Second order, linear differential equations

$$\boxed{a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = f(t)}$$

<u>GENERAL SOLUTION STRATEGY</u>:

$\boxed{1}$ Compute a FS $\{y_1, y_2\}$ for the corresponding **homogeneous problem** ($f(t) = 0$)

$$a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = 0$$

$\boxed{2}$ Bring the full problem to **standard form** for $a_2(t) \neq 0$

$$\boxed{y''(t) + \underbrace{\frac{a_1(t)}{a_2(t)}}_{=:p(t)} y'(t) + \underbrace{\frac{a_0(t)}{a_2(t)}}_{=:q(t)} y(t) = \underbrace{\frac{f(t)}{a_2(t)}}_{=:g(t)}}$$

and compute the Wronskian $W(y_1, y_2)(t)$ using your result from $\boxed{1}$.

Note: You can check the result for the Wronskian using Abel's theorem.

$\boxed{3}$ Find a particular solution via the $\boxed{\text{VoP}}$ **Variation of Parameters** formula.

$$y_p(t) = -y_1(t) \left[\int \frac{y_2(t)g(t)}{W(y_1, y_2)(t)} dt \right] + y_2(t) \left[\int \frac{y_1(t)g(t)}{W(y_1, y_2)(t)} dt \right]$$

$\boxed{4}$ The general solution is given by

$$y(t) = \underbrace{C_1 y_1(t) + C_2 y_2(t)}_{\text{general solution of hom. problem}} + \underbrace{y_p(t)}_{\text{particular solution of full problem}}$$

with $C_1, C_2 \in \mathbb{R}$.

Remarks:

- The constants $C_1, C_2 \in \mathbb{R}$ can be fixed by adding further constraints for the solution, e.g. initial conditions $y(t_0) = y_0, y'(t_0) = \tilde{y}_0$ for some $t_0, y_0, \tilde{y}_0 \in \mathbb{R}$.
- Finding a fundamental system for the homogeneous problem analytically is, in general, difficult. There are, however, special cases of coefficients a_1, a_2, a_3 where it is indeed possible to compute them (e.g. the constant-coeff. case, Euler type equations, etc.). Furthermore, in the constant-coefficient case with specific RHS f it is possible to find a particular solution also by the "Method of Undetermined Coefficients".
- With the notation $L[y](x) = y''(x) + p(x)y'(x) + q(x)y(x)$, we can write the standard form of the full ODE problem as $L[y] = g$ and the homogeneous problem as $L[y] = 0$. This is reminiscent of systems of linear equations $Ay = b$ where A is a given matrix, b is a given vector and y is the unknown vector. Look up the solution strategy for such problems in Linear Algebra.

Second order, linear differential equations

– constant coefficients, homogeneous –

ODE | characteristic equation

$$\boxed{\mathbf{ay'' + by' + cy = 0}} \text{ becomes with the "ansatz" } y = e^{\lambda t} \ \boxed{\mathbf{a\lambda^2 + b\lambda + c = 0}}$$

$\rightarrow$ gives solutions of the form

$$\boxed{\mathbf{y = e^{\lambda_{1/2} t}} \text{ with } \lambda_{\mathbf{1/2}} = \frac{\mathbf{-b \pm \sqrt{b^2 - 4ac}}}{\mathbf{2a}}}$$

<u>form of **general solution** depending on the discriminant $\mathbf{b^2 - 4ac}$</u>

(1) $\mathbf{b^2 - 4ac > 0}$: λ_1, λ_2 are real and distinct

$$\mathbf{y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}}\,, \quad C_1, C_2 \in \mathbb{R}$$

(2) $\mathbf{b^2 - 4ac = 0}$: $\lambda = -\frac{b}{2a}$ is a (real) double root

$$\mathbf{y(t) = (C_1 + C_2 t) e^{\lambda t}}\,, \quad C_1, C_2 \in \mathbb{R}$$

(3) $\mathbf{b^2 - 4ac < 0}$: $\lambda_{1/2} = \alpha \pm i\beta, \alpha, \beta \in \mathbb{R}\,, \quad C_1, C_2 \in \mathbb{R}$ complex conjugate root pair

$$\mathbf{y(t) = e^{\alpha t}\left[C_1 \cos(\beta t) + C_2 \sin(\beta t)\right]}$$

