The L-group and local Langlands parameters

Mike Daas

7th of April, 2021

Universiteit Leiden

Recalling some definitions

- An *affine group scheme* over a field k is a functor

$$k\text{-}\mathbf{Alg} \to \mathbf{Grp}$$

that is representable by some k-algebra. An *affine algebraic group* over k is an affine group scheme of finite type over k.

Recalling some definitions

- An *affine group scheme* over a field k is a functor

$$k\text{-}\mathbf{Alg} \to \mathbf{Grp}$$

  that is representable by some k-algebra. An *affine algebraic group* over k is an affine group scheme of finite type over k.
- A matrix M is said to be *unipotent* if $M - \mathrm{id}$ is nilpotent. An element g of an algebraic group G is called *unipotent* if $\phi(g)$ is so for some, and thus for any, faithful (i.e. injective) representation $\phi : G \to \mathrm{GL}_n$. The unipotent radical $R_u(G)$ of G is the maximal connected normal subgroup of G that consists of unipotent elements. We say an algebraic group is *reductive* if $R_u(G) = \{1\}$.

Recalling some definitions

- An *affine group scheme* over a field k is a functor

$$k\text{-}\mathbf{Alg} \to \mathbf{Grp}$$

  that is representable by some k-algebra. An *affine algebraic group* over k is an affine group scheme of finite type over k.
- A matrix M is said to be *unipotent* if $M - \mathrm{id}$ is nilpotent. An element g of an algebraic group G is called *unipotent* if $\phi(g)$ is so for some, and thus for any, faithful (i.e. injective) representation $\phi : G \to \mathrm{GL}_n$. The unipotent radical $R_u(G)$ of G is the maximal connected normal subgroup of G that consists of unipotent elements. We say an algebraic group is *reductive* if $R_u(G) = \{1\}$.
- An *algebraic torus* is an algebraic group T such that $T_{k^{\mathrm{sep}}} \cong \mathbb{G}_m^n$ for some n, which is called the *rank* of the torus. A *character* of an algebraic group G is an element of $X^*(G) = \mathrm{Hom}(G, \mathbb{G}_m)$. A *one-parameter subgroup* is an element from $X_*(G) = \mathrm{Hom}(\mathbb{G}_m, G)$.

Recalling more definitions

Let V be a finite dimensional $\mathbb{R}$-vector space, and Φ a subset of V. Then (Φ, V) is called a *root system* if:

- Φ is finite, does not contain 0, and spans V;
- for each $\alpha \in \Phi$ there exists a reflection operator s_α that interchanges α and $-\alpha$ and maps Φ to Φ;
- for each $\beta \in \Phi$, the vector $s_\alpha(\beta) - \beta$ is an integer multiple of α.

Recalling more definitions

Let V be a finite dimensional $\mathbb{R}$-vector space, and Φ a subset of V. Then (Φ, V) is called a *root system* if:

- Φ is finite, does not contain 0, and spans V;
- for each $\alpha \in \Phi$ there exists a reflection operator s_α that interchanges α and $-\alpha$ and maps Φ to Φ;
- for each $\beta \in \Phi$, the vector $s_\alpha(\beta) - \beta$ is an integer multiple of α.

There exists a pairing $(\ ,\ ) : V \times V \to \mathbb{C}$ such that all reflections are orthogonal transformations. If $\alpha \in \Phi$, we can find a unique α^v such that

$$\langle -, \alpha^v \rangle := \alpha^v(-) = 2(-, \alpha)/(\alpha, \alpha) \in \mathbb{Z}.$$

If $\Phi^v = \{\alpha^v | \alpha \in \Phi\}$ and $V^v = \langle \Phi^v \rangle \otimes_{\mathbb{Z}} \mathbb{R}$, then (Φ^v, V^v) is called the *dual root system*.

Recalling more definitions

Let V be a finite dimensional $\mathbb{R}$-vector space, and Φ a subset of V. Then (Φ, V) is called a *root system* if:

- Φ is finite, does not contain 0, and spans V;
- for each $\alpha \in \Phi$ there exists a reflection operator s_α that interchanges α and $-\alpha$ and maps Φ to Φ;
- for each $\beta \in \Phi$, the vector $s_\alpha(\beta) - \beta$ is an integer multiple of α.

There exists a pairing $(\ ,\ ) : V \times V \to \mathbb{C}$ such that all reflections are orthogonal transformations. If $\alpha \in \Phi$, we can find a unique α^v such that

$$\langle -, \alpha^v \rangle := \alpha^v(-) = 2(-, \alpha)/(\alpha, \alpha) \in \mathbb{Z}.$$

If $\Phi^v = \{\alpha^v | \alpha \in \Phi\}$ and $V^v = \langle \Phi^v \rangle \otimes_{\mathbb{Z}} \mathbb{R}$, then (Φ^v, V^v) is called the *dual root system*.

