Smooth morphisms

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21 February 2007

Introduction

The goal of this talk is to define smooth morphisms of schemes, which are one of the main ingredients in Néron's fundamental theorem [BLR, § 1.3, Theorem 1]:

Theorem. *Let R be a discrete valuation ring with field of fractions K, and let X_K be a smooth group scheme of finite type over K. Let R^{sh} be a strict Henselisation of R, and let K^{sh} be its field of fractions. Then X_K admits a Néron model over R if and only if $X(K^{\text{sh}})$ is bounded in X_K.*

We will not explain the boundedness condition (see [BLR, § 1.1]), but this condition is known to be satisfied in the case where X_K is proper over K. In particular, we get the following result (recall that an Abelian variety of dimension g over a scheme S is a proper smooth group scheme over S with geometrically connected fibres of dimension g):

Corollary. *Let R be a discrete valuation ring with field of fractions K, and let X_K be an Abelian variety over K. Then X_K admits a Néron model over R.*

The definition of smoothness includes two 'technical' conditions: flatness and 'locally of finite presentation'. We start by defining these; then we state the definition of smoothness and a criterion for smoothness in terms of differentials. We also summarise the different notions of smoothness found in EGA. Finally, we give some equivalent definitions of étale morphisms.

Flat modules and flat morphisms of schemes

Definition. A module M over a ring A is called a *flat* A-module if for every short exact sequence

$$0 \to N' \to N \to N'' \to 0$$

of A-modules, the sequence

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is again exact. Equivalently, since the tensor product is always right exact, an A-module M is flat if and only if the functor $M \otimes_A -$ is left exact, i.e. preserves kernels.

Proposition. *Let $A \to A'$ is a ring homomorphism. For every flat A-module M, the A'-module $M \otimes_A A'$ is also flat. Furthermore, if A' is flat as an A-module, and if M' is a flat A'-module, then M' is also flat as an A-module.*

Proof. Easy.

Examples of flat A-modules are the *locally free* (or *projective*) modules; in fact, it can be shown that if A is a Noetherian ring, the finitely generated flat A-modules are precisely the locally free A-modules of finite rank.

Definition. A module M over a ring A is called *faithfully flat* if it is flat and in addition we have the implication $M \otimes_A N = 0 \Rightarrow N = 0$ for every A-module N.

A faithfully flat module M has the useful property that a short exact sequence of A-modules is exact *if and only if* it is exact after tensoring with M. An example of this is the following lemma with its corollary, which we will need later.

Lemma. *Let A be a ring, let A' be a faithfully flat A-algebra, and let M be an A-module. Then M is flat over A if and only if $M' = A' \otimes_A M$ is flat over A'.*

Proof. If M is flat over A, then M' is flat over A' by the base change property. Conversely, suppose M' is flat over A', and let

$$0 \to N' \to N \to N'' \to 0$$

be a short exact sequence of A-modules. Then we have to show that

$$0 \to M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0$$

is again exact. Because A' is faithfully flat over A, it suffices to check this after tensoring with A'. But then we get

$$0 \to M' \otimes_A N' \to M' \otimes_A N \to M' \otimes_A N'' \to 0,$$

which is exact; namely, M' is flat over A' and A' is flat over A, hence M' is flat over A. □

Corollary. *Let A be a Noetherian ring, let A' be a faithfully flat A-algebra, and let M be a finitely generated A-module. Then M is locally free if and only if $A' \otimes_A M$ is locally free.*

Proof. This follows from the lemma since 'flat' and 'locally free' are equivalent for finitely generated modules over a Noetherian ring. □

Definition. A morphism $f: X \to Y$ of schemes is called *flat at a point* $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is flat as a module over the local ring $\mathcal{O}_{Y,f(x)}$, and f is called *flat* if it is flat at every point of X.

Proposition. *Open immersions are flat morphisms. If $f: X \to Y$ and $g: Y \to Z$ are flat, then $g \circ f: X \to Z$ is flat. Flatness is preserved under base change in the sense that in a Cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

with f flat, f' is also flat.

If Y is a locally Noetherian scheme, it follows from the above remark about flat modules that a *finite* morphism $f: X \to Y$ is flat if and only if it is locally free of finite rank.

