless this makes the question trivial. When doing so, clearly state the results that you use.
- If you are unable to answer a subquestion, you may still use the result in the remainder of the question.
- Representations are taken to be over $\mathbf{C}$, unless mentioned otherwise.

*(?? pt)* **1.** Let $\phi\colon R \to S$ be a homomorphism of rings, and let $M$ be a *simple* $S$-module. Let $\phi^*M$ be the Abelian group $M$ viewed as an $R$-module via $(r, m) \mapsto \phi(r)m$ for $r \in R$ and $m \in M$.

(a) Assume that $\phi$ is surjective. Show that $\phi^*M$ is simple.

(b) Give an example where $\phi$ is not surjective and $\phi^*M$ is not simple.

(c) Give an example where $\phi$ is not surjective, but where $\phi^*M$ is still simple.

*(?? pt)* **2.** Let $G$ be a finite group, let $[G, G]$ be the commutator subgroup of $G$, and let $G_{\mathrm{ab}} = G/[G, G]$ be the maximal Abelian quotient of $G$.

(a) Let $g$ be an element of $G$ with $g \notin [G, G]$. Show that there exists a one-dimensional representation of $G$ on which $g$ acts non-trivially. (*Hint:* one possibility is to use the group ring $\mathbf{C}[G_{\mathrm{ab}}]$.)

(b) Let $V$ be an irreducible representation of $G$. Show that for every one-dimensional representation $W$ of $G$, the representation $V \otimes_{\mathbf{C}} W$ is irreducible.

(c) Suppose that $G$ has exactly one irreducible representation of dimension $> 1$ (up to isomorphism), and let $\chi$ be the character of this representation. Show that all $g \in G$ with $g \notin [G, G]$ satisfy $\chi(g) = 0$.

*(?? pt)* **3.** Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8. (Recall the relations $(-1)^2 = 1$, $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.)

*In this question, you may only use general results about representations, as opposed to results on representations of the particular group $Q$.*

(a) Show that $Q$ has exactly four irreducible representations of dimension 1 over $\mathbf{C}$ (up to isomorphism), and give these explicitly.

Let $\zeta$ be a fixed square root of $-1$ in $\mathbf{C}$ (not denoted by $i$ to avoid confusion). There is a representation $\rho\colon Q \to \mathrm{GL}_2(\mathbf{C})$ defined by

$$\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho(j) = \begin{pmatrix} \zeta & 0 \\ 0 & -\zeta \end{pmatrix}.$$

(b) Compute $\rho(-1)$ and $\rho(k)$.

(c) Show that $\rho$ is irreducible.

(d) Show that every irreducible representation of $Q$ over $\mathbf{C}$ is either one-dimensional or isomorphic to $\rho$.

(e) Determine the decomposition of $\rho \otimes \rho \otimes \rho \otimes \rho$ as a direct sum of irreducible representations of $Q$.

(*?? pt*) **4.** Let $G$ be a finite group, and let $k$ be a field (possibly of characteristic dividing $\#G$.) Let $V = k[G]$, viewed as a $k$-linear representation of $G$ via the action

$$\begin{aligned} G \times V &\longrightarrow V \\ (g, v) &\longmapsto gvg^{-1}. \end{aligned}$$

(a) Show that the kernel of the group homomorphism $\rho\colon G \to \mathrm{Aut}_k(V)$ defined by the above action equals the centre $Z(G)$ of $G$.

Let $c$ be the number of conjugacy classes of $G$, and let $l$ be the length of $V$ as a $k[G]$-module.

(b) Prove the inequality $l \ge c$. (*Hint*: find non-trivial submodules of $V$.)

(c) *Bonus question:* Show that if $G$ is not Abelian, then $l$ is strictly larger than $c$.

(*?? pt*) **5.** Let $A_5$ be the alternating group of order 60, and let $g = (1\,2\,3\,4\,5) \in A_5$. We view the cyclic group $C_5$ of order 5 as a subgroup of $A_5$ by $C_5 = \langle g \rangle \subset A_5$. Let $\zeta = \exp(2\pi i/5) \in \mathbf{C}$, and let $V$ be the one-dimensional representation of $C_5$ on which $g$ acts as $\zeta$. Determine the decomposition of $\mathrm{Ind}_{C_5}^{A_5} V$ as a direct sum of irreducible representations of $A_5$. You may use the character table of $A_5$:

| conj. class | [(1)] | [(12)(34)] | [(123)] | [(12345)] | [(12354)] |
|---|---|---|---|---|---|
| size | 1 | 15 | 20 | 12 | 12 |
| | 1 | 1 | 1 | 1 | 1 |
| | 3 | $-1$ | 0 | $-\zeta^2 - \zeta^3$ | $-\zeta - \zeta^4$ |
| | 3 | $-1$ | 0 | $-\zeta - \zeta^4$ | $-\zeta^2 - \zeta^3$ |
| | 4 | 0 | 1 | $-1$ | $-1$ |
| | 5 | 1 | $-1$ | 0 | 0 |

(*Hint:* you may use without proof that the conjugacy classes of the powers of $g$ in $A_5$ satisfy $[g] = [g^4] = [(1\,2\,3\,4\,5)]$ and $[g^2] = [g^3] = [(1\,2\,3\,5\,4)]$.)