A *root datum* is a quadruple (X, Y, Φ, Φ^v) where X, Y are free abelian groups with a perfect pairing $\langle\ ,\ \rangle : X \times Y \to \mathbb{Z}$ and where $\Phi \subset X$ and $\Phi^v \subset Y$ are finite subsets such that $\Phi \ni \alpha \iff \alpha^v \in \Phi^v$. In addition, $\langle \alpha, \alpha^v \rangle = 2$ and for each $\alpha \in \Phi$, the reflection $s_\alpha(x) = x - \langle x, \alpha^v \rangle \alpha$ satisfies $s_\alpha(\Phi) = \Phi$ and the group $\langle s_\alpha | \alpha \in \Phi \rangle$ is finite.

Combining these concepts

Let G be a connected reductive group over a perfect field k with maximal torus T. Recall the *adjoint representation* $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ induced by the conjugation action of G onto itself. For any character $\alpha \in X^*(T)$, denote

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{Ad}(t)X = \alpha(t)X \text{ for all } t \in T(k)\}.$$

Combining these concepts

Let G be a connected reductive group over a perfect field k with maximal torus T. Recall the *adjoint representation* $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ induced by the conjugation action of G onto itself. For any character $\alpha \in X^*(T)$, denote

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{Ad}(t)X = \alpha(t)X \text{ for all } t \in T(k)\}.$$

If $\mathfrak{g}_\alpha \neq 0$, it must be 1-dimensional, and they are called the *root spaces*. We have

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha,$$

where $\mathfrak{t} = \mathrm{Lie}(T)$ and so the set of α for which $\mathfrak{g} \neq 0$ is finite and denoted by $\Phi(G, T)$. It turns out that $(X^*(T), X_*(T), \Phi, \Phi^v)$ is a root datum with the natural pairing between $X^*(T)$ and $X_*(T)$.

The dual group

Recall that a root datum (X, Y, Φ, Φ^v) is said to be *reduced* if $\alpha \in \Phi$ implies that $2\alpha \notin \Phi$. We have the following theorem:

Theorem

If k is algebraically closed, the association $G \to (X^*(T), X_*(T), \Phi, \Phi^v)$ determines a bijection between isomorphism classes of connected reductive groups and isomorphism classes of reduced root data.

The dual group

Recall that a root datum (X, Y, Φ, Φ^v) is said to be *reduced* if $\alpha \in \Phi$ implies that $2\alpha \notin \Phi$. We have the following theorem:

Theorem

If k is algebraically closed, the association $G \to (X^*(T), X_*(T), \Phi, \Phi^v)$ determines a bijection between isomorphism classes of connected reductive groups and isomorphism classes of reduced root data.

A trivial remark on one side, now leads to an interesting concept on the other. Namely, if (X, Y, Φ, Φ^v) is a root datum, then so is (Y, X, Φ^v, Φ). According to the theorem, we may thus associate to G its so-called *complex dual* $\hat{G}$ that has the corresponding dual root system.

The dual group

Recall that a root datum (X, Y, Φ, Φ^v) is said to be *reduced* if $\alpha \in \Phi$ implies that $2\alpha \notin \Phi$. We have the following theorem:

Theorem

If k is algebraically closed, the association $G \to (X^*(T), X_*(T), \Phi, \Phi^v)$ determines a bijection between isomorphism classes of connected reductive groups and isomorphism classes of reduced root data.

A trivial remark on one side, now leads to an interesting concept on the other. Namely, if (X, Y, Φ, Φ^v) is a root datum, then so is (Y, X, Φ^v, Φ). According to the theorem, we may thus associate to G its so-called *complex dual* $\hat{G}$ that has the corresponding dual root system.

Examples:

- The group GL_n is its own dual.
- If $G = \mathrm{SL}_n$, then $\hat{G} = \mathrm{PGL}_n(\mathbb{C})$.
- If $G = \mathrm{Sp}_{2n}$, then $\hat{G} = \mathrm{SO}_{2n+1}(\mathbb{C})$.

More on root data

Definition

Let Φ be a root system. A set of *positive roots* $\Phi^+ \subset \Phi$ is a subset such that for all $\alpha \in \Phi$, precisely one of α and $-\alpha$ is in Φ^+ and for any $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta \in \Phi$, we require that $\alpha + \beta \in \Phi^+$. We say a root in Φ^+ is called *simple* if it is not the sum of two positive roots.

More on root data

Definition

Let Φ be a root system. A set of *positive roots* $\Phi^+ \subset \Phi$ is a subset such that for all $\alpha \in \Phi$, precisely one of α and $-\alpha$ is in Φ^+ and for any $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta \in \Phi$, we require that $\alpha + \beta \in \Phi^+$. We say a root in Φ^+ is called *simple* if it is not the sum of two positive roots.

