Schlessinger pro-representability and hull existence criteria

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15th of October, 2020

The setup

- Let Art_k be the category of local artinian k-algebras with residue field k.
- Let $F : \mathrm{Art}_k \to \mathbf{Set}$ be a functor and let R be a complete local k-algebra with maximal ideal $\mathfrak{m}$.
- Denote $h_R : \mathrm{Art}_k \to \mathbf{Set}$ be given by $h_R(A) = \mathrm{Hom}_k(R, A)$.
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There is a bijection between $\varprojlim F(R/\mathfrak{m}^n)$ and natural transformations $h_R \to F$.

- We denote $\xi \in \varprojlim F(R/\mathfrak{m}^n)$ by $\{\xi_n\}$ where $\xi_n \in F(R/\mathfrak{m}^n)$.
- Such an element can have different properties.

Different kinds of $\{\xi_n\}$

Definition

Let F be as before. We say a pair (R, ξ) is

- a *versal family* for F if $F(k)$ has only one element and for every surjection $B \to A$, the map $\mathrm{Hom}(R, B) \to \mathrm{Hom}(R, A) \times_{F(A)} F(B)$ is also surjective.

$$\begin{array}{ccc} \mathrm{Hom}(R,B) & \longrightarrow & \mathrm{Hom}(R,A) \\ \downarrow & & \downarrow \\ \theta \in F(B) & \longrightarrow & F(A) \ni \eta \end{array}$$

  This means that given any map $R \to A$ inducing $\eta \in F(A)$ and any element $\theta \in F(B)$ mapping to η, we can lift the map $R \to A$ to a map $R \to B$ inducing θ.

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- a *miniversal* family if in addition $\mathrm{Hom}(R, k[t]/t^2) \to F(k[t]/t^2)$ is bijective. We also say that F has a *pro-representable hull*.
- a *universal family* if $h_R \to F$ is an isomorphism, so in particular, F is pro-representable by R.

Properties of versal families

Lemma

Suppose that $F : \mathrm{Art}_k \to \mathbf{Set}$ has a *versal* family. Then

- For any $A \in \mathrm{Art}_k$, the map $\mathrm{Hom}(R, A) \to F(A)$ is surjective.
- for any pair of morphisms $A', A'' \to A$, the natural map $F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$ is surjective.

Here the fibered product $A' \times_A A'' = \{(a', a'') \mid \text{equal images in A}\}$ is the same as in **Set**.

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Proof.

- Taking $A \twoheadrightarrow k$, since $F(A)$ surjects onto $F(k)$, any element from $F(A)$ can be lifted to $\mathrm{Hom}(R, A)$. In fact, it is equivalent to $F(k)$ consisting of exactly one element.
- Let $\eta' \in F(A')$ and $\eta'' \in F(A'')$ mapping to the same element in $F(A)$. By surjectivity, we can find maps $R \to A'$ and $R \to A''$ lifting η' and η'' mapping to the same $R \to A$. These lift to a map $R \to A' \times_A A''$, inducing an element in $F(A' \times_A A'')$.

Properties of miniversal families (1)

Lemma

In particular, if the family is miniversal, the map
$F(A \times_k k[t]/t^2) \to F(A) \times_{F(k)} F(k[t]/t^2)$ is even bijective.

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Proof.

First note that $A \times_k k[t]/t^2 = A[t]/t^2$ surjects onto A. Let θ_1, θ_2 map to a pair $\eta \in F(A)$ and $\xi \in F(k[t]/t^2)$ and choose $u : R \to A$ inducing η.

$$\begin{array}{ccccc}
\mathrm{Hom}(R, k[t]/t^2) & \longleftarrow & v_i \in \mathrm{Hom}(R, A[t]/t^2) & \longrightarrow & \mathrm{Hom}(R, A) \ni u \\
\downarrow{\scriptstyle \text{iso}} & & \downarrow & & \downarrow \\
\xi \in F(k[t]/t^2) & \longleftarrow & \theta_i \in F(A[t]/t^2) & \longrightarrow & F(A) \ni \eta
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Then we can find lifts v_1, v_2 inducing θ_1, θ_2 respectively.

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Then we can find lifts v_1, v_2 inducing θ_1, θ_2 respectively. But since their restrictions to $k[t]/t^2$ both induce ξ, by miniversality, these restrictions agree. Since they also agree on A, as they both induce $u : R \to A$, we conclude $v_1 = v_2$ and so $\theta_1 = \theta_2$. □

Properties of miniversal families (2)

By a *small extension* in Art_k we mean a surjection $A' \to A$ such that the kernel I is a 1-dimensional k-vector space. Note that then $I \cdot \mathfrak{m}_{A'} = 0$ and $I^2 = 0$. We have the following corollaries:

- The *tangent space* $t_F := F(k[t]/t^2)$ has the natural structure of a finite dimensional k-vector space.

Proof: Since $t_R := \mathrm{Hom}(R, k[t]/t^2) \to t_F$ is bijective, is has a k-vector space structure. We can also define it intrinsically: the map $k[t]/t^2 \to k[t]/t^2$ sending $t \mapsto \lambda t$ induces scalar multiplication $t_F \to t_F$. Also, the obvious map $k[t_1]/t_1^2 \times_k k[t_2]/t_2^2 \to k[t]/t^2$ induces addition.

