A p-adic analogue of a formula by Gross and Zagier

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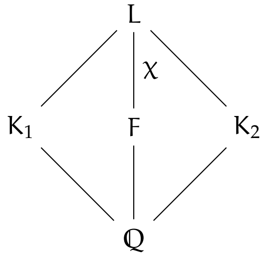
Setting up

Let $D_1, D_2 < 0$ be coprime discriminants and write $D = D_1 D_2$. Set

$$K_1 = \mathbb{Q}(\sqrt{D_1}), \quad K_2 = \mathbb{Q}(\sqrt{D_2}),$$
$$F = \mathbb{Q}(\sqrt{D}), \quad L = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}).$$



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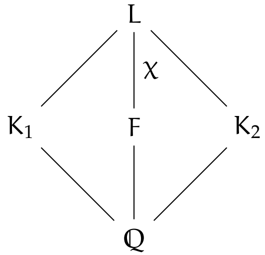

Let χ be the genus character of L/F: if $\mathfrak{l} \subset \mathcal{O}_F$ is prime, then

$$\chi(\mathfrak{l}) = \begin{cases} 1 & \text{if } \mathfrak{l} \text{ splits in } L/F; \\ -1 & \text{if } \mathfrak{l} \text{ is inert in } L/F. \end{cases}$$

The formula

Let $I \subset \mathcal{O}_F$ be an ideal. Define

$$\rho(I) = \#\{J \subset \mathcal{O}_L \mid \mathrm{Nm}_F^L(J) = I\};$$

$$\mathrm{sp}(I) = \begin{cases} \mathfrak{l} & \text{if } \mathfrak{l} \text{ is } \textit{unique} \text{ with } \chi(\mathfrak{l}) = -1 \text{ and } \nu_{\mathfrak{l}}(I) \text{ odd;} \\ 1 & \text{otherwise.} \end{cases}$$

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For $i \in \{1, 2\}$, let E_i be an elliptic curve with CM by $\mathcal{O}_i \subset K_i$. Then by CM theory, $j(E_i) \in H_i$, with H_i the Hilbert class field of K_i. Say $D_i \notin \{-3, -4\}$.

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Theorem (Gross-Zagier, 1984)

Setting $\alpha = \nu\sqrt{D}$ and $\mathcal{D}_F = (\sqrt{D})$, the following equality holds:

$$\log \mathrm{Nm}_{\mathbb{Q}}^{H_1 H_2}\big(j(E_1) - j(E_2)\big) = \sum_{\substack{\nu \in \mathcal{D}_F^{-1,+} \\ \mathrm{tr}(\nu) = 1}} \rho(\mathrm{sp}(\alpha)\alpha)(\nu_{\mathrm{sp}(\alpha)}(\alpha) + 1) \log \mathrm{Nm}(\mathrm{sp}(\alpha)).$$

Example

Let $D_1 = -7$ and $D_2 = -19$. Then

$$E_7 : y^2 + xy = x^3 - x^2 - 2x - 1, \quad j(E_7) = -3^3 5^3;$$
$$E_{19} : y^2 + y = x^3 - 38x + 90, \quad j(E_{19}) = -2^{15} 3^3.$$

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If $\nu \in \mathcal{D}_F^{-1,+}$ and $\mathrm{tr}(\nu) = 1$, then

$$\alpha = \nu\sqrt{D} = \frac{x + \sqrt{D}}{2}, \quad \text{where } x^2 < D = 133 \text{ and } x \text{ is odd.}$$

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x	± 1	± 3	± 5	± 7	± 9	± 11
$(D - x^2)/4$	$3 \cdot 11$	31	3^3	$3 \cdot 7$	13	3
$\mathrm{sp}(\alpha)$	3	31	3	3	13	3
$(\nu_{\mathrm{sp}(\alpha)}(\alpha) + 1)/2$	1	1	2	1	1	1
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Zagier's proof

First step is to rewrite the theorem as

$$\log \mathrm{Nm}_{\mathbb{Q}}^{H_1 H_2}\big(j(E_1) - j(E_2)\big) = \sum_{\substack{\nu \in \mathcal{D}_F^{-1,+} \\ \mathrm{tr}(\nu)=1}} \sum_{I \mid (\nu)\mathcal{D}_F} \chi(I) \log \mathrm{Nm}(I).$$

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This looks like the diagonal restriction of a weight k Hilbert Eisenstein series:

$$E_{1,\chi}(z, z, k) = \mathrm{const} + \sum_{\substack{\nu \in \mathcal{D}_F^{-1,+} \\ \mathrm{tr}(\nu)=n}} \Bigg(\sum_{I | (\nu)\mathcal{D}_F} \chi(I) \mathrm{Nm}(I)^{k-1} \Bigg) q^n.$$

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"One line proof:"

$$e^{\mathrm{hol}} \left(\frac{d}{ds} E_{1,\chi}(z,z,s)\Big|_{s=0} \right) \in M_2(\mathrm{SL}_2(\mathbb{Z})) = \{0\}.$$

Its Fourier coefficients involve two terms, one for each side $\implies$ equal.