Note that $\Phi^{+,v}$ is a set of positive roots in Φ^v. If Δ is a maximal set of simple roots in Φ, then so is Δ^v in Φ^v. One can show that Δ and Δ^v determine Φ and Φ^v respectively.

More on root data

Definition

Let Φ be a root system. A set of *positive roots* $\Phi^+ \subset \Phi$ is a subset such that for all $\alpha \in \Phi$, precisely one of α and $-\alpha$ is in Φ^+ and for any $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta \in \Phi$, we require that $\alpha + \beta \in \Phi^+$. We say a root in Φ^+ is called *simple* if it is not the sum of two positive roots.

Note that $\Phi^{+,v}$ is a set of positive roots in Φ^v. If Δ is a maximal set of simple roots in Φ, then so is Δ^v in Φ^v. One can show that Δ and Δ^v determine Φ and Φ^v respectively.

Definition

We call a tuple (X, Y, Δ, Δ^v) a *based root datum* if $(X^*(T), X_*(T), \Phi, \Phi^v)$ is a root datum and Δ, Δ^v are maximal sets of simple roots as above.

Recall that a Borel subgroup of G is a maximal connected solvable subgroup.

Pinnings

Lemma

On the other side of the correspondence, choosing Δ amounts to choosing a Borel subgroup $T \subset B \subset G$.

Proof: (sketch) By definition of $\mathfrak{g}_\alpha$, we have the map

$$\exp_\alpha : \mathfrak{g}_\alpha \to G(\mathbb{C})$$

that satisfies $g \exp_\alpha(x) g^{-1} = \exp(\alpha(g)x)$. Its image U_α naturally has $\mathfrak{g}_\alpha$ as Lie-algebra. Then one can show that the Borel subgroups of G are precisely those of the form

$$\langle T, \{U_\alpha\}_{\alpha \in \Delta} \rangle.$$

This would prove the claim. $\square$

Pinnings

Lemma

On the other side of the correspondence, choosing Δ amounts to choosing a Borel subgroup $T \subset B \subset G$.

Proof: (sketch) By definition of $\mathfrak{g}_\alpha$, we have the map

$$\exp_\alpha : \mathfrak{g}_\alpha \to G(\mathbb{C})$$

that satisfies $g \exp_\alpha(x) g^{-1} = \exp(\alpha(g)x)$. Its image U_α naturally has $\mathfrak{g}_\alpha$ as Lie-algebra. Then one can show that the Borel subgroups of G are precisely those of the form

$$\langle T, \{U_\alpha\}_{\alpha \in \Delta} \rangle.$$

This would prove the claim. □

This proof naturally leads to the following definition.

Definition

A *pinning* of G is a tuple $(B, T, \{u_\alpha\}_{\alpha \in \Delta})$ where $u_\alpha \in U_\alpha - 1$ for all $\alpha \in \Delta$, where Δ corresponds to the Borel subgroup B.

A bit of group theory

Proposition

Let $T \subset B \subset G$ be as before. Then

- all Borel subgroups in G are conjugate;
- all maximal tori inside B are conjugate by an element of B;
- B is its own normaliser inside G;
- T is its own normaliser inside B.

A bit of group theory

Proposition

Let $T \subset B \subset G$ be as before. Then

- all Borel subgroups in G are conjugate;
- all maximal tori inside B are conjugate by an element of B;
- B is its own normaliser inside G;
- T is its own normaliser inside B.

Let $f \in \mathrm{Aut}(G)$ and let denote $c_g \in \mathrm{Aut}(G)$ conjugation by g. Then $f(B)$ is also a Borel subgroup, so $f(B) = gBg^{-1}$ for some $g \in G$, so $(c_g \circ f)(B) = B$. Now $(c_g \circ f)(T) = bTb^{-1}$ for some $b \in B$, so $(c_{gb} \circ f)(T) = T$ and this also fixes B. This element $gb \in G$ is unique up to an element from the normaliser of T inside B; that is, from T.

A bit of group theory

Proposition

Let $T \subset B \subset G$ be as before. Then

- all Borel subgroups in G are conjugate;
- all maximal tori inside B are conjugate by an element of B;
- B is its own normaliser inside G;
- T is its own normaliser inside B.

Let $f \in \mathrm{Aut}(G)$ and let denote $c_g \in \mathrm{Aut}(G)$ conjugation by g. Then $f(B)$ is also a Borel subgroup, so $f(B) = gBg^{-1}$ for some $g \in G$, so $(c_g \circ f)(B) = B$. Now $(c_g \circ f)(T) = bTb^{-1}$ for some $b \in B$, so $(c_{gb} \circ f)(T) = T$ and this also fixes B. This element $gb \in G$ is unique up to an element from the normaliser of T inside B; that is, from T. Hence $c_{gb} \circ f$ preserves both T and B, and thus acts on $(X^*(T), X_*(T), \Delta, \Delta^v)$. Since elements from T act trivially on the root system, this defines a homomorphism

$$\mathrm{Aut}(G) \to \mathrm{Aut}(X^*(T), X_*(T), \Delta, \Delta^v).$$

It is non-trivial, but it can be shown, that its kernel is precisely $\mathrm{Inn}(G)$.