(Locally) finitely presented morphisms

Let $A \to B$ be a morphism of rings. Recall that B is *finitely generated* as an A-algebra if there exists a surjective homomorphism

$$A[x_1, \ldots, x_n] \to B$$

of A-algebras. We say that B is *finitely presented* as an A-algebra if there exists such a homomorphism with the property that its kernel is a finitely generated $A[x_1, \ldots, x_n]$-ideal. If A is Noetherian, this condition is automatic; in other words; 'finitely presented' and 'finitely generated' are equivalent for algebras over a Noetherian ring.

Definition. A morphism $f: X \to Y$ of schemes is called *locally of finite presentation at a point* $x \in X$ if there exist affine neighbourhoods $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ of $f(x)$ and x, respectively, such that $f(V) \subseteq U$ and B is a finitely presented A-algebra. The morphism f is called *locally of finite type* (resp. *locally of finite presentation*) if it is locally of finite type (resp. locally of finite presentation) at every point of X. It is *of finite type* if it is locally of finite type and quasi-compact, and it is *of finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated.

Obviously, every morphism of finite presentation is of finite type. It can be checked that for Y locally Noetherian, the two notions are equivalent [EGA IV_1, § 1.6].

Regular schemes

Definition. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue class field k. Then A is *regular* if and only if $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \operatorname{Krull}\dim A$.

Fact. *The localisation of a regular ring at a prime ideal is again regular.*

Definition. Let X be a locally Noetherian scheme. Then X is *regular* if all its local rings are regular, or equivalently (by the above fact) if all the local rings at closed points of the affine open subschemes in some affine open cover are regular.

Smooth morphisms

Smooth morphisms are the things which Néron models are all about. We use the definition from EGA since it is more general than Hartshorne's definition.

Definition. Let $f\colon X \to Y$ be a morphism of schemes. Then f is called *smooth at a point* $x \in X$ if the following conditions hold:

a) f is flat at x;

b) f is locally of finite presentation at x;

c) the fibre $X_{f(x)}$ is *geometrically regular at* x, i.e. all the localisations of the (semi-local) ring $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \overline{k(f(x))}$ are regular, where $\overline{k(f(x))}$ denotes an algebraic closure of the residue field of $\mathcal{O}_{Y,f(x)}$.

We say that f is *smooth* if it is smooth at every point of X, i.e. if

a) f is flat;

b) f is locally of finite presentation;

c) the fibres of f are geometrically regular.

It is important to note that geometric regularity is a property of schemes *over a field*, whereas regularity is a property of schemes. In general, if $\mathbf{P}$ is a property of schemes (such as regular, reduced, irreducible, connected, integral), we say that a scheme X over a field k is *geometrically* $\mathbf{P}$ if $X \times_k \bar{k}$ is $\mathbf{P}$, where $\bar{k}$ is an algebraic closure of k.

The property of a morphism being smooth is preserved under any base change, i.e. if $X \to Y$ is smooth and $Y' \to Y$ is any morphism of schemes, then $X \times_Y Y' \to Y'$ is again smooth. This follows from the fact that if X is a geometrically regular scheme over a field k, then $X \times_k K$ is geometrically regular for any field extension $k \to K$. It would *not* be true if we had only required 'regular' instead of 'geometrically regular' for the fibres.

Notice that if Y is locally Noetherian, we can replace 'locally of finite presentation' by 'locally of finite type' in the definition of smoothness.

Kähler differentials

Definition. Let $A \to B$ be a morphism of rings (as always, rings are supposed to be commutative with 1). A *derivation* from B over A is an A-linear map

$$d\colon B \to M$$

with M a B-module, such that the following product rule holds:

$$d(bb') = b\,db' + b'\,db \quad \text{for all } b, b' \in B.$$

A *universal derivation* is a B-module $\Omega_{B/A}$ together with a derivation

$$d\colon B \to \Omega_{B/A}$$

as above such that for every other derivation $d'\colon B \to M$ there is a unique B-linear map $h\colon \Omega_{B/A} \to M$ making the diagram

$$\begin{array}{ccc} B & \xrightarrow{d} & \Omega_{B/A} \\ & {\scriptstyle d'}\searrow & \downarrow{\scriptstyle h} \\ & & M \end{array}$$

commutative. The module $\Omega_{B/A}$ is also called the module of *Kähler differentials*.