Shimura curves

Let $B/\mathbb{Q}$ be an indefinite quaternion algebra with $\mathrm{disc}(B) = N > 0$. Let $R \subset B$ be a maximal order and let $R_1^\times = \{b \in R^\times \mid \mathrm{Nm}(b) = 1\}$. Choose a splitting $R_1^\times \to M_2(\mathbb{R})$ to form

$$X_N(\mathbb{C}) = R_1^\times \setminus \mathcal{H};$$

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Let $\tau_1, \tau_2 \in \mathcal{H}$ be CM points: fixed points in $\mathcal{H}$ of embeddings $\mathcal{O}_i \to R$. These exist when p and q are inert in both K_i. We want to study

$$\mathrm{Nm}\big(j_N(\tau_1) - j_N(\tau_2)\big).$$

These numbers are algebraic by Shimura reciprocity.

Čerednik-Drinfeld

The curve X_N has bad reduction at p and admits the following p-adic uniformisation. Let $B_q/\mathbb{Q}$ denote the definite quaternion algebra with $\mathrm{disc}(B) = -q$. Let $R_q \subset B_q$ be a maximal order and consider the group

$$\Gamma_q^p = R_q[1/p]_1^\times = \{b \in R_q[1/p]^\times \mid \mathrm{Nm}(b) = 1\}.$$

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where $\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$ is the *p-adic upper half plane*.

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Question

Which functions on $\Gamma_q^p \setminus \mathcal{H}_p$ correspond to j_N on the other side?

Theta functions

Let $w_1, w_2 \in \mathcal{H}_p$. Then consider the expression

$$\Theta(w_1, w_2; z) = \prod_{\gamma \in \Gamma_q^p} \frac{z - \gamma w_1}{z - \gamma w_2}.$$

If $N \in \{6, 10, 22\}$, this expression descends to a rigid analytic meromorphic function on $\Gamma_q^p \setminus \mathcal{H}_p$ with divisor $[w_1] - [w_2]$.

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$$\Theta(w_1, w_2; z) = c(w_1, w_2) \cdot \frac{j_N(z) - j_N(w_1)}{j_N(z) - j_N(w_2)}, \text{ for some } c(w_1, w_2) \in \mathbb{C}_p.$$

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Now choose $w_1 = \tau_1$ and $w_2 = \tau_1'$; its Galois conjugate. Because we don't know $c(\tau_1, \tau_1')$, we opt to study instead

$$\frac{j_N(\tau_2) - j_N(\tau_1)}{j_N(\tau_2) - j_N(\tau_1')} \frac{j_N(\tau_2') - j_N(\tau_1')}{j_N(\tau_2') - j_N(\tau_1)} = \prod_{\gamma \in \Gamma_q^p} \frac{\tau_2 - \gamma\tau_1}{\tau_2 - \gamma\tau_1'} \frac{\tau_2' - \gamma\tau_1'}{\tau_2' - \gamma\tau_1}.$$

The conjecture

One can p-adically approximate the quantity

$$J_q^p(\tau_1, \tau_2) := \prod_{\gamma \in \Gamma_q^p} \frac{\tau_2 - \gamma\tau_1}{\tau_2 - \gamma\tau_1'} \frac{\tau_2' - \gamma\tau_1'}{\tau_2' - \gamma\tau_1}$$

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There are four ideals $\mathfrak{a}$ of norm $N = pq$ in $\mathcal{O}_F$; they come in two $\mathrm{Gal}(F/\mathbb{Q})$ orbits. Assign one orbit $\delta(\mathfrak{a}) = +1$, the other $\delta(\mathfrak{a}) = -1$.