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- For any small extension $p : A' \to A$ and any $\eta \in F(A)$, the vector space t_F acts transitively on $(Fp)^{-1}(\eta) \subset F(A')$.

Proof: It is an easy exercise in commutative algebra to show that $A' \times_A A' \cong A' \times_k k[I]$. So we find

$$F(A') \times_{F(k)} t_F \cong F(A' \times_k k[t]/t^2) = F(A' \times_A A') \twoheadrightarrow F(A') \times_{F(A)} F(A').$$

So for any $\eta' \in F(A')$ over $\eta \in F(A)$, we obtain the group action by

$$\{\eta'\} \times t_F \twoheadrightarrow \{\eta'\} \times (Fp)^{-1}(\eta).$$

Schlessinger's criterion

Theorem

A functor $F : \mathrm{Art}_k \to \mathbf{Set}$ has a miniversal family if and only if

- $F(k)$ has just one element;
- for any small extension $A'' \to A$, the map $F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$ is surjective;
- the above map is bijective for $A'' = k[t]/t^2$ and $A = k$;
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Furthermore, F is pro-representable if and only if in addition for every small extension $p : A' \to A$ and every $\eta \in F(A)$ for which $(Fp)^{-1}(\eta) \neq \varnothing$, the group action of t_F is bijective.

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We have already seen that all the above properties are satisfied for miniversal families. The additional property for pro-representable F follows immediately from the construction of the action. Thus it suffices to show the converse.

Construction of (R, ξ) (1)

Let $t_1, \ldots, t_r$ be a basis for the dual vector space t_F^* and define the ring $S = k[\![t_1, \ldots, t_r]\!]$ with maximal ideal $\mathfrak{m}$. Set

$$R_1 = S/\mathfrak{m}^2 \cong k[t_1]/t_1^2 \times_k \ldots \times_k k[t_r]/t_r^2.$$

We then see that

$$t_{R_1} := \mathrm{Hom}(R_1, k[t]/t^2) \cong t_F^{**} = t_F.$$

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We also know that

$$\begin{aligned} F(R_1) &= F(k[t_1]/t_1^2 \times_k \ldots \times_k k[t_r]/t_r^2) \\ &= F(k[t_1]/t_1^2) \times \ldots \times F(k[t_r]/t_r^2) \\ &= t_F^r \\ &\cong t_F \otimes_k t_F^*, \end{aligned}$$

where the last isomorphism is naturally induced by our choice of basis. The identity element $\mathrm{id} = \sum t_i \otimes t_i^*$ thus induces some $\xi_1 \in F(R_1)$, inducing a bijection between t_{R_1} and t_F.

Construction of (R, ξ) (2)

Now suppose we have constructed (R_i, ξ_i) for $1 \leq i \leq q$ where $R_i = S/J_i$ with $\mathfrak{m}^{i+1} \subset J_i \subset J_{i-1}$ and $\xi_i \in F(R_i)$ compatible elements, meaning that $R_i \to R_{i-1}$ sends ξ_i to ξ_{i-1}. We then define R_{q+1} as S/J_{q+1} where J_{q+1} is the smallest ideal J of S satisfying

- $\mathfrak{m}J_q \subset J \subset J_q$;
- $\xi_q \in F(R_q)$ lifts to an element $\xi' \in F(S/J)$.

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To see that such a smallest ideal exists, recall from commutative algebra that given any two ideals $J, K \subset J_q$, we may enlarge either J or K to ensure that $J + K = J_q$ without changing $J \cap K$. But then

$$S/(J \cap K) = (S/J) \times_{S/J_q} (S/K),$$

and so

$$F(S/J \cap K) \twoheadrightarrow F(S/J) \times_{F(S/J_q)} F(S/K)$$

ensures the existence of a lift of ξ_q to $F(S/J \cap K)$.

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ensures the existence of a lift of ξ_q to $F(S/J \cap K)$.
In conclusion, we define $R = \varprojlim R_j$ and $\xi = \{\xi_n\}$. We claim that this is the desired miniversal family.

Verification of miniversality (1)

By construction we have $t_R \cong t_{R_1} \cong t_F$, so it suffices to show that for any small extension $p : A' \to A$, given any map $R \to A$ inducing $\eta \in F(A)$ and any element $\theta \in F(A')$ mapping to η, we can lift the map $R \to A$ to a map $R \to A'$ inducing θ.

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We claim that it suffices to lift any $R \to A$ to a map $v : R \to A'$, for v would induce an element $\theta' \in F(A')$ inducing η, and so $\theta, \theta' \in (Fp)^{-1}(\eta)$. Now t_F acts *transitively* on this preimage, so $\theta = t * \theta'$ for some $t \in t_F = t_R$. Similarly, t_R acts translitively on the maps in $\mathrm{Hom}(R, A')$ restricting to the given map $R \to A$. Then $t * v$ induces $t * \theta' = \theta$, as desired.

