Applications: cohomology of curves

Mike Daas

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Universiteit Leiden

Poincaré duality

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Theorem: Poincaré duality

Let X be a smooth projective curve over a field $k = \bar{k}$ in which $n \in k^*$. Let $U \neq \varnothing$ be an open subscheme of X. Then:

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- The canonical *cup product* pairings

$$H^r_c(U, \mathcal{F}) \times \mathrm{Ext}^{2-r}_U(\mathcal{F}, \mu_n) \to H^2_c(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

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  are nondegenerate.

Using the pairing, we have a notion of a *dual*. In particular, we may rephrase the above theorem as saying that

$$H^r_c(U, \mathcal{F}) \cong \mathrm{Ext}^{2-r}_U(\mathcal{F}, \mu_n)^\vee.$$

Localising Poincaré duality

Theorem

Let $X = \mathrm{Spec}(k[t]_t)$ where $k = \bar{k}$ and let $x \in X$ be the closed point. Let $\mathcal{F}$ be a constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules on X, where $n \in k^*$. Then there is a canonical isomorphism $\eta : H^2_x(X, \mu_n) \to \mathbb{Z}/n\mathbb{Z}$ and the canonical pairing

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Recall the notation $\check{\mathcal{F}}(m) = \mathrm{Hom}(\mathcal{F}, (\mathbb{Z}/n\mathbb{Z})(m))$, where $(\mathbb{Z}/n\mathbb{Z})(m)$ denotes the mth Tate twist. In particular, $\check{\mathcal{F}}(1) = \mathrm{Hom}(\mathcal{F}, \mu_n)$. Since Ext-groups are defined as the right derived functors of some Hom, we may naturally identify

$$\mathrm{Ext}^r_U(\mathcal{F}, \mu_n) = H^r(U, \check{\mathcal{F}}(1)).$$

Moving to $\mathbb{Q}_\ell$

Corollary

Let $U \neq \varnothing$ be an open subscheme of a smooth projective curve X over $k = \bar{k}$ with $n \in k^*$ and let $j : U \to V \subset X$ be an open immersion.

- Let $\mathcal{F}$ be a locally constant sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules. Then the following canonical pairings are nondegenerate:

$$H_c^r(V, j_*\mathcal{F}) \times H^{2-r}(V, j_*\check{\mathcal{F}}(1)) \to H_c^2(V, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

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- Now let $\mathcal{F}$ be Tate-twist of a constant, constructible sheaf of $\mathbb{Q}_\ell$-vector spaces, where $\ell \neq \mathrm{char}(k)$. Then the pairings

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  are nondegenerate pairings of finite-dimensional $\mathbb{Q}_\ell$-vector spaces.

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are nondegenerate pairings of finite-dimensional $\mathbb{Q}_\ell$-vector spaces.

Proof: The second follows from the first by taking an inverse limit over $n = \ell^m$ and tensoring with $\mathbb{Q}_\ell$. For the first, use the spectral sequence
$$H^r(V, \mathrm{Ext}^s(j_*\mathcal{F}, \mu_n)) \implies \mathrm{Ext}^{r+s}_V(j_*\mathcal{F}, \mu_n)$$
and show that $\mathrm{Hom}(j_*\mathcal{F}, \mu_n) = j_*\check{\mathcal{F}}(1)$ and $\mathrm{Ext}^s(j_*\mathcal{F}, \mu_n) = 0$ for $s > 0$. The first is easy, and the second can be checked étale locally, so we may use the previous theorem to reduce to its dual $H^{2-s}_x(\tilde{X}, j_*\mathcal{F})$. Then use
$$0 \to H^0_x(\tilde{X}, j_*\mathcal{F}) \to H^0(\tilde{X}, j_*\mathcal{F}) \xrightarrow{\sim} H^0(\tilde{X} \setminus \{x\}, j_*\mathcal{F}) \to H^1_x(\tilde{X}, j_*\mathcal{F}) \to 0.$$

Results for other fields (1/4)

It is natural to wonder what happens for non-algebraically closed fields. Milne gives the answer for finite fields.

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Proposition

Let k be a finite field and let $n \in k^*$. For any open subscheme $U \neq \varnothing$ of X, there is a canonical isomorphism $\eta(U) : H^3_c(U, \mu_n) \to \mathbb{Z}/n\mathbb{Z}$. In addition, for any locally constant constructible sheaf $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules, the canonical pairings

$$H^r_c(U, \mathcal{F}) \times H^{3-r}(U, \check{\mathcal{F}}(1)) \to H^3_c(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

are nondegenerate pairings of finite groups.

