where E is Young's modulus and ρ is the mass per unit volume. If the end $x = 0$ is fixed, then the boundary condition there is

$$u(0, t) = 0, \qquad t > 0. \tag{ii}$$

Suppose that the end $x = L$ is rigidly attached to a mass m but is otherwise unrestrained. We can obtain the boundary condition here by writing Newton's law for the mass. From the theory of elasticity it can be shown that the force exerted by the bar on the mass is given by $-EAu_x(L, t)$. Hence the boundary condition is

$$EAu_x(L, t) + mu_{tt}(L, t) = 0, \qquad t > 0. \tag{iii}$$

(a) Assume that $u(x, t) = X(x)T(t)$, and show that $X(x)$ and $T(t)$ satisfy the differential equations

$$X'' + \lambda X = 0, \tag{iv}$$
$$T'' + \lambda(E/\rho)T = 0. \tag{v}$$

(b) Show that the boundary conditions are

$$X(0) = 0, \qquad X'(L) - \gamma\lambda L X(L) = 0, \tag{vi}$$

where $\gamma = m/\rho AL$ is a dimensionless parameter that gives the ratio of the end mass to the mass of the rod.
Hint: Use the differential equation for $T(t)$ in simplifying the boundary condition at $x = L$.
(c) Determine the form of the eigenfunctions and the equation satisfied by the real eigenvalues of Eqs. (iv) and (vi). Find the first two eigenvalues λ_1 and λ_2 if $\gamma = 0.5$.

11.2 Sturm–Liouville Boundary Value Problems

We now consider two-point boundary value problems of the type obtained in Section 11.1 by separating the variables in a heat conduction problem for a bar of variable material properties and with a source term proportional to the temperature. This kind of problem also occurs in many other applications.

These boundary value problems are commonly associated with the names of Sturm and Liouville.[1] They consist of a differential equation of the form

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0 \tag{1}$$

on the interval $0 < x < 1$, together with the boundary conditions

$$a_1 y(0) + a_2 y'(0) = 0, \qquad b_1 y(1) + b_2 y'(1) = 0 \tag{2}$$

[1]Charles-François Sturm (1803–1855) and Joseph Liouville (1809–1882), in a series of papers in 1836 and 1837, set forth many properties of the class of boundary value problems associated with their names, including the results stated in Theorems 11.2.1 to 11.2.4. Sturm is also famous for a theorem on the number of real zeros of a polynomial, and in addition, did extensive work in physics and mechanics. Besides his own research in analysis, algebra, and number theory, Liouville was the founder, and for 39 years the editor, of the influential *Journal de mathématiques pures et appliquées*. One of his most important results was the proof (in 1844) of the existence of transcendental numbers.

at the endpoints. It is often convenient to introduce the linear homogeneous differential operator L defined by

$$L[y] = -[p(x)y']' + q(x)y. \tag{3}$$

Then the differential equation (1) can be written as

$$L[y] = \lambda r(x)y. \tag{4}$$

We assume that the functions p, p', q, and r are continuous on the interval $0 \le x \le 1$ and, further, that $p(x) > 0$ and $r(x) > 0$ at all points in $0 \le x \le 1$. These assumptions are necessary to render the theory as simple as possible while retaining considerable generality. It turns out that these conditions are satisfied in many significant problems in mathematical physics. For example, the equation $y'' + \lambda y = 0$, which arose repeatedly in the preceding chapter, is of the form (1) with $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$. The boundary conditions (2) are said to be **separated**; that is, each involves only one of the boundary points. These are the most general separated boundary conditions that are possible for a second order differential equation.

