The L-group and local Langlands parameters

Mike Daas

7th of April, 2021
Recalling some definitions

- An affine group scheme over a field $k$ is a functor

$$k\text{-Alg} \to \text{Grp}$$

that is representable by some $k$-algebra. An affine algebraic group over $k$ is an affine group scheme of finite type over $k$.

- A matrix $M$ is said to be unipotent if $M - \text{Id}$ is nilpotent. An element $g$ of an algebraic group $G$ is called unipotent if $\phi(g)$ is so for some, and thus for any, faithful (i.e. injective) representation $\phi: G \to \text{GL}_n$. The unipotent radical $R_u(G)$ of $G$ is the maximal connected normal subgroup of $G$ that consists of unipotent elements. We say an algebraic group is reductive if $R_u(G) = \{1\}$.

- An algebraic torus is an algebraic group $T$ such that $T_{k\text{-sep}} \cong G^m$ for some $m$, which is called the rank of the torus. A character of an algebraic group $G$ is an element of $X^*(G) = \text{Hom}(G, G^m)$. A one-parameter subgroup is an element from $X^*(G) = \text{Hom}(G_m, G)$.
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- An **algebraic torus** is an algebraic group $T$ such that $T_{k\text{sep}} \cong \mathbb{G}_m^n$ for
  some $n$, which is called the **rank** of the torus. A **character** of an
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  **one-parameter subgroup** is an element from $X_*(G) = \text{Hom}(\mathbb{G}_m, G)$. 
Recalling more definitions

Let \( V \) be a finite dimensional \( \mathbb{R} \)-vector space, and \( \Phi \) a subset of \( V \). Then \((\Phi, V)\) is called a root system if:

- \( \Phi \) is finite, does not contain \( 0 \), and spans \( V \);
- for each \( \alpha \in \Phi \) there exists a reflection operator \( s_\alpha \) that interchanges \( \alpha \) and \(-\alpha\) and maps \( \Phi \) to \( \Phi \);
- for each \( \beta \in \Phi \), the vector \( s_\alpha(\beta) - \beta \) is an integer multiple of \( \alpha \).
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There exists a pairing $(\cdot, \cdot) : V \times V \rightarrow \mathbb{C}$ such that all reflections are orthogonal transformations. If $\alpha \in \Phi$, we can find a unique $\alpha^\vee$ such that

$$\langle -, \alpha^\vee \rangle := \alpha^\vee(-) = 2(-, \alpha)/(\alpha, \alpha) \in \mathbb{Z}.$$  

If $\Phi^\vee = \{ \alpha^\vee | \alpha \in \Phi \}$ and $V^\vee = \langle \Phi^\vee \rangle \otimes_\mathbb{Z} \mathbb{R}$, then $(\Phi^\vee, V^\vee)$ is called the dual root system.
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A root datum is a quadruple $(X, Y, \Phi, \Phi^\vee)$ where $X, Y$ are free abelian groups with a perfect pairing $\langle \ , \ \rangle : X \times Y \to \mathbb{Z}$ and where $\Phi \subset X$ and $\Phi^\vee \subset Y$ are finite subsets such that $\Phi \ni \alpha \iff \alpha^\vee \in \Phi^\vee$. In addition, $\langle \alpha, \alpha^\vee \rangle = 2$ and for each $\alpha \in \Phi$, the reflection $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ satisfies $s_\alpha(\Phi) = \Phi$ and the group $\langle s_\alpha | \alpha \in \Phi \rangle$ is finite.
Combining these concepts

Let $G$ be a connected reductive group over a perfect field $k$ with maximal torus $T$. Recall the adjoint representation $\text{Ad} : G \to \text{GL}(\mathfrak{g})$ induced by the conjugation action of $G$ onto itself. For any character $\alpha \in X^*(T)$, denote

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid \text{Ad}(t)X = \alpha(t)X \text{ for all } t \in T(k) \}.$$
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$$g_\alpha = \{X \in g \mid \text{Ad}(t)X = \alpha(t)X \text{ for all } t \in T(k)\}.$$ 

If $g_\alpha \neq 0$, it must be 1-dimensional, and they are called the root spaces. We have

$$g = t \oplus \bigoplus_{\alpha} g_\alpha,$$

where $t = \text{Lie}(T)$ and so the set of $\alpha$ for which $g \neq 0$ is finite and denoted by $\Phi(G, T)$. It turns out that $(X^*(T), X_*(T), \Phi, \Phi^\vee)$ is a root datum with the natural pairing between $X^*(T)$ and $X_*(T)$.
The dual group

Recall that a root datum \((X, Y, \Phi, \Phi^v)\) is said to be *reduced* if \(\alpha \in \Phi\) implies that \(2\alpha \not\in \Phi\). We have the following theorem:

**Theorem**

If \(k\) is algebraically closed, the association \(G \to (X^*(T), X_*(T), \Phi, \Phi^v)\) determines a bijection between isomorphism classes of connected reductive groups and isomorphism classes of reduced root data.