In other words, we want to show that $H^r_c(U, \mathcal{F}) = H^{3-r}(U, \check{\mathcal{F}}(1))^\vee$. For this, we will need a few results from group cohomology.

Results for other fields (2/4)

Lemma

Let k be a finite field for which $n \in k^*$ and let $\Gamma = \mathrm{Gal}(k^{\mathrm{sep}}/k) \cong \widehat{\mathbb{Z}}$ with canonical generator σ. Then for any finite n-torsion Γ-module M, we have an exact sequence

$$0 \to H^0(\Gamma, M) \to M \xrightarrow{\sigma - 1} M \to H^1(\Gamma, M) \to 0$$

and $H^r(\Gamma, M) = 0$ for all $r \geq 2$.

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We will denote $H^0(\Gamma, M)$ by M^Γ and $H^1(\Gamma, M)$ by M_Γ.

Lemma

Let Γ and M be as above, and write $\check{M} = \mathrm{Hom}(M, \mathbb{Z}/n\mathbb{Z})$. Then the canonical pairings

$$H^r(\Gamma, M) \times H^{1-r}(\Gamma, \check{M}) \to H^1(\Gamma, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$$

for $r = 0$ sending (m, ψ) to $\tau \mapsto \psi(\tau)(m)$ and for $r = 1$ sending (ϕ, θ) to $\theta \circ \phi$ are nondegenerate.

Results for other fields (3/4)

Just as with Poincaré duality, we may rewrite the previous lemma by saying

$$M_\Gamma = H^1(\Gamma, M) = H^0(\Gamma, \check{M})^\vee = (\check{M}^\Gamma)^\vee.$$

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Recall the Hochschild-Serre spectral sequence: for a Galois covering $X' \to X$ with Galois group G and $\mathcal{F}$ a sheaf for the étale topology on X, there is a spectral sequence

$$H^p(G, H^q(X'_{\text{et}}, \mathcal{F})) \implies H^{p+q}(X_{\text{et}}, \mathcal{F}).$$

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We apply this to $\bar{X} = X \otimes_k k^{\text{sep}}$ and $\bar{U} = U \otimes_k k^{\text{sep}}$. Since we know that $H^r(G, -) = 0$ for $r \geq 2$ always in our situation, we have a spectral sequence with only two non-zero *columns*. This will degenerate immediately, so we obtain exact sequences

$$0 \to H_c^{r-1}(\bar{U}, \mathcal{F})_\Gamma \to H_c^r(U, \mathcal{F}) \to H_c^r(\bar{U}, \mathcal{F})^\Gamma \to 0;$$
$$0 \to H^{r-1}(\bar{U}, \mathcal{F})_\Gamma \to H^r(U, \mathcal{F}) \to H^r(\bar{U}, \mathcal{F})^\Gamma \to 0.$$

For $r = 3$, the top one gives $H_c^3(U, \mu_n) \cong H_c^2(\bar{U}, \mu_n)_\Gamma \cong \mathbb{Z}/n\mathbb{Z}$.

Results for other fields (4/4)

From Poincaré duality we obtain

$$H_c^r(\bar{U}, \mathcal{F}) \cong \mathrm{Ext}_U^{2-r}(\mathcal{F}, \mu_n)^\vee \cong H^{2-r}(\bar{U}, \check{\mathcal{F}}(1))^\vee$$

and so by the previous lemma,

$$H_c^r(\bar{U}, \mathcal{F})^\Gamma \cong (H^{2-r}(U, \check{\mathcal{F}}(1))_\Gamma)^\vee.$$

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Now, we can construct a commutative diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & H_c^{r-1}(\bar{U}, \mathcal{F})_\Gamma \arrow[r] \arrow[d, "\text{iso}"] & H_c^r(U, \mathcal{F}) \arrow[r] \arrow[d] & H_c^r(\bar{U}, \mathcal{F})^\Gamma \arrow[r] \arrow[d, "\text{iso}"] & 0 \\
0 \arrow[r] & (H^{3-r}(\bar{U}, \check{\mathcal{F}}(1))^\Gamma)^\vee \arrow[r] & H^{3-r}(U, \check{\mathcal{F}}(1))^\vee \arrow[r] & (H^{2-r}(\bar{U}, \check{\mathcal{F}}(1))_\Gamma)^\vee \arrow[r] & 0
\end{tikzcd}$$

Since the outer two maps are isomorphisms, the five lemma tells us that the middle one is also. This is precisely the duality statement that we sought to prove.