Before proceeding to establish some of the properties of the Sturm–Liouville problem (1), (2), it is necessary to derive an identity, known as **Lagrange's identity**, which is basic to the study of linear boundary value problems. Let u and v be functions having continuous second derivatives on the interval $0 \le x \le 1$. Then[2]

$$\int_0^1 L[u]v\,dx = \int_0^1 [-(pu')'v + quv]\,dx.$$

Integrating the first term on the right side twice by parts, we obtain

$$\begin{aligned}\int_0^1 L[u]v\,dx &= -p(x)u'(x)v(x)\Big|_0^1 + p(x)u(x)v'(x)\Big|_0^1 + \int_0^1 [-u(pv')' + uqv]\,dx \\ &= -p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_0^1 + \int_0^1 uL[v]\,dx.\end{aligned}$$

Hence, on transposing the integral on the right side, we have

$$\int_0^1 \{L[u]v - uL[v]\}\,dx = -p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_0^1, \tag{5}$$

which is Lagrange's identity.

Now let us suppose that the functions u and v in Eq. (5) also satisfy the boundary conditions (2). Then, if we assume that $a_2 \ne 0$ and $b_2 \ne 0$, the right side of Eq. (5) becomes

$$\begin{aligned}&-p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_0^1 \\ &= -p(1)[u'(1)v(1) - u(1)v'(1)] + p(0)[u'(0)v(0) - u(0)v'(0)] \\ &= -p(1)\left[-\frac{b_1}{b_2}u(1)v(1) + \frac{b_1}{b_2}u(1)v(1)\right] + p(0)\left[-\frac{a_1}{a_2}u(0)v(0) + \frac{a_1}{a_2}u(0)v(0)\right] \\ &= 0.\end{aligned}$$

[2]For brevity we sometimes use the notation $\int_0^1 f\,dx$ rather than $\int_0^1 f(x)\,dx$ in this chapter.

The same result holds if either a_2 or b_2 is zero; the proof in this case is even simpler, and is left for you. Thus, if the differential operator L is defined by Eq. (3), and if the functions u and v satisfy the boundary conditions (2), Lagrange's identity reduces to

$$\int_0^1 \{L[u]v - uL[v]\}\,dx = 0. \tag{6}$$

Let us now write Eq. (6) in a slightly different way. In Eq. (4) of Section 10.2 we introduced the inner product (u, v) of two real-valued functions u and v on a given interval; using the interval $0 \le x \le 1$, we have

$$(u, v) = \int_0^1 u(x)v(x)\,dx. \tag{7}$$

In this notation Eq. (6) becomes

$$(L[u], v) - (u, L[v]) = 0. \tag{8}$$

In proving Theorem 11.2.1 below it is necessary to deal with complex-valued functions. By analogy with the definition in Section 7.2 for vectors, we define the inner product of two complex-valued functions on $0 \le x \le 1$ as

$$(u, v) = \int_0^1 u(x)\overline{v}(x)\,dx, \tag{9}$$

where $\overline{v}$ is the complex conjugate of v. Clearly, Eq. (9) coincides with Eq. (7) if $u(x)$ and $v(x)$ are real. It is important to know that Eq. (8) remains valid under the stated conditions if u and v are complex-valued functions and if the inner product (9) is used. To see this, one can start with the quantity $\int_0^1 L[u]\overline{v}\,dx$ and retrace the steps leading to Eq. (6), making use of the fact that $p(x)$, $q(x)$, a_1, a_2, b_1, and b_2 are all real quantities (see Problem 22).

We now consider some of the implications of Eq. (8) for the Sturm–Liouville boundary value problem (1), (2). We assume without proof[3] that this problem actually has eigenvalues and eigenfunctions. In Theorems 11.2.1 to 11.2.4 below, we state several of their important, but relatively elementary, properties. Each of these properties is illustrated by the basic Sturm–Liouville problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(1) = 0, \tag{10}$$

whose eigenvalues are $\lambda_n = n^2\pi^2$, with the corresponding eigenfunctions $\phi_n(x) = \sin n\pi x$.

Theorem 11.2.1 All the eigenvalues of the Sturm–Liouville problem (1), (2) are real.