A note about F-forms

Note that elements from the Galois group need not induce *algebraic* automorphisms, i.e. if G is a matrix group, conjugating all entries by some $\sigma \in \mathrm{Gal}(\bar{F}/F)$, an action commonly denoted τ_0, is not an algebraic automorphism of G. However, if τ is another action, then we do have a map

$$\alpha : \mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}_{\mathrm{alg}}(G) : \gamma \mapsto \tau(\gamma) \circ \tau_0(\gamma)^{-1}.$$

A note about F-forms

Note that elements from the Galois group need not induce *algebraic* automorphisms, i.e. if G is a matrix group, conjugating all entries by some $\sigma \in \mathrm{Gal}(\bar{F}/F)$, an action commonly denoted τ_0, is not an algebraic automorphism of G. However, if τ is another action, then we do have a map

$$\alpha : \mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}_{\mathrm{alg}}(G) : \gamma \mapsto \tau(\gamma) \circ \tau_0(\gamma)^{-1}.$$

However, this is not generally a homomorphism, but it is a 1-cocycle. We call this a *rational structure* and they are characterised by the first group cohomology of $\mathrm{Gal}(\bar{F}/F)$ with values in $\mathrm{Aut}_{\mathrm{alg}}(G)$. The invariants for any given τ are denoted $G_\tau(F)$ and two such groups define the same dual groups if and only if the τ's differ by an *inner* twist, i.e. something from $H^1(\mathrm{Gal}(\bar{F}/F), \mathrm{Int}_{\mathrm{alg}}(G))$. For more details, ask Eric.

A note about F-forms

Note that elements from the Galois group need not induce *algebraic* automorphisms, i.e. if G is a matrix group, conjugating all entries by some $\sigma \in \mathrm{Gal}(\bar{F}/F)$, an action commonly denoted τ_0, is not an algebraic automorphism of G. However, if τ is another action, then we do have a map

$$\alpha : \mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}_{\mathrm{alg}}(G) : \gamma \mapsto \tau(\gamma) \circ \tau_0(\gamma)^{-1}.$$

However, this is not generally a homomorphism, but it is a 1-cocycle. We call this a *rational structure* and they are characterised by the first group cohomology of $\mathrm{Gal}(\bar{F}/F)$ with values in $\mathrm{Aut}_{\mathrm{alg}}(G)$. The invariants for any given τ are denoted $G_\tau(F)$ and two such groups define the same dual groups if and only if the τ's differ by an *inner* twist, i.e. something from $H^1(\mathrm{Gal}(\bar{F}/F), \mathrm{Int}_{\mathrm{alg}}(G))$. For more details, ask Eric.

Examples:

- Consider GL_1 over $\mathbb{R}$. We can make the non-trivial automorphism act by $z \mapsto \bar{z}$ or by $z \mapsto 1/z$. Combining these gives the algebraic action $z \mapsto 1/\bar{z}$. Its fixed points are those on the unit circle.
- Consider SU_3 over $\mathbb{R}$. We now have an algebraic automorphism $A \mapsto (A^T)^{-1}$. Unlike in the SL_2-case, now SU_3 is not an inner twist from SL_3, by considering the action on the Dynkin diagram.

The Langland dual group

Suppose that all groups are defined over a local or global field F. Let $\hat{T} \subset \hat{B} \subset \hat{G}$ be the complex dual group associated with the based root datum $(X_*(T), X^*(T), \Delta^v, \Delta)$. From the previous slide, we obtain the exact sequence

$$1 \to \mathrm{Inn}(G) \to \mathrm{Aut}(G) \to \mathrm{Aut}(X^*(T), X_*(T), \Delta, \Delta^v) \to 1$$

The Langland dual group

Suppose that all groups are defined over a local or global field F. Let $\hat{T} \subset \hat{B} \subset \hat{G}$ be the complex dual group associated with the based root datum $(X_*(T), X^*(T), \Delta^v, \Delta)$. From the previous slide, we obtain the exact sequence

$$1 \to \mathrm{Inn}(G) \to \mathrm{Aut}(G) \to \mathrm{Aut}(X^*(T), X_*(T), \Delta, \Delta^v) \to 1$$

Proving surjectivity shows that it is split; choosing a pinning $\{u_\alpha\}_{\alpha \in \Delta}$ we see that any automorphism of the root datum induces an action on the u_α. These elements and their *negatives* generate G, and one can show that this defines a unique automorphism of G.