Because of the universal property, a universal derivation is unique up to unique isomorphism, if it exists. Let us give two constructions of it:

1) Choose a presentation for B as an A-algebra. This means to choose a set $S \subseteq B$ of generators for B as an A-algebra, so that we have a surjective homomorphism

$$A[S] \to B,$$

where $A[S]$ is the polynomial algebra over A with generators labelled by S; write I for the ideal of $A[S]$ which is the kernel of this homomorphism, and choose a set $T \subseteq A[S]$ of generators of I as an $A[S]$-module. For each $f \in A[S]$ and each $x \in S$, we let f_x denote the partial derivative of the polynomial f with respect to x. Then we put

$$\Omega_{B/A} = \Big(\bigoplus_{x \in S} B\Big)\Big/\Big\langle (f_x \bmod I)_{x \in S} \;\Big|\; f \in T \Big\rangle;$$

the homomorphism

$$\begin{aligned} A[S] &\to \bigoplus_{x \in S} A[S] \\ g &\mapsto (g_x)_{x \in S} \end{aligned}$$

is a derivation and induces a derivation

$$d\colon B \to \Omega_{B/A},$$

which is universal, as one can easily check. In particular, if $B = A[x_1, \ldots, x_n]$ is a finitely generated polynomial algebra, we see (taking $S = \{x_1, \ldots, x_n\}$ and $T = \emptyset$) that

$$\Omega_{B/A} = \bigoplus_{i=1}^{n} B\,dx_i.$$

2) Consider B as an algebra over the tensor product $B \otimes_A B$ via the *multiplication homomorphism*

$$\begin{aligned} m\colon B \otimes_A B &\to B \\ b \otimes b' &\mapsto bb' \end{aligned}$$

of A-algebras, and let I be its kernel. Then I is a $B \otimes_A B$-module, and

$$I/I^2 = I \otimes_{B \otimes_A B} ((B \otimes_A B)/I) = I \otimes_{B \otimes_A B} B$$

is a B-module. We put

$$\Omega_{B/A} = I/I^2$$

and

$$\begin{aligned} d\colon B &\to \Omega_{B/A} \\ b &\mapsto b \otimes 1 - 1 \otimes b. \end{aligned}$$

Again, one can check that this has the required universal property.

The construction of the module of Kähler differentials is compatible with localisation. This implies that if $f\colon X \to Y$ is a morphism of schemes, there exists a sheaf $\Omega_{B/A}$ on X and a morphism of $f^{-1}\mathcal{O}_Y$-modules

$$d\colon \mathcal{O}_X \to \Omega_{X/Y}$$

having the expected universal property.

Proposition. *For every composed morphism*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

of schemes, there is an canonical exact sequence

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

of $\mathcal{O}_X$-modules. Furthermore, the formation of $\Omega_{X/Y}$ is compatible with base change in the sense that for every Cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

there is a canonical isomorphism

$$\Omega_{X'/Y'} = (g')^*\Omega_{X/Y}.$$

Smoothness and regularity for schemes over a field

In practice, it is convenient to have a more explicit condition for geometric regularity. Such a condition is provided by the Kähler differentials which we have just seen. We start with the algebraic analogue of what we want to prove.

Theorem. *Let B be a local ring containing a field k isomorphic to its residue field. Assume furthermore that B is a localisation of a finitely generated k-algebra. Then $\Omega_{B/k}$ is a free B-module of rank equal to $\dim B$ if and only if B is a regular local ring.*

Proof. [Hartshorne, Theorem II.8.8] (according to a remark of Bas Edixhoven during the talk, Hartshorne's assumption that k is perfect is unnecessary).

Proposition. *Let k be a field, let X be a scheme which is locally of finite type over k. Let x be a closed point of X, and let d be the dimension of X at x (i.e. the Krull dimension of $\mathcal{O}_{X,x}$). Then the following are equivalent:*
(1) X is geometrically regular at x;
(2) the stalk $\Omega_{X/k,x}$ is free of rank d.

Proof. Fix an algebraic closure $\bar{k}$ of k, and put

$$B = \mathcal{O}_{X,x} \otimes_k \bar{k}.$$

We have to prove that $\operatorname{Spec} B$ is a regular scheme if and only if $\Omega_{X/k,x}$ is free of rank d.