To prove this theorem let us suppose that λ is a (possibly complex) eigenvalue of the problem (1), (2) and that ϕ is a corresponding eigenfunction, also possibly complex-valued. Let us write $\lambda = \mu + i\nu$ and $\phi(x) = U(x) + iV(x)$, where μ, ν, $U(x)$, and $V(x)$ are real. Then, if we let $u = \phi$ and also $v = \phi$ in Eq. (8), we have

$$(L[\phi], \phi) = (\phi, L[\phi]). \tag{11}$$

[3] The proof may be found, for example, in the references by Sagan (Chapter 5) or Birkhoff and Rota (Chapter 10).

However, we know that $L[\phi] = \lambda r\phi$, so Eq. (11) becomes

$$(\lambda r\phi, \phi) = (\phi, \lambda r\phi). \tag{12}$$

Writing out Eq. (12) in full, using the definition (9) of the inner product, we obtain

$$\int_0^1 \lambda r(x)\phi(x)\overline{\phi}(x)\, dx = \int_0^1 \phi(x)\overline{\lambda}\overline{r}(x)\overline{\phi}(x)\, dx. \tag{13}$$

Since $r(x)$ is real, Eq. (13) reduces to

$$(\lambda - \overline{\lambda}) \int_0^1 r(x)\phi(x)\overline{\phi}(x)\, dx = 0,$$

or

$$(\lambda - \overline{\lambda}) \int_0^1 r(x)[U^2(x) + V^2(x)]\, dx = 0. \tag{14}$$

The integrand in Eq. (14) is nonnegative and not identically zero. Since the integrand is also continuous, it follows that the integral is positive. Therefore, the factor $\lambda - \overline{\lambda} = 2i\nu$ must be zero. Hence $\nu = 0$ and λ is real, so the theorem is proved.

An important consequence of Theorem 11.2.1 is that in finding eigenvalues and eigenfunctions of a Sturm–Liouville boundary value problem, one need look only for real eigenvalues. Recall that this is what we did in Chapter 10. It is also possible to show that the eigenfunctions of the boundary value problem (1), (2) are real. A proof is sketched in Problem 23.

Theorem 11.2.2 If ϕ_1 and ϕ_2 are two eigenfunctions of the Sturm–Liouville problem (1), (2) corresponding to eigenvalues λ_1 and λ_2, respectively, and if $\lambda_1 \neq \lambda_2$, then

$$\int_0^1 r(x)\phi_1(x)\phi_2(x)\, dx = 0. \tag{15}$$

This theorem expresses the property of **orthogonality** of the eigenfunctions with respect to the weight function r. To prove the theorem we note that ϕ_1 and ϕ_2 satisfy the differential equations

$$L[\phi_1] = \lambda_1 r\phi_1 \tag{16}$$

and

$$L[\phi_2] = \lambda_2 r\phi_2, \tag{17}$$

respectively. If we let $u = \phi_1$, $v = \phi_2$, and substitute for $L[u]$ and $L[v]$ in Eq. (8), we obtain

$$(\lambda_1 r\phi_1, \phi_2) - (\phi_1, \lambda_2 r\phi_2) = 0,$$

or, using Eq. (9),

$$\lambda_1 \int_0^1 r(x)\phi_1(x)\overline{\phi}_2(x)\, dx - \overline{\lambda}_2 \int_0^1 \phi_1(x)\overline{r}(x)\overline{\phi}_2(x)\, dx = 0.$$

Because λ_2, $r(x)$, and $\phi_2(x)$ are real, this equation becomes

$$(\lambda_1 - \lambda_2) \int_0^1 r(x)\phi_1(x)\phi_2(x)\, dx = 0. \tag{18}$$

Since by hypothesis $\lambda_1 \neq \lambda_2$, it follows that ϕ_1 and ϕ_2 must satisfy Eq. (15), and the theorem is proved.

Theorem 11.2.3 The eigenvalues of the Sturm–Liouville problem (1), (2) are all simple; that is, to each eigenvalue there corresponds only one linearly independent eigenfunction. Further, the eigenvalues form an infinite sequence, and can be ordered according to increasing magnitude so that

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots.$$

Moreover, $\lambda_n \to \infty$ as $n \to \infty$.

