Gross and Zagier's Fascinating Formula

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Elliptic Curves

Definition

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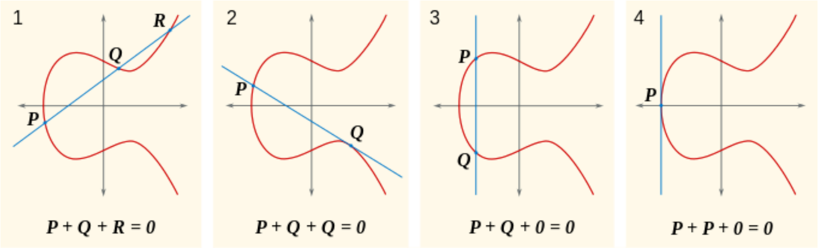

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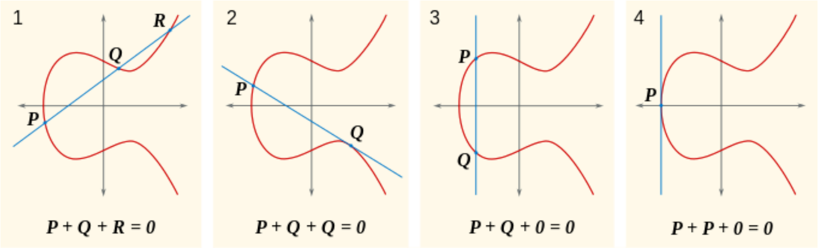

- Define the discriminant $\Delta = -16(4a^3 + 27b^2)$.
- An elliptic curve must be *non-singular*: $\Delta \neq 0$.

The j-invariant

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The j-*invariant* of an elliptic curve $E : y^2 = x^3 + ax + b$ is defined as

$$j(E) = -1728\frac{(4a)^3}{\Delta} = 1728\frac{4a^3}{4a^3 + 27b^2}.$$

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Theorem

It holds that $E_1 \cong E_2$ over $\overline{k}$ if and only if $j(E_1) = j(E_2)$.

Endomorphisms

Observation

We have infinitely many rational maps $E \to E$: namely, for $n \in \mathbb{Z}$, consider $[n] : E \to E$ sending $P \in E$ to $n \cdot P \in E$.

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For $E_1 : y^2 = x^3 + 1$, the map $[2] : E \to E$ is explicitly given by

$$P = (x, y) \mapsto 2 \cdot P = \left(\frac{9x^4}{4y^2} - 2x, \frac{3x^2}{2y} \left(3x - \frac{9x^4}{4y^2} \right) - y \right).$$

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Question

Are there more such *endomorphisms* $E \to E$?
What are all possible structures of the ring $\mathrm{End}(E)$?

Complex Multiplication

Theorem

Let E/k be an elliptic curve with $\mathrm{char}(k) = 0$. Then:

- Either $\mathrm{End}(E) = \mathbb{Z}$;
- Or $\mathrm{End}(E)$ is isomorphic to an order in an imaginary quadratic number field. We say that E *has CM*.

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For $E_1 : y^2 = x^3 + x$, we have a map $[i] : E \to E$ given by

$$P = (x, y) \mapsto (-x, iy) \implies \text{End}(E) \cong \mathbb{Z}[i].$$

Gross and Zagier's discovery (1/2)

Most curves do *not* have CM. Examples:

$$E_3 : y^2 = x^3 - 2835x - 71442 \quad \text{has CM by} \quad \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right];$$

$$E_4 : y^2 = x^3 - 9504x + 365904 \quad \text{has CM by} \quad \mathbb{Z}\left[\frac{1+\sqrt{-11}}{2}\right];$$

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However, the following is striking:

$$j(E_3) - j(E_4) = 7 \cdot 13 \cdot 17 \cdot 19;$$
$$j(E_3) - j(E_5) = 3^7 \cdot 13 \cdot 31;$$
$$j(E_4) - j(E_5) = 2^{16} \cdot 13.$$

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More examples:

$$E_6 : y^2 = x^3 - 13760x + 621264 \quad \text{has CM by} \quad \mathbb{Z}\left[\frac{1+\sqrt{-43}}{2}\right];$$

$$E_7 : y^2 = x^3 - 117920x + 15585808 \quad \text{has CM by} \quad \mathbb{Z}\left[\frac{1+\sqrt{-67}}{2}\right];$$

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The following is even more striking:

$$j(E_6) - j(E_7) = 2^{15} \cdot 3^6 \cdot 5^3 \cdot 7^2;$$
$$j(E_6) - j(E_8) = 2^{19} \cdot 3^6 \cdot 5^3 \cdot 7^3 \cdot 37 \cdot 433;$$
$$j(E_7) - j(E_8) = 2^{15} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 139 \cdot 331.$$

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x	± 1	± 3	± 5	± 7	± 9	± 11
$(D - x^2)/4$	$3 \cdot 11$	31	3^3	$3 \cdot 7$	13	3

Three colours of primes

Suppose we have E_1 and E_2 with CM by $\mathbb{Z}\left[\frac{1+\sqrt{D_1}}{2}\right]$ and $\mathbb{Z}\left[\frac{1+\sqrt{D_2}}{2}\right]$.

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Example: let $D_1 = -7$ and $D_2 = -43$. Then:

x	± 1	± 3	± 5	± 7	± 9	± 11	± 13	± 15	± 17
$\frac{D-x^2}{4}$	$3 \cdot 5^2$	73	$3 \cdot 23$	$3^2 \cdot 7$	$5 \cdot 11$	$3^2 \cdot 5$	$3 \cdot 11$	19	3

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Recipe

- For each integer $(D - x^2)/4 > 0$, colour its primes.

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$p^{k/2}$	3	73	3^2	7	5^2	5	3^2	19	3

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Final step: multiplying all these $p^{k/2}$ together gives the right answer!

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Then indeed,

$$j(E_3) - j(E_6) = 2^{18} 3^3 5^3 - 3^3 5^3 = 884732625$$

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More generally, when $j(E) \notin \mathbb{Z}$ but in a bigger number field, then it still holds for the *absolute norm* of the difference between j-values.

What is the j-function really?

Theorem

The $\mathbb{C}$-points of an elliptic curve are given by $E(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ for some $\tau \in \mathcal{H}$. Now CM-curves correspond to actual CM-points:

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$$j : \mathrm{SL}_2(\mathbb{Z}) \setminus \mathcal{H} \to \mathbb{C}$$

where $\mathrm{SL}_2(\mathbb{Z}) = \{A \in M_2(\mathbb{Z}) \mid \det(A) = 1\}$ acts on $\mathcal{H}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

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The quotient $SL_2(\mathbb{Z}) \setminus \mathcal{H}$ is called a *modular curve*.

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this is known as a Shimura curve. Sometimes, there exists a generator J of the function field. This choice is not unique, but the *cross-ratio* is:

$$\frac{J(x) - J(z)}{J(x) - J(w)} \frac{J(y) - J(z)}{J(y) - J(w)}.$$

My research

Theorem, D. (2023)

For *CM-points* τ_1, τ_1', τ_2, τ_2', the algebraic norm of the cross-ratio

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Thank you for your attention!