If X is geometrically regular, then for all maximal ideals $\mathfrak{m} \subset B$ the local ring $B_{\mathfrak{m}}$ is regular by the definition of geometric regularity, and its dimension equals d [Hartshorne, Exercise II.3.20]. By the previous theorem we see that $\Omega_{B/\bar{k}}$ is locally free of rank d. Therefore $\Omega_{B/\bar{k}}$ is a *flat* B-module. Furthermore, it follows from the fact that $\bar{k}$ is faithfully flat over k that B is faithfully flat over $\mathcal{O}_{X,x}$. The above lemma implies that $\Omega_{X/k,x}$ is flat over $\mathcal{O}_{X,x}$. Now we are done, because a finitely generated flat module over a Noetherian ring is projective, and a projective module over a local ring is free.

Conversely, suppose $\Omega_{X/k,x}$ is free of rank d. Then $\Omega_{B/\bar{k}} = \Omega_{X/k} \otimes_k \bar{k}$ is also free of rank d, so by the previous theorem we see that $B_{\mathfrak{m}}$ is regular for all maximal ideals $\mathfrak{m} \subset B$. This implies that $\operatorname{Spec} B$ is a regular scheme. □

Example. Let k be an imperfect field of characteristic p, where p is an odd prime number, and let $t \in k$ be an element which is not a p-th power. We put

$$A = k[x,y]/(y^2 - x^p + t),$$

which is a 1-dimensional k-algebra, and we consider the morphism of schemes

$$f\colon \operatorname{Spec} A \to \operatorname{Spec} k.$$

Let $\mathfrak{m}$ be the maximal ideal $(x^p - t, y)$ of A. Then $A/\mathfrak{m}^2 = k[x,y]/(y^2, x^p - t)$ is a k-vector space of dimension $p+1$, with basis $\{1, x, \ldots, x^{p-1}, y\}$, and $\mathfrak{m}/\mathfrak{m}^2$ is the 1-dimensional subspace generated by y. A similar computation at the other maximal ideals shows that $\operatorname{Spec} A$ is regular. Now consider the base extension $k \to k(u)$, where $u^p = t$. Denoting by B the $k(u)$-algebra $A \otimes_k k(u)$, we have

$$B = k(u)[x,y]/(y^2 - (x-u)^p).$$

For the maximal ideal $\mathfrak{n} = (x-u, y)$ we see that $B/\mathfrak{n}^2$ is of dimension 3 over $k(u)$, with basis $\{1, x, y\}$, and $\mathfrak{n}/\mathfrak{n}^2$ is the 2-dimensional subspace generated by $x - u$ and y. Therefore $\operatorname{Spec} B$ is not regular at $(x-u, y)$, so $\operatorname{Spec} A$ is not geometrically regular over k and f is not smooth. Using the above proposition, we can show this more easily by computing $\Omega_{A/k}$, the module of Kähler differentials; with $g = y^2 - x^p + t$, this gives

$$\begin{aligned}\Omega_{A/k} &= (A \cdot dx \oplus A \cdot dy)/(A \cdot g_x\, dx + A \cdot g_y\, dy) \\ &\cong A + A/(y).\end{aligned}$$

This module is locally free of rank 1 outside $\mathfrak{m} = (x^p - t, y)$, whereas $\Omega_{A/k}/\mathfrak{m}\Omega_{A/k}$ is isomorphic to $A/\mathfrak{m} \oplus A/\mathfrak{m}$.

Equivalent definitions of smoothness

Besides the 'fibre-by-fibre' criterion for smoothness given in the previous section, there are several equivalent definitions of smoothness to be found in EGA IV. The first one is related to the property of *formal smoothness* [EGA IV_4, définition 17.1.1.]

Definition. Let $f\colon X \to Y$ be a morphism of schemes. Then f is said to be *formally smooth*, or to possess the *infinitesimal lifting property*, if for every ring A, every nilpotent ideal J of A and every morphism $\operatorname{Spec} A \to Y$, the canonical map

$$\operatorname{Hom}_Y(\operatorname{Spec} A, X) \to \operatorname{Hom}_Y(\operatorname{Spec}(A/J), X)$$

is surjective.

It can be shown that formal smoothness of a morphism can be checked on open coverings of X or Y, so in this sense it is a local property. For the proof, see [EGA IV_4, proposition 17.1.6].