The proof of this theorem is somewhat more advanced than those of the two previous theorems, and will be omitted. However, a proof that the eigenvalues are simple is indicated in Problem 20.

Again we note that all the properties stated in Theorems 11.2.1 to 11.2.3 are exemplified by the eigenvalues $\lambda_n = n^2\pi^2$ and eigenfunctions $\phi_n(x) = \sin n\pi x$ of the example problem (10). Clearly, the eigenvalues are real. The eigenfunctions satisfy the orthogonality relation

$$\int_0^1 \phi_m(x)\phi_n(x)\, dx = \int_0^1 \sin m\pi x \, \sin n\pi x \, dx = 0, \qquad m \neq n, \tag{19}$$

which was established in Section 10.2 by direct integration. Further, the eigenvalues can be ordered so that $\lambda_1 < \lambda_2 < \cdots$, and $\lambda_n \to \infty$ as $n \to \infty$. Finally, to each eigenvalue there corresponds a single linearly independent eigenfunction.

We will now assume that the eigenvalues of the Sturm–Liouville problem (1), (2) are ordered as indicated in Theorem 11.2.3. Associated with the eigenvalue λ_n is a corresponding eigenfunction ϕ_n, determined up to a multiplicative constant. It is often convenient to choose the arbitrary constant multiplying each eigenfunction so as to satisfy the condition

$$\int_0^1 r(x)\phi_n^2(x)\, dx = 1, \qquad n = 1, 2, \ldots. \tag{20}$$

Equation (20) is called a normalization condition, and eigenfunctions satisfying this condition are said to be **normalized**. Indeed, in this case, the eigenfunctions are said to form an **orthonormal set** (with respect to the weight function r) since they already satisfy the orthogonality relation (15). It is sometimes useful to combine Eqs. (15) and (20) into a single equation. To this end we introduce the symbol δ_{mn}, known as the Kronecker (1823–1891) delta and defined by

$$\delta_{mn} = \begin{cases} 0, & \text{if } m \neq n, \\ 1, & \text{if } m = n. \end{cases} \tag{21}$$

Making use of the Kronecker delta, we can write Eqs. (15) and (20) as

$$\int_0^1 r(x)\phi_m(x)\phi_n(x)\,dx = \delta_{mn}. \tag{22}$$

EXAMPLE 1

Determine the normalized eigenfunctions of the problem (10):

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y(1) = 0.$$

The eigenvalues of this problem are $\lambda_1 = \pi^2$, $\lambda_2 = 4\pi^2, \dots, \lambda_n = n^2\pi^2, \dots,$ and the corresponding eigenfunctions are $k_1 \sin \pi x$, $k_2 \sin 2\pi x, \dots, k_n \sin n\pi x, \dots,$ respectively. In this case the weight function is $r(x) = 1$. To satisfy Eq. (20) we must choose k_n so that

$$\int_0^1 (k_n \sin n\pi x)^2\,dx = 1 \tag{23}$$

for each value of n. Since

$$k_n^2 \int_0^1 \sin^2 n\pi x\,dx = k_n^2 \int_0^1 (\tfrac{1}{2} - \tfrac{1}{2}\cos 2n\pi x)\,dx = \tfrac{1}{2}k_n^2,$$

Eq. (23) is satisfied if k_n is chosen to be $\sqrt{2}$ for each value of n. Hence the normalized eigenfunctions of the given boundary value problem are

$$\phi_n(x) = \sqrt{2}\sin n\pi x, \qquad n = 1, 2, 3, \dots. \tag{24}$$

EXAMPLE 2

Determine the normalized eigenfunctions of the problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \quad y'(1) + y(1) = 0. \tag{25}$$