Examples:

- The group \(\text{GL}_n\) is its own dual.
- If \(G = \text{SL}_n\), then \(\hat{G} = \text{PGL}_n(C)\).
- If \(G = \text{Sp}_{2n}\), then \(\hat{G} = \text{SO}_{2n+1}(C)\).
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A trivial remark on one side, now leads to an interesting concept on the other. Namely, if \((X, Y, \Phi, \Phi^v)\) is a root datum, then so is \((Y, X, \Phi^v, \Phi)\). According to the theorem, we may thus associate to \(G\) its so-called *complex dual* \(\hat{G}\) that has the corresponding dual root system.

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More on root data

Definition
Let $\Phi$ be a root system. A set of positive roots $\Phi^+ \subset \Phi$ is a subset such that for all $\alpha \in \Phi$, precisely one of $\alpha$ and $-\alpha$ is in $\Phi^+$ and for any $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta \in \Phi$, we require that $\alpha + \beta \in \Phi^+$. We say a root in $\Phi^+$ is called simple if it is not the sum of two positive roots.
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Note that $\Phi^+, v$ is a set of positive roots in $\Phi^v$. If $\Delta$ is a maximal set of simple roots in $\Phi$, then so is $\Delta^v$ in $\Phi^v$. One can show that $\Delta$ and $\Delta^v$ determine $\Phi$ and $\Phi^v$ respectively.
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Definition
We call a tuple $(X, Y, \Delta, \Delta^v)$ a based root datum if $(X^*(T), X^*_*(T), \Phi, \Phi^v)$ is a root datum and $\Delta, \Delta^v$ are maximal sets of simple roots as above.

Recall that a Borel subgroup of $G$ is a maximal connected solvable subgroup.
Pinnings

Lemma
On the other side of the correspondence, choosing $\Delta$ amounts to choosing a Borel subgroup $T \subset B \subset G$.

Proof: (sketch) By definition of $g_\alpha$, we have the map

$$\exp_\alpha : g_\alpha \to G(\mathbb{C})$$

that satisfies $g\exp_\alpha(x)g^{-1} = \exp(\alpha(g)x)$. Its image $U_\alpha$ naturally has $g_\alpha$ as Lie-algebra. Then one can show that the Borel subgroups of $G$ are precisely those of the form

$$\langle T, \{U_\alpha\}_{\alpha \in \Delta} \rangle.$$

This would prove the claim. \qed
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 Lemma
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This would prove the claim. □

This proof naturally leads to the following definition.

 Definition
 A pinning of $G$ is a tuple $(B, T, \{u_{\alpha}\}_{\alpha \in \Delta})$ where $u_{\alpha} \in U_{\alpha} - 1$ for all $\alpha \in \Delta$, where $\Delta$ corresponds to the Borel subgroup $B$. 
A bit of group theory

Proposition
Let $T \subset B \subset G$ be as before. Then

- all Borel subgroups in $G$ are conjugate;
- all maximal tori inside $B$ are conjugate by an element of $B$;
- $B$ is its own normaliser inside $G$;
- $T$ is its own normaliser inside $B$. 
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Let $f \in \text{Aut}(G)$ and let denote $c_g \in \text{Aut}(G)$ conjugation by $g$. Then $f(B)$ is also a Borel subgroup, so $f(B) = gBg^{-1}$ for some $g \in G$, so $(c_g \circ f)(B) = B$. Now $(c_g \circ f)(T) = bTb^{-1}$ for some $b \in B$, so $(c_{gb} \circ f)(T) = T$ and this also fixes $B$. This element $gb \in G$ is unique up to an element from the normaliser of $T$ inside $B$; that is, from $T$. 
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$$
\text{Aut}(G) \to \text{Aut}(X^*(T), X_*(T), \Delta, \Delta^\vee).
$$

It is non-trivial, but it can be shown, that its kernel is precisely $\text{Inn}(G)$. 
A note about $F$-forms

