Applications: cohomology of curves

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22nd of June, 2021
Poincaré duality

Recall that the big theorem we talked about last time is the following.

**Theorem: Poincaré duality**

Let $X$ be a smooth projective curve over a field $k = \bar{k}$ in which $n \in k^*$. Let $U \neq \emptyset$ be an open subscheme of $X$. Then:

- There is a canonical isomorphism $\eta(U) : H^2_c(U, \mu_n) \to \mathbb{Z}/n\mathbb{Z}$;
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- For any constructible sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on $U$, the groups $H^r_c(U, \mathcal{F})$ and $\text{Ext}^r_U(\mathcal{F}, \mu_n) := \text{Ext}_{\text{Sh}(U_{\text{et}}, \mathbb{Z}/n\mathbb{Z})}(\mathcal{F}, \mu_n)$ are finite for all $r$ and zero for $r > 2$;
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- The canonical cup product pairings

  $$H^r_c(U, \mathcal{F}) \times \text{Ext}^{2-r}_U(\mathcal{F}, \mu_n) \to H^2_c(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

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are nondegenerate.

Using the pairing, we have a notion of a *dual*. In particular, we may rephrase the above theorem as saying that

$$H^r_c(U, \mathcal{F}) \cong \left(\text{Ext}^{2-r}_U(\mathcal{F}, \mu_n)\right)^\vee.$$
Localising Poincaré duality

Theorem
Let $X = \text{Spec}(k[t])$ where $k = \bar{k}$ and let $x \in X$ be the closed point. Let $\mathcal{F}$ be a constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on $X$, where $n \in k^*$. Then there is a canonical isomorphism $\eta : H^2_x(X, \mu_n) \to \mathbb{Z}/n\mathbb{Z}$ and the canonical pairing

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Proof: View $X$ as the local ring at some geometric point of a smooth projective curve as in Poincaré duality and check that everything works nicely when localising. Milne also gives a direct proof, see Prop V.2.2a.
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Recall the notation $\tilde{\mathcal{F}}(m) = \text{Hom}(\mathcal{F}, (\mathbb{Z}/n\mathbb{Z})(m))$, where $(\mathbb{Z}/n\mathbb{Z})(m)$ denotes the $m$th Tate twist. In particular, $\tilde{\mathcal{F}}(1) = \text{Hom}(\mathcal{F}, \mu_n)$. Since Ext-groups are defined as the right derived functors of some Hom, we may naturally identify

$$\text{Ext}^r_U(\mathcal{F}, \mu_n) = H^r(U, \tilde{\mathcal{F}}(1)).$$
Corollary

Let $U \neq \emptyset$ be an open subscheme of a smooth projective curve $X$ over $k = \bar{k}$ with $n \in k^*$ and let $j : U \to V \subset X$ be an open immersion.

Let $\mathcal{F}$ be a locally constant sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules. Then the following canonical pairings are nondegenerate:

$$H^r_c(V, j_* \mathcal{F}) \times H^{2-r}(V, j_* \mathcal{F}(1)) \to H^2_c(V, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$
Corollary

Moving to $\mathcal{Q}_\ell$

Let $U \neq \emptyset$ be an open subscheme of a smooth projective curve $X$ over $k = \bar{k}$ with $n \in k^*$ and let $j : U \to V \subset X$ be an open immersion.

- Let $\mathcal{F}$ be a locally constant sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules. Then the following canonical pairings are nondegenerate:
  \[
  H_c^r(V, j_\ast \mathcal{F}) \times H^{2-r}(V, j_\ast \mathcal{F}(1)) \to H^2_c(V, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}
  \]

- Now let $\mathcal{F}$ be Tate-twist of a constant, constructible sheaf of $\mathcal{Q}_\ell$-vector spaces, where $\ell \neq \text{char}(k)$. Then the pairings
  \[
  H_c^r(V, j_\ast \mathcal{F}) \times H^{2-r}(V, j_\ast \mathcal{F}(1)) \to H^2_c(V, \mathcal{Q}_\ell(1)) \cong \mathcal{Q}_\ell
  \]
  are nondegenerate pairings of finite-dimensional $\mathcal{Q}_\ell$-vector spaces.