In Example 1 of Section 11.1 we found that the eigenvalues λ_n satisfy the equation

$$\sin\sqrt{\lambda_n} + \sqrt{\lambda_n}\cos\sqrt{\lambda_n} = 0, \tag{26}$$

and that the corresponding eigenfunctions are

$$\phi_n(x) = k_n \sin\sqrt{\lambda_n}\,x, \tag{27}$$

where k_n is arbitrary. We can determine k_n from the normalization condition (20). Since $r(x) = 1$ in this problem, we have

$$\begin{aligned}
\int_0^1 \phi_n^2(x)\,dx &= k_n^2 \int_0^1 \sin^2\sqrt{\lambda_n}\,x\,dx \\
&= k_n^2 \int_0^1 \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\sqrt{\lambda_n}\,x\right)dx = k_n^2 \left(\frac{x}{2} - \frac{\sin 2\sqrt{\lambda_n}\,x}{4\sqrt{\lambda_n}}\right)\Bigg|_0^1 \\
&= k_n^2 \frac{2\sqrt{\lambda_n} - \sin 2\sqrt{\lambda_n}}{4\sqrt{\lambda_n}} = k_n^2 \frac{\sqrt{\lambda_n} - \sin\sqrt{\lambda_n}\cos\sqrt{\lambda_n}}{2\sqrt{\lambda_n}} \\
&= k_n^2 \frac{1 + \cos^2\sqrt{\lambda_n}}{2},
\end{aligned}$$

where in the last step we have used Eq. (26). Hence, to normalize the eigenfunctions ϕ_n we must choose

$$k_n = \left(\frac{2}{1 + \cos^2 \sqrt{\lambda_n}} \right)^{1/2}. \tag{28}$$

The normalized eigenfunctions of the given problem are

$$\phi_n(x) = \frac{\sqrt{2} \sin \sqrt{\lambda_n}\, x}{(1 + \cos^2 \sqrt{\lambda_n})^{1/2}}; \qquad n = 1, 2, \ldots. \tag{29}$$

We now turn to the question of expressing a given function f as a series of eigenfunctions of the Sturm–Liouville problem (1), (2). We have already seen examples of such expansions in Sections 10.2 to 10.4. For example, it was shown there that if f is continuous and has a piecewise continuous derivative on $0 \le x \le 1$, and satisfies the boundary conditions $f(0) = f(1) = 0$, then f can be expanded in a Fourier sine series of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x. \tag{30}$$

The functions $\sin n\pi x$, $n = 1, 2, \ldots$, are precisely the eigenfunctions of the boundary value problem (10). The coefficients b_n are given by

$$b_n = 2 \int_0^1 f(x) \sin n\pi x \, dx \tag{31}$$

and the series (30) converges for each x in $0 \le x \le 1$. In a similar way f can be expanded in a Fourier cosine series using the eigenfunctions $\cos n\pi x$, $n = 0, 1, 2, \ldots$, of the boundary value problem $y'' + \lambda y = 0$, $y'(0) = 0$, $y'(1) = 0$.

Now suppose that a given function f, satisfying suitable conditions, can be expanded in an infinite series of eigenfunctions of the more general Sturm–Liouville problem (1), (2). If this can be done, then we have

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \tag{32}$$

where the functions $\phi_n(x)$ satisfy Eqs. (1), (2), and also the orthogonality condition (22). To compute the coefficients in the series (32) we multiply Eq. (32) by $r(x)\phi_m(x)$, where m is a fixed positive integer, and integrate from $x = 0$ to $x = 1$. Assuming that the series can be integrated term by term we obtain

$$\int_0^1 r(x) f(x) \phi_m(x) \, dx = \sum_{n=1}^{\infty} c_n \int_0^1 r(x) \phi_m(x) \phi_n(x) \, dx = \sum_{n=1}^{\infty} c_n \delta_{mn}. \tag{33}$$

Hence, using the definition of δ_{mn}, we have

$$c_m = \int_0^1 r(x) f(x) \phi_m(x) \, dx = (f, r\phi_m), \qquad m = 1, 2, \ldots. \tag{34}$$

The coefficients in the series (32) have thus been formally determined. Equation (34) has the same structure as the Euler–Fourier formulas for the coefficients in a Fourier series, and the eigenfunction series (32) also has convergence properties similar to those of Fourier series. The following theorem is analogous to Theorem 10.3.1.