Note that elements from the Galois group need not induce algebraic automorphisms, i.e. if $G$ is a matrix group, conjugating all entries by some $\sigma \in \text{Gal}(\bar{F}/F)$, an action commonly denoted $\tau_0$, is not an algebraic automorphism of $G$. However, if $\tau$ is another action, then we do have a map

$$\alpha : \text{Gal}(\bar{F}/F) \to \text{Aut}_{\text{alg}}(G) : \gamma \mapsto \tau(\gamma) \circ \tau_0(\gamma)^{-1}.$$
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However, this is not generally a homomorphism, but it is a 1-cocycle. We call this a \textit{rational structure} and they are characterised by the first group cohomology of $\text{Gal}(\bar{F}/F)$ with values in $\text{Aut}_{\text{alg}}(G)$. The invariants for any given $\tau$ are denoted $G_\tau(F)$ and two such groups define the same dual groups if and only if the $\tau$’s differ by an \textit{inner} twist, i.e. something from $H^1(\text{Gal}(\bar{F}/F), \text{Int}_{\text{alg}}(G))$. For more details, ask Eric.
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**Examples:**

- Consider $\GL_1$ over $\mathbb{R}$. We can make the non-trivial automorphism act by $z \mapsto \bar{z}$ or by $z \mapsto 1/z$. Combining these gives the algebraic action $z \mapsto 1/\bar{z}$. Its fixed points are those on the unit circle.

- Consider $\SU_3$ over $\mathbb{R}$. We now have an algebraic automorphism $A \mapsto (A^T)^{-1}$. Unlike in the $\SL_2$-case, now $\SU_3$ is not an inner twist from $\SL_3$, by considering the action on the Dynkin diagram.
The Langland dual group

Suppose that all groups are defined over a local or global field $F$. Let $\hat{T} \subset \hat{B} \subset \hat{G}$ be the complex dual group associated with the based root datum $(X_*(T), X^*(T), \Delta^\vee, \Delta)$. From the previous slide, we obtain the exact sequence

$$1 \rightarrow \text{Inn}(G) \rightarrow \text{Aut}(G) \rightarrow \text{Aut}(X^*(T), X_*(T), \Delta, \Delta^\vee) \rightarrow 1$$
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Proving surjectivity shows that it is split; choosing a pinning $\{u_\alpha\}_{\alpha \in \Delta}$ we see that any automorphism of the root datum induces an action on the $u_\alpha$. These elements and their negatives generate $G$, and one can show that this defines a unique automorphism of $G$. 

Since $T$ and $B$ are defined over $F$, the group $\text{Gal}(\bar{F}/F)$ acts on the root datum of $G$ and so also on the root datum of $\hat{G}$. The above section for $\hat{G}$ gives us an induced map $\text{Gal}(\bar{F}/F) \to \text{Aut}(\hat{G})$. We use it to define the Langlands dual group $L_G = \hat{G}(C) \rtimes \text{Gal}(\bar{F}/F)$. This is simply a direct product if $G$ is split, i.e. if $T \sim \hat{G}$ over $F$, because then the Galois action will be trivial.
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Class field theory

Recall the following facts about \textbf{local} class field theory:

- For any finite field extension $E/F$, there is an isomorphism
  \[ \theta_{E/F} : F^\times / \mathcal{N}_{E/F} E^\times \xrightarrow{\sim} \text{Gal}(E/F)^{ab}. \]

- These maps are compatible and define a homomorphism
  \[ \theta_F : F^\times \rightarrow \text{Gal}(\bar{F}/F)^{ab} \]

  called the \textit{local reciprocity map}.

- If $H \subset G$ and $[G : H] < \infty$, we have a \textit{transfer map} $G \rightarrow H^{ab}$. 

Also recall the following facts about \textbf{global} class field theory:

- There exists a surjective continuous homomorphism
  \[ A \times F \rightarrow \text{Gal}(\bar{F}/F)^{ab} : s \mapsto [s, F] \]

  called the \textit{global reciprocity map}.

- It has the property that for $s \in A \times F$ whose ideal is coprime to all the ramified places in a certain finite extension $E/F$,
  \[ [s, F] \mid E = (s), \]

  where the latter denotes the \textit{Artin symbol}, extended multiplicatively from $(p, E/F) = \text{Frob}_p$ for all primes $p$ in $E$. 

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Recall the following facts about **local** class field theory:

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  called the **global reciprocity map**.

