Representation Theory of Finite Groups, spring 2019

Problem Sheet 4

25 Februari

In the following exercises, **Sets** (resp. **Groups**, **Rings**, ...) denotes the category of sets (resp. groups, rings, ...), where the morphisms are the "standard" ones (i.e. maps of sets, group homomorphisms, ring homomorphisms, ...), and composition is defined in the "standard" way, i.e. $(g \circ f)(x) = g(f(x))$ if $f\colon X \to Y$ and $g\colon Y \to Z$ are maps of sets (resp. group homomorphisms, ring homomorphisms, ...).

1. Let G be a group. Recall that a (left) G-set is a pair (X, α), where X is a set and $\alpha\colon G \times X \to X$ is a left action of G on X. (Often, one does not mention α explicitly and abbreviates $\alpha(g, x)$ to gx.) A G-equivariant map from a G-set (X, α) to a G-set (Y, β) is a map of sets $f\colon X \to Y$ such that for all $g \in G$ and $x \in X$ we have $f(\alpha(g, x)) = \beta(g, f(x))$. Show that there is a category ${}_G\mathbf{Sets}$ in which the objects are the G-sets and the morphisms are the G-equivariant maps.

2. (Some examples of functors.) In each case, to show that there is a functor $F\colon \mathcal{C} \to \mathcal{D}$ with the given effect on objects of $\mathcal{C}$, start by defining $F(f)$ for morphisms f in $\mathcal{C}$.

(a) For every ring R, let $R^\times$ be the unit group of R. Show that there is a functor $U\colon \mathbf{Rings} \to \mathbf{Groups}$ such that $U(R) = R^\times$ for every ring R.

(b) For every ring R, let $R[x]$ be the polynomial ring in one variable over R. Show that there is a functor $P\colon \mathbf{Rings} \to \mathbf{Rings}$ such that $P(R) = R[x]$ for every ring R.

(c) Let k be a field. Show that there is a functor $R\colon \mathbf{Groups} \to \mathbf{Rings}$ such that $R(G) = k[G]$ for every group G.

(d) Let G be a group. For every G-set X (see Exercise 1), let X^G be the set of fixed points, i.e. $X^G = \{x \in X \mid gx = x \text{ for all } g \in G\}$. Show that there is a functor $F\colon {}_G\mathbf{Sets} \to \mathbf{Sets}$ such that $F(X) = X^G$ for every G-set X.

3. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories, and let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors. For every object X of $\mathcal{C}$, define an object $H(X)$ of $\mathcal{E}$ by $H(X) = G(F(X))$. For every morphism $f\colon X \to Y$ in $\mathcal{C}$, define a morphism $H(f)$ in $\mathrm{Mor}_{\mathcal{E}}(X, Y)$ by $H(f) = G(F(f))$. Show that H is a functor from $\mathcal{C}$ to $\mathcal{D}$. (The functor H is called the *composition* of G and F and is denoted by GF or $G \circ F$.)

4. Let $\mathcal{C}$ be a category, and let X be an object of $\mathcal{C}$.

(a) For all objects Y of $\mathcal{C}$, define a set $h_X(Y)$ by

$$h_X(Y) = \mathrm{Mor}_{\mathcal{C}}(X, Y).$$

For all morphisms $f\colon Y \to Y'$ in $\mathcal{C}$, define a map of sets

$$\begin{aligned} h_X(f)\colon h_X(Y) &\longrightarrow h_X(Y') \\ g &\longmapsto f \circ g. \end{aligned}$$

Show that h_X is a functor from $\mathcal{C}$ to **Sets**.

(b) For all objects Y of $\mathcal{C}$, define a set $h^X(Y)$ by

$$h^X(Y) = \mathrm{Mor}_{\mathcal{C}}(Y, X).$$

For all morphisms $f: Y \to Y'$ in $\mathcal{C}$, define a map of sets

$$\begin{aligned} h^X(f): h^X(Y') &\longrightarrow h^X(Y) \\ g &\longmapsto g \circ f. \end{aligned}$$

Show that h^X is a contravariant functor from $\mathcal{C}$ to **Sets** (equivalently, a functor from $\mathcal{C}^{\mathrm{op}}$ to **Sets**).

5. Let G be a group, and let k be a field. For every G-set X (see Exercise 1), let k^X be the set of functions $v: X \to k$, viewed as a k-vector space under pointwise addition and scalar multiplication.

(a) For $g \in G$ and $v \in k^X$, define $gv \in k^X$ by

$$(gv)(x) = v(g^{-1}x).$$

Show that this gives k^X the structure of a k-linear representation of G, hence of a $k[G]$-module.

(b) Let $f: X \to Y$ be a G-equivariant map. Show that the map

$$\begin{aligned} f^*: k^Y &\to k^X \\ w &\mapsto w \circ f \end{aligned}$$

is $k[G]$-linear.