Theorem 11.2.4 Let $\phi_1, \phi_2, \dots, \phi_n, \dots$ be the normalized eigenfunctions of the Sturm–Liouville problem (1), (2):

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0,$$

$$a_1 y(0) + a_2 y'(0) = 0, \qquad b_1 y(1) + b_2 y'(1) = 0.$$

Let f and f' be piecewise continuous on $0 \le x \le 1$. Then the series (32) whose coefficients c_m are given by Eq. (34) converges to $[f(x+) + f(x-)]/2$ at each point in the open interval $0 < x < 1$.

If f satisfies further conditions, then a stronger conclusion can be established. Suppose that, in addition to the hypotheses of Theorem 11.2.4, the function f is continuous on $0 \le x \le 1$. If $a_2 = 0$ in the first of Eqs. (2) [so that $\phi_n(0) = 0$], then assume that $f(0) = 0$. Similarly, if $b_2 = 0$ in the second of Eqs. (2), assume that $f(1) = 0$. Otherwise no boundary conditions need be prescribed for f. Then the series (32) converges to $f(x)$ at each point in the closed interval $0 \le x \le 1$.

EXAMPLE 3

Expand the function

$$f(x) = x, \qquad 0 \le x \le 1 \tag{35}$$

in terms of the normalized eigenfunctions $\phi_n(x)$ of the problem (25).

In Example 2 we found the normalized eigenfunctions to be

$$\phi_n(x) = k_n \sin\sqrt{\lambda_n}\, x, \tag{36}$$

where k_n is given by Eq. (28) and λ_n satisfies Eq. (26). To find the expansion for f in terms of the eigenfunctions ϕ_n we write

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \tag{37}$$

where the coefficients are given by Eq. (34). Thus

$$c_n = \int_0^1 f(x)\phi_n(x)\, dx = k_n \int_0^1 x \sin\sqrt{\lambda_n}\, x\, dx.$$

Integrating by parts, we obtain

$$c_n = k_n \left(\frac{\sin\sqrt{\lambda_n}}{\lambda_n} - \frac{\cos\sqrt{\lambda_n}}{\sqrt{\lambda_n}} \right) = k_n \frac{2\sin\sqrt{\lambda_n}}{\lambda_n},$$

where we have used Eq. (26) in the last step. Upon substituting for k_n from Eq. (28) we obtain

$$c_n = \frac{2\sqrt{2}\sin\sqrt{\lambda_n}}{\lambda_n(1+\cos^2\sqrt{\lambda_n})^{1/2}}. \tag{38}$$

Thus

$$f(x) = 4\sum_{n=1}^{\infty} \frac{\sin\sqrt{\lambda_n}\sin\sqrt{\lambda_n}\,x}{\lambda_n(1+\cos^2\sqrt{\lambda_n})}. \tag{39}$$

Observe that although the right side of Eq. (39) is a series of sines, it is not included in the discussion of Fourier sine series in Section 10.4.

Self-adjoint Problems. Sturm–Liouville boundary value problems are of great importance in their own right, but they can also be viewed as belonging to a much more extensive class of problems that have many of the same properties. For example, there are many similarities between Sturm–Liouville problems and the algebraic system

$$\mathbf{Ax} = \lambda\mathbf{x}, \tag{40}$$

where the $n \times n$ matrix $\mathbf{A}$ is real symmetric or Hermitian. Comparing the results mentioned in Section 7.3 with those of this section, we note that in both cases the eigenvalues are real and the eigenfunctions or eigenvectors form an orthogonal set. Further, the eigenfunctions or eigenvectors can be used as the basis for expressing an essentially arbitrary function or vector, respectively, as a sum. The most important difference is that a matrix has only a finite number of eigenvalues and eigenvectors, while a Sturm–Liouville problem has infinitely many. It is interesting and of fundamental importance in mathematics that these seemingly different problems—the matrix problem (40) and the Sturm–Liouville problem (1), (2)—which arise in different ways, are actually part of a single underlying theory. This theory is usually referred to as linear operator theory, and is part of the subject of functional analysis.