The Langland dual group

Suppose that all groups are defined over a local or global field F. Let $\hat{T} \subset \hat{B} \subset \hat{G}$ be the complex dual group associated with the based root datum $(X_*(T), X^*(T), \Delta^v, \Delta)$. From the previous slide, we obtain the exact sequence

$$1 \to \mathrm{Inn}(G) \to \mathrm{Aut}(G) \to \mathrm{Aut}(X^*(T), X_*(T), \Delta, \Delta^v) \to 1$$

Proving surjectivity shows that it is split; choosing a pinning $\{u_\alpha\}_{\alpha \in \Delta}$ we see that any automorphism of the root datum induces an action on the u_α. These elements and their *negatives* generate G, and one can show that this defines a unique automorphism of G.
Since T and B are defined over F, the group $\mathrm{Gal}(\bar{F}/F)$ acts on the root datum of G and so also on the root datum of $\hat{G}$. The above section for $\hat{G}$ gives us an induced map $\mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}(\hat{G})$. We use it to define the *Langlands dual group*

$${}^L G = \hat{G}(\mathbb{C}) \rtimes \mathrm{Gal}(\bar{F}/F).$$

This is simply a direct product if G is split, i.e. if $T \cong \mathbb{G}_m^n$ over F, because then the Galois action will be trivial.

Class field theory

Recall the following facts about **local** class field theory:

- For any finite field extension E/F, there is an isomorphism
$$\theta_{E/F} : F^\times/N_{E/F}E^\times \xrightarrow{\sim} \mathrm{Gal}(E/F)^{\mathrm{ab}}.$$
- These maps are compatible and define a homomorphism
$$\theta_F : F^\times \to \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$$
called the *local reciprocity map*.
- If $H \subset G$ and $[G : H] < \infty$, we have a *transfer map* $G \to H^{\mathrm{ab}}$.

Class field theory

Recall the following facts about **local** class field theory:

- For any finite field extension E/F, there is an isomorphism
$$\theta_{E/F} : F^\times / N_{E/F} E^\times \xrightarrow{\sim} \mathrm{Gal}(E/F)^{\mathrm{ab}}.$$
- These maps are compatible and define a homomorphism
$$\theta_F : F^\times \to \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}}$$
called the *local reciprocity map*.
- If $H \subset G$ and $[G : H] < \infty$, we have a *transfer map* $G \to H^{\mathrm{ab}}$.

Also recall the following facts about **global** class field theory:

- There exists a surjective continuous homomorphism
$$\mathbb{A}_F^\times \to \mathrm{Gal}(\bar{F}/F)^{\mathrm{ab}} : \ s \mapsto [s, F]$$
called the *global reciprocity map*.
- It has the property that for $s \in \mathbb{A}_F^\times$ whose ideal is coprime to all the ramified places in a certain finite extension E/F,
$$[s, F]|_E = ((s), E/F),$$
where the latter denotes the *Artin symbol*, extended multiplicatively from $(\mathfrak{p}, E/F) = \mathrm{Frob}_\mathfrak{p}$ for all primes $\mathfrak{p}$ in E.

The Weil group

Let F be a local or a global field. Then a *Weil group* for F is a topological group W_F along with a continuous homomorphism $\phi : W_F \to \mathrm{Gal}(\bar{F}/F)$ with dense image, and for each finite field extension E/F, the group $W_E = \phi^{-1}(\mathrm{Gal}(\bar{E}/E))$ admits an isomorphism $r_E : C_E \to W_E^{\mathrm{ab}}$, where

$$C_E = \begin{cases} E^\times & \text{if } F \text{ is local;} \\ E^\times \setminus \mathbb{A}_E^\times & \text{if } F \text{ is global.} \end{cases}$$

The Weil group

Let F be a local or a global field. Then a *Weil group* for F is a topological group W_F along with a continuous homomorphism $\phi : W_F \to \mathrm{Gal}(\bar{F}/F)$ with dense image, and for each finite field extension E/F, the group $W_E = \phi^{-1}(\mathrm{Gal}(\bar{E}/E))$ admits an isomorphism $r_E : C_E \to W_E^{\mathrm{ab}}$, where

$$C_E = \begin{cases} E^\times & \text{if } F \text{ is local;} \\ E^\times \setminus \mathbb{A}_E^\times & \text{if } F \text{ is global.} \end{cases}$$

In addition, these groups and maps must satisfy that

$$C_E \xrightarrow{r_E} W_E^{\mathrm{ab}} \xrightarrow{\phi} \mathrm{Gal}(\bar{E}/E)^{\mathrm{ab}}$$

is the reciprocity map from class field theory, that

$$W_F = \varprojlim W_F/\overline{W_E^{\mathrm{ab}}},$$

The Weil group

Let F be a local or a global field. Then a *Weil group* for F is a topological group W_F along with a continuous homomorphism $\phi : W_F \to \mathrm{Gal}(\bar{F}/F)$ with dense image, and for each finite field extension E/F, the group $W_E = \phi^{-1}(\mathrm{Gal}(\bar{E}/E))$ admits an isomorphism $r_E : C_E \to W_E^{\mathrm{ab}}$, where

$$C_E = \begin{cases} E^\times & \text{if } F \text{ is local;} \\ E^\times \backslash \mathbb{A}_E^\times & \text{if } F \text{ is global.} \end{cases}$$

