CM-values of p-adic Θ-functions

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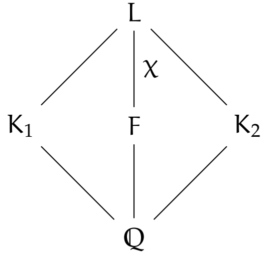
Setting up

Let $D_1, D_2 < 0$ be coprime discriminants and write $D = D_1 D_2$. Set

$$K_1 = \mathbb{Q}(\sqrt{D_1}), \quad K_2 = \mathbb{Q}(\sqrt{D_2}),$$
$$F = \mathbb{Q}(\sqrt{D}), \quad L = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}).$$



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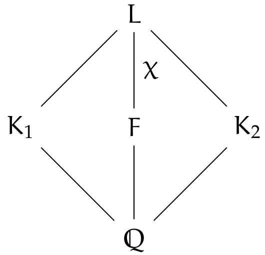

Let χ be the genus character of L/F: if $\mathfrak{p} \subset \mathcal{O}_F$ is prime, then

$$\chi(\mathfrak{p}) = \begin{cases} 1 & \text{if } \mathfrak{p} \text{ splits in } L/F; \\ -1 & \text{if } \mathfrak{p} \text{ is inert in } L/F. \end{cases}$$

The formula

Let $I \subset \mathcal{O}_F$ be an ideal. Define

$$\rho(I) = \#\{J \subset \mathcal{O}_L \mid \mathrm{Nm}_F^L(J) = I\};$$

$$\mathrm{sp}(I) = \begin{cases} \mathfrak{p} & \text{if } \mathfrak{p} \text{ is } \textit{unique} \text{ with } \chi(\mathfrak{p}) = -1 \text{ and } \nu_{\mathfrak{p}}(I) \text{ odd;} \\ 1 & \text{otherwise.} \end{cases}$$

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Let E_1 be an elliptic curve with CM by $\mathcal{O}_1$ and E_2 an elliptic curve with CM by $\mathcal{O}_2$. Then by CM theory, $j(E_i) \in H_i$ for $i = 1, 2$, where H_i is the Hilbert class field of K_i. For simplicity, assume $D_i \neq -3, -4$.

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Theorem (Gross-Zagier, 1984)

Setting $\alpha = \nu\sqrt{D}$ and $\mathcal{D}_F = (\sqrt{D})$, the following equality holds:

$$\log \mathrm{Nm}_{\mathbb{Q}}^{H_1 H_2}\left(j(E_1) - j(E_2)\right) = \sum_{\substack{\nu \in \mathcal{D}_F^{-1,+} \\ \mathrm{tr}(\nu) = 1}} \rho(\mathrm{sp}(\alpha)\alpha)(v_{\mathrm{sp}(\alpha)}(\alpha) + 1) \log \mathrm{Nm}(\mathrm{sp}(\alpha)).$$

Example

Let $D_1 = -7$ and $D_2 = -19$. Then

$$E_1 : y^2 + xy = x^3 - x^2 - 2x - 1, \quad j(E_1) = -3^3 5^3;$$
$$E_2 : y^2 + y = x^3 - 38x + 90, \quad j(E_2) = -2^{15} 3^3.$$

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If $\nu \in \mathcal{D}_F^{-1,+}$ and $\mathrm{tr}(\nu) = 1$, then

$$\alpha = \nu\sqrt{D} = \frac{x + \sqrt{D}}{2}, \quad \text{where } x^2 < D = 133 \text{ and } x \text{ is odd.}$$

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x	± 1	± 3	± 5	± 7	± 9	± 11
$(D - x^2)/4$	$3 \cdot 11$	31	3^3	$3 \cdot 7$	13	3
$\mathrm{sp}(\alpha)$	3	31	3	3	13	3
$(\nu_{\mathrm{sp}(\alpha)}(\alpha) + 1)/2$	1	1	2	1	1	1
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Zagier's proof

First step is to rewrite the task at hand to proving

$$\log \mathrm{Nm}_{\mathbb{Q}}^{H_1 H_2}\big(j(E_1) - j(E_2)\big) = \sum_{\substack{\nu \in \mathcal{D}_F^{-1,+} \\ \mathrm{tr}(\nu)=1}} \sum_{I \mid (\nu)\mathcal{D}_F} \chi(I) \log \mathrm{Nm}(I).$$

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This reminds one of a diagonal restriction of a weight k Hilbert Eisenstein series:

$$E_{k,\chi}(z,z) = \mathrm{const} + \sum_{\substack{\nu \in \mathcal{D}_F^{-1,+} \\ \mathrm{tr}(\nu)=n}} \Bigg(\sum_{I \mid (\nu)\mathcal{D}_F} \chi(I)\mathrm{Nm}(I)^{k-1} \Bigg) q^n.$$

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This must be in $M_2(\mathrm{SL}_2(\mathbb{Z})) = 0$. The explicit formula for its Fourier coefficients involves two terms, one for each side $\implies$ equal. **Hard**.