We now point out some ways in which Sturm–Liouville problems can be generalized, while still preserving the main results of Theorems 11.2.1 to 11.2.4—the existence of a sequence of real eigenvalues tending to infinity, the orthogonality of the eigenfunctions, and the possibility of expressing an arbitrary function as a series of eigenfunctions. In making these generalizations it is essential that the crucial relation (8) remain valid.

Let us consider the boundary value problem consisting of the differential equation

$$L[y] = \lambda r(x)y, \qquad 0 < x < 1, \tag{41}$$

where

$$L[y] = P_n(x)\frac{d^n y}{dx^n} + \cdots + P_1(x)\frac{dy}{dx} + P_0(x)y, \tag{42}$$

and n linear homogeneous boundary conditions at the endpoints. If Eq. (8) is valid for every pair of sufficiently differentiable functions that satisfy the boundary conditions, then the given problem is said to be **self-adjoint**. It is important to observe that Eq. (8) involves restrictions on both the differential equation and the boundary conditions. The differential operator L must be such that the same operator appears in both terms of

Eq (8). This requires that L be of even order. Further, a second order operator must have the form (3), a fourth order operator must have the form

$$L[y] = [p(x)y'']'' - [q(x)y']' + s(x)y, \tag{43}$$

and higher order operators must have an analogous structure. In addition, the boundary conditions must be such as to eliminate the boundary terms that arise during the integration by parts used in deriving Eq. (8). For example, in a second order problem this is true for the separated boundary conditions (2) and also in certain other cases, one of which is given in Example 4 below.

Let us suppose that we have a self-adjoint boundary value problem for Eq. (41), where $L[y]$ is given now by Eq. (43). We assume that p, q, r, and s are continuous on $0 \le x \le 1$, and that the derivatives of p and q indicated in Eq. (43) are also continuous. If in addition $p(x) > 0$ and $r(x) > 0$ for $0 \le x \le 1$, then there is an infinite sequence of real eigenvalues tending to $+\infty$, the eigenfunctions are orthogonal with respect to the weight function r, and an arbitrary function can be expressed as a series of eigenfunctions. However, the eigenfunctions may not be simple in these more general problems.

We turn now to the relation between Sturm–Liouville problems and Fourier series. We have noted previously that Fourier sine and cosine series can be obtained by using the eigenfunctions of certain Sturm–Liouville problems involving the differential equation $y'' + \lambda y = 0$. This raises the question of whether we can obtain a full Fourier series, including both sine and cosine terms, by choosing a suitable set of boundary conditions. The answer is provided by the following example, which also serves to illustrate the occurrence of nonseparated boundary conditions.

EXAMPLE 4

Find the eigenvalues and eigenfunctions of the boundary value problem

$$y'' + \lambda y = 0, \tag{44}$$

$$y(-L) - y(L) = 0, \qquad y'(-L) - y'(L) = 0. \tag{45}$$

This is not a Sturm–Liouville problem because the boundary conditions are not separated. The boundary conditions (45) are called **periodic boundary conditions** since they require that y and y' assume the same values at $x = L$ as at $x = -L$. Nevertheless, it is straightforward to show that the problem (44), (45) is self-adjoint. A simple calculation establishes that $\lambda_0 = 0$ is an eigenvalue and that the corresponding eigenfunction is $\phi_0(x) = 1$. Further, there are additional eigenvalues $\lambda_1 = (\pi/L)^2$, $\lambda_2 = (2\pi/L)^2, \ldots, \lambda_n = (n\pi/L)^2, \ldots$. To each of these nonzero eigenvalues there correspond *two* linearly independent eigenfunctions; for example, corresponding to λ_n are the two eigenfunctions $\phi_n(x) = \cos(n\pi x/L)$ and $\psi_n(x) = \sin(n\pi x/L)$. This illustrates that the eigenvalues may not be simple when the boundary conditions are not separated. Further, if we seek to expand a given function f of period $2L$ in a series of eigenfunctions of the problem (44), (45), we obtain the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right),$$

which is just the Fourier series for f.