- It has the property that for $s \in \mathbb{A}_F^\times$ whose ideal is coprime to all the ramified places in a certain finite extension $E/F$,
  \[ [s, F]|_E = ((s), E/F), \]

  where the latter denotes the **Artin symbol**, extended multiplicatively from $(p, E/F) = \text{Frob}_p$ for all primes $p$ in $E$. 
The Weil group

Let $F$ be a local or a global field. Then a *Weil group* for $F$ is a topological group $W_F$ along with a continuous homomorphism $\phi : W_F \to \text{Gal}(\bar{F}/F)$ with dense image, and for each finite field extension $E/F$, the group $W_E = \phi^{-1}(\text{Gal}(\bar{E}/E))$ admits an isomorphism $r_E : C_E \to W_E^{\text{ab}}$, where

$$C_E = \begin{cases} E^\times & \text{if } F \text{ is local;} \\ E^\times \setminus \mathbb{A}_E^\times & \text{if } F \text{ is global.} \end{cases}$$
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In addition, these groups and maps must satisfy that

$$C_E \xrightarrow{r_E} W_E^{ab} \xrightarrow{\phi} \text{Gal}(\overline{E}/E)^{ab}$$

is the reciprocity map from class field theory, that

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and that for all $w \in W_F$, $\sigma = \phi(w)$ and $E'/E/F$ these commute:
Examples of the Weil group

Let $F = \mathbb{C}$. Then the map $\phi : W_\mathbb{C} \to \text{Gal}(\mathbb{C}/\mathbb{C}) = \{1\}$ must be trivial. Since $\mathbb{C}/\mathbb{C}$ is the only finite field extension, considering $\mathbb{C}$ as a local field (as it is complete), we must have an isomorphism $r_\mathbb{C} : \mathbb{C}^\times \to W_\mathbb{C}$. All other conditions are now trivially satisfied.
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Let $F = \mathbb{R}$. Then $\phi : W_\mathbb{R} \to \{1, \sigma\}$ where $\sigma$ denotes complex conjugation. Also, $r_\mathbb{C} : \mathbb{C}^\times \to W_\mathbb{C}$ must be an isomorphism, and be the kernel of $\phi$. One can show that now $W_\mathbb{R} = \mathbb{C}^\times \cup j\mathbb{C}^\times$ where $j^2 = -1$ and $jzj^{-1} = \sigma(z)$ for all $z \in \mathbb{C}^\times$. 

For number fields, it is highly non-trivial to show that the Weil group exists and there is no easy description of it. More on this in a few weeks.
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- Let $F$ be a non-archimedian local field with finite residue field $k$. Then recall that we have a maximal unramified extension $F^{unr}$ satisfying

\[ 1 \to I_F \to \text{Gal}(\bar{F}/F) \to \text{Gal}(F^{unr}/F) \cong \text{Gal}(\bar{k}/k) \cong \hat{\mathbb{Z}} \to 1, \]

where $I_F$ denotes the inertia group. In this case, one can show that $W_F$ is the dense subgroup of $\text{Gal}(\bar{F}/F)$ that maps to $\mathbb{Z} \subset \hat{\mathbb{Z}}$. For number fields, it is highly non-trivial to show that the Weil group exists and there is no easy description of it. More on this in a few weeks.
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- Let $F = \mathbb{C}$. Then the map $\phi : W_{\mathbb{C}} \to \text{Gal}(\mathbb{C}/\mathbb{C}) = \{1\}$ must be trivial. Since $\mathbb{C}/\mathbb{C}$ is the only finite field extension, considering $\mathbb{C}$ as a local field (as it is complete), we must have an isomorphism $r_\mathbb{C} : \mathbb{C}^\times \to W_{\mathbb{C}}$. All other conditions are now trivially satisfied.

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For number fields, it is highly non-trivial to show that the Weil group exists and there is no easy description of it. More on this in a few weeks.
The Weil-Deligne group

We first record the following theorem about the Weil group.

**Langlands for GL$_1$**

There is a bijection between isomorphism classes of irreducible automorphic representations of $GL_1(\mathbb{A}_F)$ and continuous representations $W_F \rightarrow GL_1(\mathbb{C})$.