Verification of miniversailty (2)

Let us be given a map $R \to A$ and a small extension $A' \to A$. Because A is Artin, we can factor $R \to R_q \to A$ for some q. Also, because S is just a power series ring, we may lift this factorisation to form the following diagram.

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\begin{array}{ccccc}
S & \xrightarrow{w} & R_q \times_A A' & \longrightarrow & A' \\
\downarrow & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\
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Note that p' is a small extension, and if it has a section, we are done, so assume it does not. Note that the image of w surjects onto R_q, and so its kernel cannot be trivial (for then $\mathrm{im}(w) \cong R_q$, giving a section). So $0 \neq \ker(\mathrm{im}(w) \to R_q) \subset \ker(p')$. But p' is a small extension, so we must have equality. Therefore w is surjective.

Verification of miniversality (3)

$$\begin{array}{ccccc} S & \xrightarrow{w} & R_q \times_A A' & \longrightarrow & A' \\ \downarrow & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ R & \longrightarrow & R_q & \longrightarrow & A \end{array}$$

Let $J = \ker(w)$. Then clearly $J \subset J_q$, but also $\mathfrak{m}J_q \subset J$ because $m \cdot \ker(p') = 0$. Because

$$F(S/J) = F(R_q \times_A A') \twoheadrightarrow F(R_q) \times_{F(A)} \times F(A'),$$

we can lift $\xi_q \in F(R_q)$ to some $\xi' \in F(S/J)$ inducing ξ_q and $\theta \in F(A')$.

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$$w : S \to R_{q+1} \to R_q \times_A A' \to A',$$

so the canonical map $R \to R_{q+1}$ induces a lift $R \to A'$, as desired.

Verification of pro-representability

Recall, now we assume that for every small extension $p : A' \to A$ and every $\eta \in F(A)$ for which $(Fp)^{-1}(\eta) \neq \varnothing$, the group action of t_F is bijective. We must show that $h_R(A) \to F(A)$ is bijective for all A. Note that if $h_R(A) \to F(A)$ is bijective and $A' \to A$ is a small extension such that

$$\mathrm{Hom}(R, A') \to \mathrm{Hom}(R, A) \times_{F(A)} F(A')$$

is bijective, then $h_R(A') \to F(A')$ must be bijective too.

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Thus we will prove the unique lifting property. Let a map $R \to A$ be given inducing $\eta \in F(A)$. If $(Fp)^{-1}(\eta) = \varnothing$, there is nothing to show. Otherwise, t_F acts bijectively on $(Fp)^{-1}(\eta)$. Since h_R is pro-represented by itself, t_R acts bijectively on the set of morphisms $R \to A'$ inducting the given map $R \to A$. But since $t_R = t_F$, we obtain the bijection.

Pro-representability of deformation functors

- Recall that a deformation of a scheme X_0/k is a pair (X, i) where X is a scheme flat over $\mathrm{Spec}(A)$ with $A \in \mathrm{Art}_k$ and $i : X_0 \to X$ is a closed immersion with the property that the induced map $X_0 \to X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k)$ is an isomorphism.
- Define the *deformation functor* Def_X of X_0 by associating to $A \in \mathrm{Art}_k$ the set of deformations (X, i) up to equivalence, where we say $(X_1, i_1) \sim (X_2, i_2)$ when there exists an isomorphism $\phi : X_1 \to X_2$ such that $\phi \circ i_1 = i_2$.

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Theorem: The deformation functor Def_X of a smooth variety X over a field k is pro-representable if and only if for any small extension $B \to A$ and $\mathcal{X}_A \in F_X(A)$ and $\mathcal{X}_B \in F_X(B)$, any automorphism of $\mathcal{X}_A$ extends to an automorphism of $\mathcal{X}_B$.

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- Recall that a deformation of a scheme X_0/k is a pair (X, i) where X is a scheme flat over $\mathrm{Spec}(A)$ with $A \in \mathrm{Art}_k$ and $i : X_0 \to X$ is a closed immersion with the property that the induced map $X_0 \to X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(k)$ is an isomorphism.
- Define the *deformation functor* Def_X of X_0 by associating to $A \in \mathrm{Art}_k$ the set of deformations (X, i) up to equivalence, where we say $(X_1, i_1) \sim (X_2, i_2)$ when there exists an isomorphism $\phi : X_1 \to X_2$ such that $\phi \circ i_1 = i_2$.

Theorem: The deformation functor Def_X of a smooth variety X over a field k is pro-representable if and only if for any small extension $B \to A$ and $\mathcal{X}_A \in F_X(A)$ and $\mathcal{X}_B \in F_X(B)$, any automorphism of $\mathcal{X}_A$ extends to an automorphism of $\mathcal{X}_B$.

Counter-example: Let X be the blowup of many points lying on a line ℓ in $\mathbb{P}^2_k$. Then the automorphisms of X can be viewed as automorphisms of $\mathbb{P}^2_k$ that preserve the line ℓ. If we deform X to make the points in general position, there are no automorphisms fixing all the points anymore. Hence Def_X is not pro-representable by the previous theorem.