In addition, these groups and maps must satisfy that

$$C_E \xrightarrow{r_E} W_E^{\mathrm{ab}} \xrightarrow{\phi} \mathrm{Gal}(\bar{E}/E)^{\mathrm{ab}}$$

is the reciprocity map from class field theory, that

$$W_F = \varprojlim W_F/\overline{W_E^{\mathrm{ab}}},$$

and that for all $w \in W_F$, $\sigma = \phi(w)$ and $E'/E/F$ these commute:

$$\begin{array}{ccc} C_E & \xrightarrow{r_E} & W_E^{\mathrm{ab}} \\ \sigma \downarrow & & \downarrow \\ C_{E^\sigma} & \xrightarrow{r_{E^\sigma}} & W_{E^\sigma}^{\mathrm{ab}} \end{array} \qquad \begin{array}{ccc} C_E & \xrightarrow{r_E} & W_E^{\mathrm{ab}} \\ \text{incl.} \downarrow & & \downarrow \text{transfer} \\ C_{E'} & \xrightarrow{r_{E'}} & W_{E'}^{\mathrm{ab}} \end{array}$$

Examples of the Weil group

- Let $F = \mathbb{C}$. Then the map $\phi : W_{\mathbb{C}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{C}) = \{1\}$ must be trivial. Since $\mathbb{C}/\mathbb{C}$ is the only finite field extension, considering $\mathbb{C}$ as a local field (as it is complete), we must have an isomorphism $r_{\mathbb{C}} : \mathbb{C}^{\times} \to W_{\mathbb{C}}$. All other conditions are now trivially satisfied.

Examples of the Weil group

- Let $F = \mathbb{C}$. Then the map $\phi : W_{\mathbb{C}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{C}) = \{1\}$ must be trivial. Since $\mathbb{C}/\mathbb{C}$ is the only finite field extension, considering $\mathbb{C}$ as a local field (as it is complete), we must have an isomorphism $r_{\mathbb{C}} : \mathbb{C}^{\times} \to W_{\mathbb{C}}$. All other conditions are now trivially satisfied.
- Let $F = \mathbb{R}$. Then $\phi : W_{\mathbb{R}} \to \{1, \sigma\}$ where σ denotes complex conjugation. Also, $r_{\mathbb{C}} : \mathbb{C}^{\times} \to W_{\mathbb{C}}$ must be an isomorphism, and be the kernel of ϕ. One can show that now $W_{\mathbb{R}} = \mathbb{C}^{\times} \cup j\mathbb{C}^{\times}$ where $j^2 = -1$ and $jzj^{-1} = \sigma(z)$ for all $z \in \mathbb{C}^{\times}$.

Examples of the Weil group

- Let $F = \mathbb{C}$. Then the map $\phi : W_{\mathbb{C}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{C}) = \{1\}$ must be trivial. Since $\mathbb{C}/\mathbb{C}$ is the only finite field extension, considering $\mathbb{C}$ as a local field (as it is complete), we must have an isomorphism $r_{\mathbb{C}} : \mathbb{C}^\times \to W_{\mathbb{C}}$. All other conditions are now trivially satisfied.
- Let $F = \mathbb{R}$. Then $\phi : W_{\mathbb{R}} \to \{1, \sigma\}$ where σ denotes complex conjugation. Also, $r_{\mathbb{C}} : \mathbb{C}^\times \to W_{\mathbb{C}}$ must be an isomorphism, and be the kernel of ϕ. One can show that now $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$ where $j^2 = -1$ and $jzj^{-1} = \sigma(z)$ for all $z \in \mathbb{C}^\times$.
- Let F be a non-archimedean local field with finite residue field k. Then recall that we have a maximal unramified extension F^{unr} satisfying

$$1 \to I_F \to \mathrm{Gal}(\bar{F}/F) \to \mathrm{Gal}(F^{\mathrm{unr}}/F) \cong \mathrm{Gal}(\bar{k}/k) \cong \hat{\mathbb{Z}} \to 1,$$

where I_F denotes the inertia group. In this case, one can show that W_F is the dense subgroup of $\mathrm{Gal}(\bar{F}/F)$ that maps to $\mathbb{Z} \subset \hat{\mathbb{Z}}$.