The proof follows from identifying a representation with its associated character of $F^\times \backslash \mathbb{A}_F^\times$, which is isomorphic to $W_F^{ab}$ by definition of $W_F$. 
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Definition

The *Weil-Deligne group* for a local field $F$ is defined as

$$W'_F = W_F \times \text{SL}_2(\mathbb{C}).$$

It is interesting to remark that the correct analogue of this for global fields, the so-called Langlands group, is currently still only hypothetical.
Representations of $W'_F$

Recall the Weil group for a local field. It fit naturally into an exact sequence of the form

$$1 \rightarrow I_F \rightarrow W_F \rightarrow \mathbb{Z} \rightarrow 1,$$

and so we may write $W_F \cong I_F \rtimes \langle \text{Fr} \rangle$. Let $G$ be a reductive group over $\mathbb{C}$. Recall that some $g \in G$ is said to be semi-simple if $\phi(g)$ is for some, and thus for any, faithful representation $\phi : G \rightarrow \text{GL}_n$. 

Definition

A representation/admissible homomorphism of $W'_F$ into $G(\mathbb{C})$ is a homomorphism $\phi : W'_F \rightarrow G(\mathbb{C})$ such that $\phi$ is trivial on an open subgroup of $I_F$, such that $\phi(\text{Fr})$ is semi-simple in $G$, and $\phi|_{\text{SL}_2(\mathbb{C})}$ is induced by a morphism of algebraic groups $\text{SL}_2(\mathbb{C}) \rightarrow G$. 

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**Local Langlands for GL$_n$**

There is a bijection between representations of $W_F'$ into $\text{GL}_n(\mathbb{C})$ and irreducible admissible representations of $\text{GL}_n(F)$. 
L-parameters

Recall that the Weil group $W_F$ comes with a map $W_F \to \text{Gal} (\bar{F}/F)$ and that the Langlands dual of a reductive group $G$, denoted $^LG$, is defined by taking a suitable semi-direct product of $\text{Gal} (\bar{F}/F)$ and $\hat{G}$.

Definition

An $L$-parameter is a representation of $W_F'$ into $^LG$ that commutes with the projections to $\text{Gal} (\bar{F}/F)$. We say that two $L$-parameters are equivalent if they differ only by conjugation by some element of $\hat{G}(\mathbb{C})$. 
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**Vague local Langlands correspondence conjectures**

There is a bijection between *L-packets* of admissible $G(F)$-representations and equivalence classes of L-parameters satisfying certain conditions. Given a map $^L H \to ^L G$ that commutes with the projections to $\text{Gal}(\bar{F}/F)$, there is a corresponding transfer of L-packets compatible with the natural transfer of L-parameters.
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There is a bijection between *L-packets* of admissible $G(F)$-representations and equivalence classes of L-parameters satisfying certain conditions. Given a map $^LH \to ^LG$ that commutes with the projections to $\text{Gal}(\bar{F}/F)$, there is a corresponding transfer of L-packets compatible with the natural transfer of L-parameters.

What are L-packets? In the case of $GL_n$, they are just singletons. In general? Hard to say. Everything is still only conjectural. Many definitions of L-packets are ad-hoc, and assume the conjectures to define the L-packets instead...
Why do we study all of this? Turns out, these representations are closely related to \textit{L-functions}. Let \( v \) be a place of a number field \( F \) and suppose that we have an L-parameter \( \phi : W'_F \to ^L G \) and a representation \( r : ^L G_{F_v} \to \text{GL}(V) \) for some \( F \)-vector space \( V \). Let \( q_v \) denote the cardinality of the residue field of \( F \). Define the \textit{local factor} by

\[
L(s, r \circ \phi) = \det(1 - r(F_{r,v})q_v^{-s}|V^{l_{F_v}})^{-1}.
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Why do we study all of this? Turns out, these representations are closely related to \textit{L-functions}. Let \( v \) be a place of a number field \( F \) and suppose that we have an L-parameter \( \phi : W'_{F_v} \to L^G \) and a representation \( r : L^G_{F_v} \to \text{GL}(V) \) for some \( F \)-vector space \( V \). Let \( q_v \) denote the cardinality of the residue field of \( F \). Define the \textit{local factor} by

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The L-function is defined by multiplying all the local factors at all the places together. It is conjecturally meromorphic and satisfies a functional equation relating the value at \( s \) to the value at \( 1 - s \) of the \textit{adjoint} L-function, i.e. the one defined by the dual representation. Many properties of the representation can be recovered from the L-function. For example, the image of the representation has finite centraliser if and only if the dual L-function is regular at \( s = 0 \). More on L-functions will be treated in the near future.
Thanks for listening!