Proof: The second follows from the first by taking an inverse limit over $n = \ell^m$ and tensoring with $\mathcal{Q}_\ell$. For the first, use the spectral sequence $H_r(V, \text{Ext}^s(j_\ast \mathcal{F}, \mu_n)) \Rightarrow \text{Ext}^{r+s}_V(V, j_\ast \mathcal{F}, \mu_n)$ and show that $\text{Hom}(j_\ast \mathcal{F}, \mu_n) = j_\ast \check{\mathcal{F}}(1)$ and $\text{Ext}^s(j_\ast \mathcal{F}, \mu_n) = 0$ for $s > 0$. The first is easy, and the second can be checked étale locally, so we may use the previous theorem to reduce to its dual $H^{2-r}(\check{X}, j_\ast \mathcal{F})$. Then use
  \[
  0 \to H^0(\check{X}, j_\ast \mathcal{F}) \to H^0(\check{X}, j_\ast \mathcal{F}) \to H^1(\check{X} \setminus \{x\}, j_\ast \mathcal{F}) \to 0
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Let $U \neq \emptyset$ be an open subscheme of a smooth projective curve $X$ over $k = \overline{k}$ with $n \in k^*$ and let $j : U \to V \subset X$ be an open immersion.

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and show that $\text{Hom}(j_*\mathcal{F}, \mu_n) = j_*\check{\mathcal{F}}(1)$ and $\text{Ext}^s(j_*\mathcal{F}, \mu_n) = 0$ for $s > 0$. The first is easy, and the second can be checked étale locally, so we may use the previous theorem to reduce to its dual $H^{2-s}_x(\tilde{X}, j_*\mathcal{F})$. Then use

$$0 \to H^0_x(\tilde{X}, j_*\mathcal{F}) \to H^0(\tilde{X}, j_*\mathcal{F}) \to H^0(\tilde{X} \setminus \{x\}, j_*\mathcal{F}) \to H^1_x(\tilde{X}, j_*\mathcal{F}) \to 0.$$
Results for other fields (1/4)

It is natural to wonder what happens for non-algebraically closed fields. Milne gives the answer for finite fields.
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**Proposition**

Let $k$ be a finite field and let $n \in k^*$. For any open subscheme $U \neq \emptyset$ of $X$, there is a canonical isomorphism $\eta(U) : H^3_c(U, \mu_n) \to \mathbb{Z}/n\mathbb{Z}$. In addition, for any locally constant constructible sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules, the canonical pairings

$$H^r_c(U, \mathcal{F}) \times H^{3-r}(U, \check{\mathcal{F}}(1)) \to H^3_c(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

are nondegenerate pairings of finite groups.

In other words, we want to show that $H^r_c(U, \mathcal{F}) = H^{3-r}(U, \check{\mathcal{F}}(1))^\vee$. For this, we will need a few results from group cohomology.
Results for other fields (2/4)

Lemma
Let $k$ be a finite field for which $n \in k^*$ and let $\Gamma = \text{Gal}(k_{\text{sep}}/k) \cong \hat{\mathbb{Z}}$ with canonical generator $\sigma$. Then for any finite $n$-torsion $\Gamma$-module $M$, we have an exact sequence

$$0 \rightarrow H^0(\Gamma, M) \rightarrow M \xrightarrow{\sigma^{-1}} M \rightarrow H^1(\Gamma, M) \rightarrow 0$$

and $H^r(\Gamma, M) = 0$ for all $r \geq 2$. 

We will denote $H^0(\Gamma, M)$ by $M^\Gamma$ and $H^1(\Gamma, M)$ by $M^\Gamma$. 

Lemma
Let $\Gamma$ and $M$ be as above, and write $\check{M} = \text{Hom}(M, \mathbb{Z}/n\mathbb{Z})$. Then the canonical pairings $H^r(\Gamma, M) \times H^{1-r}(\Gamma, \check{M}) \rightarrow H^1(\Gamma, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ for $r = 0$ sending $(m, \psi) \rightarrow \tau \mapsto \psi(\tau)(m)$ and for $r = 1$ sending $(\phi, \theta) \rightarrow \theta \circ \phi$ are nondegenerate.
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Results for other fields (3/4)