We will not give further consideration to problems that have nonseparated boundary conditions, nor will we deal with problems of higher than second order, except in a few problems. There is, however, one other kind of generalization that we do wish to discuss. That is the case in which the coefficients p, q, and r in Eq. (1) do not quite satisfy the rather strict continuity and positivity requirements laid down at the beginning of this section. Such problems are called singular Sturm–Liouville problems, and are the subject of Section 11.4.

PROBLEMS

In each of Problems 1 through 5 determine the normalized eigenfunctions of the given problem.

1. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0$
2. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = 0$
3. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(1) = 0$
4. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(1) + y(1) = 0;$ see Section 11.1, Problem 8.
5. $y'' - 2y' + (1 + \lambda)y = 0, \quad y(0) = 0, \quad y(1) = 0;$ see Section 11.1, Problem 17.

In each of Problems 6 through 9 find the eigenfunction expansion $\sum_{n=1}^{\infty} a_n\phi_n(x)$ of the given function, using the normalized eigenfunctions of Problem 1.

6. $f(x) = 1, \quad 0 \le x \le 1$
7. $f(x) = x, \quad 0 \le x \le 1$
8. $f(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ 0, & \frac{1}{2} \le x \le 1 \end{cases}$
9. $f(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ 1, & \frac{1}{2} \le x \le 1 \end{cases}$

In each of Problems 10 through 13 find the eigenfunction expansion $\sum_{n=1}^{\infty} a_n\phi_n(x)$ of the given function, using the normalized eigenfunctions of Problem 4.

10. $f(x) = 1, \quad 0 \le x \le 1$
11. $f(x) = x, \quad 0 \le x \le 1$
12. $f(x) = 1 - x, \quad 0 \le x \le 1$
13. $f(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ 0, & \frac{1}{2} \le x \le 1 \end{cases}$

In each of Problems 14 through 18 determine whether the given boundary value problem is self-adjoint.

14. $y'' + y' + 2y = 0, \quad y(0) = 0, \quad y(1) = 0$
15. $(1 + x^2)y'' + 2xy' + y = 0, \quad y'(0) = 0, \quad y(1) + 2y'(1) = 0$
16. $y'' + y = \lambda y, \quad y(0) - y'(1) = 0, \quad y'(0) - y(1) = 0$
17. $(1 + x^2)y'' + 2xy' + y = \lambda(1 + x^2)y, \quad y(0) - y'(1) = 0, \quad y'(0) + 2y(1) = 0$
18. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) + y'(\pi) = 0$
19. Show that if the functions u and v satisfy Eqs. (2), and either $a_2 = 0$ or $b_2 = 0$, or both, then
$$p(x)[u'(x)v(x) - u(x)v'(x)]\Big|_0^1 = 0.$$
20. In this problem we outline a proof of the first part of Theorem 11.2.3: that the eigenvalues of the Sturm–Liouville problem (1), (2) are simple. For a given λ suppose that ϕ_1 and ϕ_2 are two linearly independent eigenfunctions. Compute the Wronskian $W(\phi_1, \phi_2)(x)$ and use the boundary conditions (2) to show that $W(\phi_1, \phi_2)(0) = 0$. Then use Theorems 3.3.2 and 3.3.3 to conclude that ϕ_1 and ϕ_2 cannot be linearly independent as assumed.
21. Consider the Sturm–Liouville problem
$$-[p(x)y']' + q(x)y = \lambda r(x)y,$$
$$a_1 y(0) + a_2 y'(0) = 0, \qquad b_1 y(1) + b_2 y'(1) = 0,$$