Examples of the Weil group

- Let $F = \mathbb{C}$. Then the map $\phi : W_{\mathbb{C}} \to \mathsf{Gal}(\mathbb{C}/\mathbb{C}) = \{1\}$ must be trivial. Since $\mathbb{C}/\mathbb{C}$ is the only finite field extension, considering $\mathbb{C}$ as a local field (as it is complete), we must have an isomorphism $r_{\mathbb{C}} : \mathbb{C}^{\times} \to W_{\mathbb{C}}$. All other conditions are now trivially satisfied.
- Let $F = \mathbb{R}$. Then $\phi : W_{\mathbb{R}} \to \{1, \sigma\}$ where σ denotes complex conjugation. Also, $r_{\mathbb{C}} : \mathbb{C}^{\times} \to W_{\mathbb{C}}$ must be an isomorphism, and be the kernel of ϕ. One can show that now $W_{\mathbb{R}} = \mathbb{C}^{\times} \cup j\mathbb{C}^{\times}$ where $j^2 = -1$ and $jzj^{-1} = \sigma(z)$ for all $z \in \mathbb{C}^{\times}$.
- Let F be a non-archimedian local field with finite residue field k. Then recall that we have a maximal unramified extension F^{unr} satisfying

  $$1 \to I_F \to \mathsf{Gal}(\bar{F}/F) \to \mathsf{Gal}(F^{\mathsf{unr}}/F) \cong \mathsf{Gal}(\bar{k}/k) \cong \hat{\mathbb{Z}} \to 1,$$

  where I_F denotes the inertia group. In this case, one can show that W_F is the dense subgroup of $\mathsf{Gal}(\bar{F}/F)$ that maps to $\mathbb{Z} \subset \hat{\mathbb{Z}}$.

For number fields, it is highly non-trivial to show that the Weil group exists and there is no easy description of it. More on this in a few weeks.

The Weil-Deligne group

We first record the following theorem about the Weil group.

Langlands for GL_1

There is a bijection between isomorphism classes of irreducible automorphic representations of $\mathrm{GL}_1(\mathbb{A}_F)$ and continuous representations $W_F \to \mathrm{GL}_1(\mathbb{C})$.

The proof follows from identifying a representation with its associated character of $F^\times \setminus \mathbb{A}_F^\times$, which is isomorphic to W_F^{ab} by definition of W_F.

The Weil-Deligne group

We first record the following theorem about the Weil group.

Langlands for GL_1

There is a bijection between isomorphism classes of irreducible automorphic representations of $\mathrm{GL}_1(\mathbb{A}_F)$ and continuous representations $W_F \to \mathrm{GL}_1(\mathbb{C})$.

The proof follows from identifying a representation with its associated character of $F^\times \setminus \mathbb{A}_F^\times$, which is isomorphic to W_F^{ab} by definition of W_F.

Definition

The *Weil-Deligne group* for a local field F is defined as

$$W_F' = W_F \times \mathrm{SL}_2(\mathbb{C}).$$

It is interesting to remark that the correct analogue of this for global fields, the so-called Langlands group, is currently still only hypothetical.

Representations of W_F'

Recall the Weil group for a local field. It fit naturally into an exact sequence of the form

$$1 \to I_F \to W_F \to \mathbb{Z} \to 1,$$

and so we may write $W_F \cong I_F \rtimes \langle \mathrm{Fr} \rangle$. Let G be a reductive group over $\mathbb{C}$. Recall that some $g \in G$ is said to be semi-simple if $\phi(g)$ is for some, and thus for any, faithful representation $\phi : G \to \mathrm{GL}_n$.

Representations of W_F'

Recall the Weil group for a local field. It fit naturally into an exact sequence of the form

$$1 \to I_F \to W_F \to \mathbb{Z} \to 1,$$

and so we may write $W_F \cong I_F \rtimes \langle \mathrm{Fr} \rangle$. Let G be a reductive group over $\mathbb{C}$. Recall that some $g \in G$ is said to be semi-simple if $\phi(g)$ is for some, and thus for any, faithful representation $\phi : G \to \mathrm{GL}_n$.

Definition

A *representation* / *admissible homomorphism* of W_F' into $G(\mathbb{C})$ is a homomorphism

$$\phi : W_F' \to G(\mathbb{C})$$

such that ϕ is trivial on an open subgroup of I_F, such that $\phi(\mathrm{Fr})$ is semi-simple in G, and $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is induced by a morphism of algebraic groups $\mathrm{SL}_2 \to G$.

Representations of W_F'

Recall the Weil group for a local field. It fit naturally into an exact sequence of the form

$$1 \to I_F \to W_F \to \mathbb{Z} \to 1,$$

and so we may write $W_F \cong I_F \rtimes \langle \mathrm{Fr} \rangle$. Let G be a reductive group over $\mathbb{C}$. Recall that some $g \in G$ is said to be semi-simple if $\phi(g)$ is for some, and thus for any, faithful representation $\phi : G \to \mathrm{GL}_n$.