Artin-Verdier duality

Theorem

Let K be a totally imaginary number field with ring of integers $\mathcal{O}_K$ and let $X = \mathrm{Spec}(\mathcal{O}_K)$. Then there is a canonical isomorphism $\eta : H^3(X, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}$. In addition, for any constructible étale abelian sheaf $\mathcal{F}$ on X, and for any r, the pairings

$$H^r(X, \mathcal{F}) \times \mathrm{Ext}_X^{3-r}(\mathcal{F}, \mathbb{G}_m) \to H^3(X, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$$

are all nondegenerate pairings of finite abelian groups.

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Remark 2: Morally, this theorem shows that on the étale site, the ring of integers in a number field behaves like a 3-dimensional mathematical object. According to Milne, the proof of this is a *refinement* of the proof of Poincaré duality, but using a lot of class field theory.

We will not discuss this difficult proof.

Explicit description of the cohomology ring (1/4)

Recall the definition of an *abelian variety*, i.e. a projective algebraic variety that is also an algebraic group, e.g. an elliptic curve. More generally, for a lattice $L \subset \mathbb{C}^n$ of maximal rank, the variety $\mathbb{C}^n/L$ is an abelian complex variety, but not all of these are *algebraic*. In general, if A is an abelian variety, we define the *dual* abelian variety by $\hat{A} = \mathrm{Pic}^0(A)$.

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$$n_A^* \mathrm{div}(f) = n_A^*(nD) = n \cdot n_A^* D = \mathrm{div}(g^n).$$

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Since $f \circ n_A$ and g^n have the same divisor, so $g = \lambda \cdot f \circ n_A$. But then

$$g(x)^n = \lambda f(n \cdot x) = \lambda f(n \cdot (x + a)) = g(x + a)^n$$

for all $a \in A_n$. Hence $g(x)/g(x + a)$ must be a constant n-th root of unity. This defines the pairing $e_n : A_n \times \hat{A}_n \to \mu_n$.

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The pairing e_n defined above enjoys many beautiful properties. It is bilinear in both variables, it is alternating and thus skew-symmetric, it is nondegenerate, and most importantly, it is Galois invariant.

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Let X be a smooth projective curve over a field $k = \bar{k}$. Then there exists a moduli space of all degree 0 line bundles on X, sometimes denoted $\mathrm{Pic}^0(X)$. It is also the connected component of the identity of the Picard group $\mathrm{Pic}(X)$ of X, and is also called the *Jacobian* of X, denoted $J(X)$.

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Remark: What does Poincaré duality say for $\mathcal{F} = \mu_n$? Well, in that case $\check{\mathcal{F}}(1) = \mathrm{Hom}(\mathcal{F}, \mu_n) = \mu_n$, so

$$\mathrm{Ext}_U^{2-r}(\mu_n, \mu_n) = H^{2-r}(X, \mu_n).$$

Hence for $r = 1$, Poincaré duality reduces to the ordinary cup product

$$H^1(X, \mu_n) \times H^1(X, \mu_n) \to H^2(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

Explicit description of the cohomology ring (3/4)

Recall that $H^1(X, \mathbb{G}_m) \cong \mathrm{Pic}(X)$. Using this, we can show that

$$H^1(X, \mu_n) \cong J(X)_n.$$

Namely, use the Kummer sequence $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$ and then:

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 & 0 \arrow[r] & \mathbb{Z} \arrow[r, "n"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
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The cup product pairing

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Remark 1: The proof can be found in SGA $4\frac{1}{2}$: Cohomologie Etale, Section 5.3. It is technical, so we will not discuss it.

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The cup product pairing

$$H^1(X, \mu_n) \times H^1(X, \mu_n) \to H^2(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

under the isomorphism from before agrees with the e_n-pairing

$$e_n : J(X)_n \times \widehat{J(X)}_n \to \mathbb{Z}/n\mathbb{Z}.$$

Remark 1: The proof can be found in SGA $4\frac{1}{2}$: Cohomologie Etale, Section 5.3. It is technical, so we will not discuss it.
Remark 2: Since the e_n-pairings are compatible for varying n, we can combine some of them into an inverse system to obtain maps

$$T_\ell(X) \times T_\ell(\hat{X}) \to \varprojlim \mu_{\ell^m}$$

from the *Tate module* of X for each prime $\ell \in k^*$. This pairing is even $\mathbb{Z}_\ell$-bilinear and nondegenerate, and thus shows how one can naturally start to consider ℓ-adic cohomology. This is the realm of for instance the *Lefschetz trace formula*... but more on that later.

Fin

Thanks for listening!

Figure: Étale is a mountain of Savoie and Haute-Savoie, France. It lies in the Aravis Range of the French Prealps and reaches 2,484 metres above sea level.