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Conjecture (Giampietro, Darmon)

The expression

$$\log \mathrm{Nm}_{\mathbb{Q}}^{H_1H_2} J_q^p(\tau_1,\tau_2)$$

is up to sign given by the formula

$$\sum_{\mathrm{Nm}(\mathfrak{a})=N} \delta(\mathfrak{a}) \sum_{\substack{\nu\in\mathcal{D}_F^{-1,+}\\ \mathrm{tr}(\nu)=1}} \rho(\mathrm{sp}(\alpha\mathfrak{a}^{-1})\alpha\mathfrak{a}^{-1})(v_{\mathrm{sp}(\alpha\mathfrak{a}^{-1})}(\alpha\mathfrak{a}^{-1})+1)\log \mathrm{Nm}\,(\mathrm{sp}(\alpha\mathfrak{a}^{-1})).$$

Idea of the proof (1/2)

The prime p splits in $F/\mathbb{Q}$ and since $\chi(\mathfrak{p}_i) = -1$, these primes are thus *regular*, i.e. the Hecke polynomial of $E_{1,\chi}$ has the two distinct roots ± 1.
We consider a p-stabilisation of the Hilbert Eisenstein series $E_{1,\chi}$:

$$E_{1,\chi}^{(p)} = (1 - V_{\mathfrak{p}_1})(1 + V_{\mathfrak{p}_2})E_{1,\chi}.$$

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We look for a cuspidal family of Hilbert modular forms through $E_{1,\chi}^{(p)}$; then

$$e^{\mathrm{ord}} \left(\frac{d}{ds} E_{1,\chi}^{(p)}(z, z, s) \Big|_{s=0} \right) \in S_2(\Gamma_0(N)) \approx 0.$$

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Complication: the $s = 0$-specialisation does not vanish.

Solution: use a family in the *anti-parallel* weight direction, i.e. $(1 - s, 1 + s)$. Then the diagonal restriction is of constant weight 2.

Idea of the proof (2/2)

Suppose the family has Fourier coefficients $a_\nu(s)$ for $\nu \gg 0$. Then

$$a_n\left(\frac{d}{ds}E_{1,\chi}^{(p)}(z,z,s)\Big|_{s=0}\right) = \sum_{\substack{\nu\in(\mathcal{D}_F^{-1}\mathfrak{q})^+\\ \mathrm{tr}(\nu)=n}} \frac{d}{ds}a_\nu(s)\big|_{s=0}.$$

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Compute the ordinary projection:

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Writing down families of p-adic cusp forms is hard. Idea:

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- Compute its Fourier coefficients $a_\nu(\epsilon)$ for $\nu \gg 0$ and take the ϵ-part.

The deformation

Let $\tilde{\rho}$ be the deformation of $\rho = \mathbb{1} \oplus \chi$ to $\mathrm{GL}_2(\mathbb{Q}_p[\epsilon])$ given by

$$\tilde{\rho}(\tau) = \begin{pmatrix} 1 + \phi_p^{\mathrm{cyc}}\epsilon & 0 \\ 0 & \chi - \chi\phi_p^{\mathrm{cyc}}\epsilon \end{pmatrix},$$

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We recover φ from

$$\varphi(T_{\mathfrak{l}}) = \mathrm{tr}(\tilde{\rho}(\mathrm{Frob}_{\mathfrak{l}})) = \begin{cases} 2 & \text{if } \chi(\mathfrak{l}) = 1; \\ 2\log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon & \text{if } \chi(\mathfrak{l}) = -1. \end{cases}$$

Further, note that

$$\varphi(\langle \mathfrak{l} \rangle \mathrm{Nm}(\mathfrak{l})) = \det(\tilde{\rho}(\mathrm{Frob}_{\mathfrak{l}})) = \chi(\mathfrak{l}).$$

Connection with the formula

We still have the essential recursion relation

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The factor $1 - \chi(I^n) = 0$ unless I is a special prime of J, and if J/I^n still has another special prime, $\rho(J/I^n) = 0$. So we must have exactly one special prime.

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The expression

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is up to sign given by the formula

$$\sum_{\mathrm{Nm}(\mathfrak{a})=N} \delta(\mathfrak{a}) \sum_{\substack{\nu\in\mathcal{D}_F^{-1,+}\\ \mathrm{tr}(\nu)=1}} \rho(\mathrm{sp}(\alpha\mathfrak{a}^{-1})\alpha\mathfrak{a}^{-1})(\nu_{\mathrm{sp}(\alpha\mathfrak{a}^{-1})}(\alpha\mathfrak{a}^{-1})+1)\log\mathrm{Nm}\,(\mathrm{sp}(\alpha\mathfrak{a}^{-1})).$$

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CM-values of p-adic Θ-functions: https://arxiv.org/abs/2309.17251.