Just as with Poincaré duality, we may rewrite the previous lemma by saying

\[ M_\Gamma = H^1(\Gamma, M) = H^0(\Gamma, \tilde{M})^\vee = (\tilde{M}^\Gamma)^\vee. \]
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Recall the Hochschild-Serre spectral sequence: for a Galois covering \( X' \to X \) with Galois group \( G \) and \( \mathcal{F} \) a sheaf for the étale topology on \( X \), there is a spectral sequence

\[ H^p(G, H^q(X'_\text{et}, \mathcal{F})) \Rightarrow H^{p+q}(X_{\text{et}}, \mathcal{F}). \]
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We apply this to \( \tilde{X} = X \otimes_k k^{\text{sep}} \) and \( \tilde{U} = U \otimes_k k^{\text{sep}} \). Since we know that \( H^r(G, -) = 0 \) for \( r \geq 2 \) always in our situation, we have a spectral sequence with only two non-zero columns. This will degenerate immediately, so we obtain exact sequences

\[
0 \to H^{-1}_c(\tilde{U}, \mathcal{F})_\Gamma \to H^r_c(U, \mathcal{F}) \to H^r_c(\tilde{U}, \mathcal{F})_\Gamma \to 0;
0 \to H^{-1}(\tilde{U}, \mathcal{F})_\Gamma \to H^r(U, \mathcal{F}) \to H^r(\tilde{U}, \mathcal{F})_\Gamma \to 0.
\]

For \( r = 3 \), the top one gives \( H^3_c(U, \mu_n) \cong H^2_c(\tilde{U}, \mu_n)_\Gamma \cong \mathbb{Z}/n\mathbb{Z} \).
Results for other fields (4/4)

From Poincaré duality we obtain

\[ H^r_c(\bar{U}, \mathcal{F}) \cong \text{Ext}^{2-r}(\mathcal{F}, \mu_n)^\vee \cong H^{2-r}(\bar{U}, \check{\mathcal{F}}(1))^\vee \]

and so by the previous lemma,

\[ H^r_c(\bar{U}, \mathcal{F})^\Gamma \cong (H^{2-r}(U, \check{\mathcal{F}}(1))^\Gamma)^\vee. \]

Similarly,

\[ H^{r-1}_c(\bar{U}, \mathcal{F})^\Gamma \cong (H^{3-r}(U, \check{\mathcal{F}}(1))^\Gamma)^\vee. \]
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Similarly,

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Now, we can construct a commutative diagram as follows:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & H^{-1}_c(\bar{U}, \mathcal{F}) & \rightarrow & H^r_c(U, \mathcal{F}) & \rightarrow & H^r_c(\bar{U}, \mathcal{F}) & \rightarrow & 0 \\
\downarrow\text{iso} & & \downarrow & & \downarrow & & \downarrow\text{iso} & & \\
0 & \rightarrow & (H^{3-r}(\bar{U}, \check{\mathcal{F}}(1)))^\vee & \rightarrow & H^{3-r}(U, \check{\mathcal{F}}(1))^\vee & \rightarrow & (H^{2-r}(\bar{U}, \check{\mathcal{F}}(1)))^\vee & \rightarrow & 0
\end{array}
$$

Since the outer two maps are isomorphisms, the five lemma tells us that the middle one is also. This is precisely the duality statement that we sought to prove.
Artin-Verdier duality

Theorem
Let $K$ be a totally imaginary number field with ring of integers $\mathcal{O}_K$ and let $X = \text{Spec}(\mathcal{O}_K)$. Then there is a canonical isomorphism $\eta : H^3(X, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$. In addition, for any constructible étale abelian sheaf $\mathcal{F}$ on $X$, and for any $r$, the pairings

$$H^r(X, \mathcal{F}) \times \text{Ext}_X^{3-r}(\mathcal{F}, \mathbb{G}_m) \to H^3(X, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$$

are all nondegenerate pairings of finite abelian groups.
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**Remark 1:** This theorem is not true for not totally imaginary number fields, as their will be problems with 2-torsion elements. However, modulo 2-torsion the statements will remain true.
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Remark 1: This theorem is not true for not totally imaginary number fields, as their will be problems with 2-torsion elements. However, modulo 2-torsion the statements will remain true.

Remark 2: Morally, this theorem shows that on the étale site, the ring of integers in a number field behaves like a 3-dimensional mathematical object. According to Milne, the proof of this is a refinement of the proof of Poincaré duality, but using a lot of class field theory. We will not discuss this difficult proof.
Explicit description of the cohomology ring (1/4)