(c) For every G-set X, define a $k[G]$-module $F(X)$ by $F(X) = k^X$. For every G-equivariant map $f: X \to Y$, let $F(f): F(Y) \to F(X)$ be the $k[G]$-linear map f^* defined in (b). Show that F is a contravariant functor from ${}_G\mathbf{Sets}$ to ${}_{k[G]}\mathbf{Mod}$.

Definition. Let $\mathcal{C}$ be a category, and let X and Y be two objects of $\mathcal{C}$.

A (categorical) *product* of X and Y in $\mathcal{C}$ is an object P of $\mathcal{C}$, together with morphisms $p: P \to X$ and $q: P \to Y$, with the following property: for every object Z of $\mathcal{C}$ and every pair of morphisms $f: Z \to X$ and $g: Z \to Y$, there exists a unique morphism $h: Z \to P$ such that $p \circ h = f$ and $q \circ h = g$. Equivalently, (P, p, q) is a product of X and Y if and only if for every object Z of $\mathcal{C}$, the map of sets

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(Z, P) &\longrightarrow \mathrm{Mor}_{\mathcal{C}}(Z, X) \times \mathrm{Mor}_{\mathcal{C}}(Z, Y) \\ h &\longmapsto (p \circ h, q \circ h) \end{aligned}$$

is a bijection.

A (categorical) *sum* or *coproduct* of X and Y in $\mathcal{C}$ is an object S of $\mathcal{C}$, together with morphisms $i: X \to S$ and $j: Y \to S$, with the following property: for every object Z of $\mathcal{C}$ and every pair of morphisms $f: X \to Z$ and $g: Y \to Z$, there exists a unique morphism $h: S \to Z$ such that $h \circ i = f$ and $h \circ j = g$. Equivalently, (S, i, j) is a sum of X and Y if and only if for every object Z of $\mathcal{C}$, the map of sets

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(S, Z) &\longrightarrow \mathrm{Mor}_{\mathcal{C}}(X, Z) \times \mathrm{Mor}_{\mathcal{C}}(Y, Z) \\ h &\longmapsto (h \circ i, h \circ j) \end{aligned}$$

is a bijection.

6. Let X and Y be sets.

(a) Show that the disjoint union $X \sqcup Y$, together with the canonical maps $i: X \to X \sqcup Y$ and $j: Y \to X \sqcup Y$, is a (categorical) sum of X and Y in **Sets**

(b) Show that the Cartesian product $X \times Y$, together with the canonical maps $p: X \times Y \to X$ and $q: X \times Y \to Y$, is a (categorical) product of X and Y in **Sets**.

7. (a) Let G and H be groups, and let $G \times H$ be their product (according to the usual definition, i.e. $G \times H$ is the product set with coordinatewise operations). Show that $G \times H$, together with the canonical projection maps $p: G \times H \to G$ and $q: G \times H \to H$, is a categorical product of G and H in **Groups**.

(b) Same question for the category of rings.

8. Let m and n be positive integers, and let d be their greatest common divisor. Let $i: \mathbf{Z}/m\mathbf{Z} \to \mathbf{Z}/d\mathbf{Z}$ and $j: \mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}/d\mathbf{Z}$ be the canonical ring homomorphisms. Show that $(\mathbf{Z}/d\mathbf{Z}, i, j)$ is a (categorical) sum of $\mathbf{Z}/m\mathbf{Z}$ and $\mathbf{Z}/n\mathbf{Z}$ in the category **Rings**. (Taking $m, n > 1$ coprime, this shows that the sum of two non-zero rings can be the zero ring.)

9. Let $G = H = \mathbf{Z}$.

(a) Show that $\mathbf{Z}^2$, together with the group homomorphisms $i: G \to \mathbf{Z}^2$ and $j: H \to \mathbf{Z}^2$ defined by $i(m) = (m, 0)$ and $j(n) = (0, n)$, is a sum of G and H in the category of Abelian groups.

(b) Let $F = \langle g, h \rangle$ be the (non-Abelian) free group on two generators. Show that F, together with the group homomorphisms $i: G \to F$ and $j: H \to F$ defined by $i(m) = g^m$ and $j(n) = h^n$, is a sum of G and H in the category of groups.

(This shows that categorical notions like sums can depend heavily on the category.)

10. Let **FinGrp** be the category of *finite* groups (objects are finite groups, morphisms and composition are as in **Groups**.) Let $G = H = \mathbf{Z}/2\mathbf{Z}$. For every positive integer n, let D_n be the dihedral group of order $2n$, defined using generators and relations by

$$D_n = \langle \rho, \sigma \mid \rho^n, \sigma^2, (\sigma\rho)^2 \rangle.$$

(a) Show that for every $n \geq 1$ there exist homomorphisms $f: G \to D_n$ and $g: H \to D_n$ such that D_n is generated by the union of the images of f and g.

(b) Suppose that there exists a finite group S, together with group homomorphisms $i: G \to S$ and $j: H \to S$, such that (S, i, j) is a sum of G and H in **FinGrp**. Show that for all $n \geq 1$ there exists a surjective group homomorphism $S \to D_n$.

(c) Conclude that G and H do not have a sum in **FinGrp**.