Definition

A *representation / admissible homomorphism* of W_F' into $G(\mathbb{C})$ is a homomorphism

$$\phi : W_F' \to G(\mathbb{C})$$

such that ϕ is trivial on an open subgroup of I_F, such that $\phi(\mathrm{Fr})$ is semi-simple in G, and $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is induced by a morphism of algebraic groups $\mathrm{SL}_2 \to G$.

Local Langlands for GL_n

There is a bijection between representations of W_F' into $\mathrm{GL}_n(\mathbb{C})$ and irreducible admissible representations of $\mathrm{GL}_n(F)$.

L-parameters

Recall that the Weil group W_F comes with a map $W_F \to \mathrm{Gal}(\bar{F}/F)$ and that the Langlands dual of a reductive group G, denoted ${}^L G$, is defined by taking a suitable semi-direct product of $\mathrm{Gal}(\bar{F}/F)$ and $\hat{G}$.

Definition

An *L-parameter* is a representation of W_F' into ${}^L G$ that commutes with the projections to $\mathrm{Gal}(\bar{F}/F)$. We say that two L-parameters are equivalent if they differ only by conjugation by some element of $\hat{G}(\mathbb{C})$.

L-parameters

Recall that the Weil group W_F comes with a map $W_F \to \mathrm{Gal}(\bar{F}/F)$ and that the Langlands dual of a reductive group G, denoted ${}^L G$, is defined by taking a suitable semi-direct product of $\mathrm{Gal}(\bar{F}/F)$ and $\hat{G}$.

Definition

An *L-parameter* is a representation of W_F' into ${}^L G$ that commutes with the projections to $\mathrm{Gal}(\bar{F}/F)$. We say that two L-parameters are equivalent if they differ only by conjugation by some element of $\hat{G}(\mathbb{C})$.

Vague local Langlands corresponcence conjectures

There is a bijection between *L-packets* of admissible $G(F)$-representations and equivalence classes of L-parameters satisfying certain conditions. Given a map ${}^L H \to {}^L G$ that commutes with the projections to $\mathrm{Gal}(\bar{F}/F)$, there is a corresponding transfer of L-packets compatible with the natural transfer of L-parameters.

L-parameters

Recall that the Weil group W_F comes with a map $W_F \to \mathrm{Gal}(\bar{F}/F)$ and that the Langlands dual of a reductive group G, denoted ${}^L G$, is defined by taking a suitable semi-direct product of $\mathrm{Gal}(\bar{F}/F)$ and $\hat{G}$.

Definition

An *L-parameter* is a representation of W_F' into ${}^L G$ that commutes with the projections to $\mathrm{Gal}(\bar{F}/F)$. We say that two L-parameters are equivalent if they differ only by conjugation by some element of $\hat{G}(\mathbb{C})$.

Vague local Langlands corresponcence conjectures

There is a bijection between *L-packets* of admissible $G(F)$-representations and equivalence classes of L-parameters satisfying certain conditions. Given a map ${}^L H \to {}^L G$ that commutes with the projections to $\mathrm{Gal}(\bar{F}/F)$, there is a corresponding transfer of L-packets compatible with the natural transfer of L-parameters.

What are L-packets? In the case of GL_n, they are just singletons. In general? Hard to say. Everything is still only conjectural. Many definitions of L-packets are ad-hoc, and assume the conjectures to define the L-packets instead...

L-functions

Why do we study all of this? Turns out, these representations are closely related to *L-functions*. Let v be a place of a number field F and suppose that we have an L-parameter $\phi : W'_{F_v} \to {}^L G$ and a representation $r : {}^L G_{F_v} \to \mathrm{GL}(V)$ for some F-vector space V. Let q_v denote the cardinality of the residue field of F. Define the *local factor* by

$$L(s, r \circ \phi) = \det(1 - r(\mathrm{Fr}_v) q_v^{-s} | V^{I_{F_v}})^{-1}.$$

L-functions

Why do we study all of this? Turns out, these representations are closely related to *L-functions*. Let v be a place of a number field F and suppose that we have an L-parameter $\phi : W'_{F_v} \to {}^L G$ and a representation $r : {}^L G_{F_v} \to \mathrm{GL}(V)$ for some F-vector space V. Let q_v denote the cardinality of the residue field of F. Define the *local factor* by

$$L(s, r \circ \phi) = \det(1 - r(\mathrm{Fr}_v) q_v^{-s} | V^{I_{F_v}})^{-1}.$$

The L-function is defined by multiplying all the local factors at all the places together. It is conjecturally meromorphic and satisfies a functional equation relating the value at s to the value at $1 - s$ of the *adjoint* L-function, i.e. the one defined by the dual representation. Many properties of the representation can be recovered from the L-function. For example, the image of the representation has finite centraliser if and only if the dual L-function is regular at $s = 0$.
More on L-functions will be treated in the near future.

Fin

Thanks for listening!