Recall the definition of an *abelian variety*, i.e. a projective algebraic variety that is also an algebraic group, e.g. an elliptic curve. More generally, for a lattice $L \subset \mathbb{C}^n$ of maximal rank, the variety $\mathbb{C}^n/L$ is an abelian complex variety, but not all of these are *algebraic*. In general, if $A$ is an abelian variety, we define the *dual* abelian variety by $\hat{A} = \text{Pic}^0(A)$. 
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Let $A_n$ denote the $n$-torsion of an abelian variety $A$. There exists a pairing generalising the Weil-pairing $A_n \times \hat{A}_n \to \mu_n(k)$ defined as follows. Let $\mathcal{L} \cong \mathcal{O}_X(D) \in \hat{A}_n$ for some $n \in k^*$. Then since $\mathcal{L}^n$ is trivial, one can show that also $n_A^* \mathcal{L}$ is trivial, where $n_A^*$ denotes the map induced by multiplication by $n$ on $A$. Hence we can find functions $f, g$ on $A$ such that $\text{div}(f) = nD$ and $\text{div}(g) = n_A^* D$. We see that

$$n_A^* \text{div}(f) = n_A^*(nD) = n \cdot n_A^* D = \text{div}(g^n).$$
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$$n^*_A \text{div}(f) = n^*_A (nD) = n \cdot n^*_A D = \text{div}(g^n).$$

Since $f \circ n_A$ and $g^n$ have the same divisor, so $g = \lambda \cdot f \circ n_A$. But then

$$g(x)^n = \lambda f(n \cdot x) = \lambda f(n \cdot (x + a)) = g(x + a)^n$$

for all $a \in A_n$. Hence $g(x)/g(x + a)$ must be a constant $n$-th root of unity. This defines the pairing $e_n : A_n \times \hat{A}_n \to \mu_n$. 
Explicit description of the cohomology ring (2/4)

The pairing $e_n$ defined above enjoys many beautiful properties. It is bilinear in both variables, it is alternating and thus skew-symmetric, it is nondegenerate, and most importantly, it is Galois invariant.
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Let $X$ be a smooth projective curve over a field $k = \bar{k}$. Then there exists a moduli space of all degree 0 line bundles on $X$, sometimes denoted $\text{Pic}^0(X)$. It is also the connected component of the identity of the Picard group $\text{Pic}(X)$ of $X$, and is also called the Jacobian of $X$, denoted $J(X)$.
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Remark: What does Poincaré duality say for $\mathcal{F} = \mu_n$? Well, in that case $\check{\mathcal{F}}(1) = \text{Hom}(\mathcal{F}, \mu_n) = \mu_n$, so

$$\text{Ext}^{2-r}_{U}(\mu_n, \mu_n) = H^{2-r}(X, \mu_n).$$

Hence for $r = 1$, Poincaré duality reduces to the ordinary cup product

$$H^1(X, \mu_n) \times H^1(X, \mu_n) \to H^2(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$
Explicit description of the cohomology ring (3/4)

Recall that $H^1(X, \mathbb{G}_m) \cong \text{Pic}(X)$. Using this, we can show that

$$H^1(X, \mu_n) \cong J(X)_n.$$ 

Namely, use the Kummer sequence $0 \rightarrow \mu_n \rightarrow \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \rightarrow 0$ and then:
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\[
\begin{array}{cccccc}
0 & \to & J(X)_n & \to & J(X) & \to & 0 \\
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0 & \to & H^1(X, \mu_n) & \to & \text{Pic}(X) & \xrightarrow{n} & \text{Pic}(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathbb{Z} & \xrightarrow{n} & \mathbb{Z} & \to & \mathbb{Z}/n\mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
\]
Explicit description of the cohomology ring (4/4)

Theorem
The cup product pairing

\[ H^1(X, \mu_n) \times H^1(X, \mu_n) \to H^2(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z} \]

under the isomorphism from before agrees with the \(e_n\)-pairing

\[ e_n : J(X)_n \times \widehat{J(X)}_n \to \mathbb{Z}/n\mathbb{Z}. \]
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Remark 1: The proof can be found in SGA 4 1/2: Cohomologie Etale, Section 5.3. It is technical, so we will not discuss it.
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Remark 1: The proof can be found in SGA 4$_1$2: Cohomologie Etale, Section 5.3. It is technical, so we will not discuss it.

Remark 2: Since the $e_n$-pairings are compatible for varying $n$, we can combine some of them into an inverse system to obtain maps

$$T_\ell(X) \times T_\ell(\hat{X}) \to \lim_{\leftarrow} \mu_{\ell^m}$$

from the Tate module of $X$ for each prime $\ell \in k^*$. This pairing is even $\mathbb{Z}_\ell$-bilinear and nondegenerate, and thus shows how one can naturally start to consider $\ell$-adic cohomology. This is the realm of for instance the Lefschetz trace formula... but more on that later.
Thanks for listening!

Figure: Étale is a mountain of Savoie and Haute-Savoie, France. It lies in the Aravis Range of the French Prealps and reaches 2,484 metres above sea level.