Notice that formal smoothness can be seen as a property of the functor on Y-schemes which the scheme X represents. This means that we can in principle check whether a functor, if it is representable, will be represented by a smooth scheme, before we even know that it is representable.

Definition. Let $f\colon X \to Y$ be a morphism of schemes which is locally of finite type. The *relative dimension* of f at x, denoted by $\dim_x f$, is the dimension of the topological space underlying the fibre X_f at the point x [EGA IV_1, définition 14.1.2].

Proposition. *Let $f\colon X \to Y$ be a morphism of schemes, and let x be a point of X. Then the following are equivalent:*

(1) f is smooth at x;

(2) f is locally of finite presentation at x, and there is an open neighbourhood $U \subset X$ of x such that $f|_U\colon U \to Y$ is formally smooth;

(3) f is flat at x, locally of finite presentation at x, and the $\mathcal{O}_X$-module $\Omega_{X/Y}$ is locally free in a neighbourhood of x in X, of rank $\dim_x f$ at x.

Proof. The equivalence of (1) and (2) is [EGA IV_4, corollaire 17.5.2]. The equivalence of (2) and (3) is [EGA IV_4, proposition 17.15.15]. □

Corollary. *The following are equivalent for a morphism $f\colon X \to Y$:*

(1) f is smooth;

(2) f is locally of finite type and formally smooth;

(3) f is flat and locally of finite presentation, and the $\mathcal{O}_X$-module $\Omega_{X/Y}$ is locally free of rank equal to the relative dimension of f at all points of X.

Because the rank of a locally free module is locally constant, we see from the last characterisation of smoothness that the relative dimension of a smooth morphism is locally constant. This is not the case for arbitrary flat morphisms which are locally of finite presentation.

Example. To see how the infinitesimal lifting property fails for a non-smooth morphism, consider again the example $\operatorname{Spec} A \to \operatorname{Spec} k$, where k is a field of characteristic $p \geq 3$ and

$$A = k[x,y]/(y^2 - x^p + t)$$

for some $t \in k$. Let B be the k-algebra

$$B = k[\xi,\epsilon]/(\xi^p - t, \epsilon^3),$$

and let J be the ideal $\epsilon^2 B$; then $J^2 = 0$ and $B/J = k[\xi,\epsilon]/(\xi^p - t, \epsilon^2)$. We claim that the homomorphism $A \to B/J$ given by $x \mapsto \xi$ and $y \mapsto \epsilon$ cannot be lifted to a homomorphism $A \to B$. Namely, such a homomorphism has to satisfy

$$\begin{aligned} x &\mapsto \xi + a\epsilon^2 \\ y &\mapsto \epsilon + b\epsilon^2 \end{aligned} \qquad (a, b \in B),$$

but then y^2 maps to ϵ^2 and $x^p - t$ maps to 0, a contradiction since $y^2 = x^p - t$ in A and $\epsilon^2 \neq 0$ in B.

Étale morphisms

Definition. Let $f\colon X \to Y$ be a morphism of schemes. Then f is called *étale* if it is smooth with fibres of dimension 0.

Let $f\colon X \to Y$ be an étale morphism. Let y be a point of Y, let $k(y)$ be its residue class field, and let $\overline{k(y)}$ be an algebraic closure of $k(y)$. Then the geometric fibre $X_{\bar{y}} = X \times_Y \operatorname{Spec} \overline{k(y)}$ is a regular scheme of dimension 0, i.e. a disjoint union of spectra of fields. These fields have to be finite extensions of $\overline{k(y)}$, but since $\overline{k(y)}$ is algebraically closed this means that f is étale if and only if f is flat, locally of finite presentation and the geometric fibre over every point y of Y is a disjoint union of copies of $\overline{k(y)}$.

Equivalently, a morphism $f\colon X \to Y$ is étale if and only if it is flat, locally of finite presentation and $\Omega_{X/Y} = 0$. Yet another definition: f is étale if and only if it is flat and unramified, where 'unramified' means that f is locally of finite presentation and for every point $x \in X$, we have $\mathfrak{m}_{f(x)}\mathcal{O}_{X,x} = \mathfrak{m}_x$ and $k(x)/k(f(x))$ is a finite separable field extension.

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