Remarks:

- The real part α of λ gives exponential decay or growth, while the imaginary part β gives oscillations.
- By defining $v = y'$ we can rewrite $y'' + by' + cy = 0$ as a first order system for the vector $Y = (y, v)$ given by

$$Y'(t) = \underbrace{\begin{pmatrix} 0 & 1 \\ -c & -b \end{pmatrix}}_{=:A} Y(t)\,.$$

  The characteristic equation of the matrix is $\lambda^2 + b\lambda + c = 0$ and the solution can be given in terms of the matrix exponential $Y(t) = Ce^{At}$ where $C \in \mathbb{R}^{2\times 2}$.

Repeated roots

Understanding why the GS of $\mathbf{ay'' + by' + cy = 0}$ with $\mathbf{b^2 - 4ac = 0}$ is of the form

$$\mathbf{y(t) = (C_2 t + C_1) e^{-\frac{b}{2a}t}}.$$

- One solution is obtained via the characteristic equation: $y_1(t) = e^{-\frac{b}{2a}t}$.
- A second solution is obtained by making the ansatz $y(t) = v(t)e^{-\frac{b}{2a}t}$:

$$\begin{aligned}
0 = ay'' + by' + cy &= a\left(ve^{-\frac{b}{2a}t}\right)'' + b\left(ve^{-\frac{b}{2a}t}\right)' + c\left(ve^{-\frac{b}{2a}t}\right) \\
&= a\left[v'' - \frac{b}{a}v' + \frac{b^2}{4a^2}\right]e^{-\frac{b}{2a}t} + b\left[v' - \frac{b}{2a}v\right]e^{-\frac{b}{2a}t} + cve^{-\frac{b}{2a}t} \\
&= \left[av'' + \underbrace{(-b+b)}_{=0}v' + \underbrace{(-\frac{b^2}{4a} + c)}_{=0}v\right]e^{-\frac{b}{2a}t} \\
&= av''e^{-\frac{b}{2a}t}
\end{aligned}$$

- Since $a \neq 0$ and $e^{-\frac{b}{2a}t} \neq 0$ the equation for v is given by $v'' = 0$, so $v(t) = C_2 t + C_1$.
- $W(e^{-\frac{b}{2a}t}, te^{-\frac{b}{2a}t})(t) \neq 0 \to \{e^{-\frac{b}{2a}t}, te^{-\frac{b}{2a}t}\}$ is a FS
- $y(t) = (C_2 t + C_1)e^{-\frac{b}{2a}t}$ is the general solution

NOTE: The presented strategy is a special case of a **reduction of order**: Suppose you know a solution $y_1 \neq 0$ of the **second order** equation $y'' + p(t)y' + q(t)y = 0$. Making the **ansatz** $y(t) = v(t)y_1(t)$ gives an equation for v reading $y_1 v'' + (2y_1' + py_1)v' = 0$, which is a **first order** equation for $w = v'$.

Const.- coeff., non-homogeneous ODEs with special RHS f

– Method of Undetermined Coefficients (MoUC) –

$$\boxed{ay''(t) + by'(t) + cy(t) = f(t)}$$

ansatz $y_p(t)$ for a particular solution of the non-homogeneous problem

RHS $f(t)$	**ANSATZ** $y_p(t)$
$P_n(t) = a_n t^n + \ldots + a_0$	$t^s(A_n t^n + \ldots + A_0)$
$P_n(t)e^{rt}$	$t^s(A_n t^n + \ldots + A_0)e^{rt}$
$P_n(t)e^{rt}\sin(mt)$, $P_n(t)e^{rt}\cos(mt)$	$t^s[(A_n t^n + \ldots + A_0)e^{rt}\sin(mt)$ $+(B_n t^n + \ldots + B_0)e^{rt}\cos(mt)]$

The exponent s is the smallest integer $\in \{0, 1, 2\}$ such that the ansatz $y_p(t)$ does not contain solutions of the homogeneous problem.