What is the j-function really?

Consider $M_2(\mathbb{Q})$; this is a quaternion algebra with norm det. Here, a maximal order is given by

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Question

What happens if we change $M_2(\mathbb{Q})$ to a different quaternion algebra?

Shimura curves

Choose two primes $p \neq q$ and let $N = pq$. Let B_N denote the quaternion algebra ramified at p and q. Let R_N be a maximal order and let $R_{N,1}^\times$ denote the subgroup of units of norm 1. We may choose an embedding $R_{N,1}^\times \to M_2(\mathbb{R})$ to form the quotient

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Let $\tau_1, \tau_2 \in \mathcal{H}$ be CM points: fixed points in $\mathbb{C}$ of embeddings $\mathcal{O}_i \to R_N$. These exist when p and q are inert in both K_i. We want to study

$$\mathrm{Nm}\big(j_N(\tau_1) - j_N(\tau_2)\big).$$

They are algebraic by Shimura reciprocity.

Cerednik-Drinfeld

Let B_q denote the quaternion algebra ramified at q and ∞. Let R_q be a maximal order. Now B_q is definite, so consider the group

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$$\Gamma_q^p \setminus \mathcal{H}_p,$$

where $\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$ is the *p-adic upper half plane*.

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Question

Which functions on $\Gamma_q^p \setminus \mathcal{H}_p$ correspond to j_N on the other side?

Theta functions

Let $w_1, w_2 \in \mathcal{H}_p$. Then consider the expression

$$\Theta(w_1, w_2; z) = \prod_{\gamma \in \Gamma_q^p} \frac{z - \gamma w_1}{z - \gamma w_2}.$$

If $N \in \{6, 10, 22\}$, this expression descends to a rigid analytic meromorphic function on $\Gamma_q^p \setminus \mathcal{H}_p$ with divisor $[w_1] - [w_2]$.

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$$\Theta(w_1, w_2; z) = c(w_1, w_2) \cdot \frac{j_N(z) - j_N(w_1)}{j_N(z) - j_N(w_2)}, \text{ for some } c(w_1, w_2) \in \mathbb{C}_p.$$

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Now choose $w_1 = \tau_1$ and $w_2 = \tau_1'$; its Galois conjugate. Because we don't know $c(\tau_1, \tau_1')$, we opt to study instead

$$\frac{j_N(\tau_2) - j_N(\tau_1)}{j_N(\tau_2) - j_N(\tau_1')} \frac{j_N(\tau_2') - j_N(\tau_1')}{j_N(\tau_2') - j_N(\tau_1)} = \prod_{\gamma \in \Gamma_q^p} \frac{\tau_2 - \gamma\tau_1}{\tau_2 - \gamma\tau_1'} \frac{\tau_2' - \gamma\tau_1'}{\tau_2' - \gamma\tau_1}.$$

The conjecture

One can p-adically approximate the quantity

$$J_q^p(\tau_1, \tau_2) := \prod_{\gamma \in \Gamma_q^p} \frac{\tau_2 - \gamma\tau_1}{\tau_2 - \gamma\tau_1'} \frac{\tau_2' - \gamma\tau_1'}{\tau_2' - \gamma\tau_1}$$

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There are four ideals $\mathfrak{a}$ of norm $N = pq$ in $\mathcal{O}_F$; they come in two $\mathrm{Gal}(F/\mathbb{Q})$ orbits. Assign one orbit $\delta(\mathfrak{a}) = +1$, the other $\delta(\mathfrak{a}) = -1$.

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Conjecture (Giampietro, Darmon)

The expression

$$\log \mathrm{Nm}_{\mathbb{Q}}^{H_1 H_2} J_q^p(\tau_1, \tau_2)$$

is up to sign explicitly equal to

$$\sum_{\mathrm{Nm}(\mathfrak{a})=N} \delta(\mathfrak{a}) \sum_{\substack{\nu \in \mathcal{D}_F^{-1,+} \\ \mathrm{tr}(\nu)=1}} \rho(\mathrm{sp}(\alpha\mathfrak{a}^{-1})\alpha\mathfrak{a}^{-1})(\nu_{\mathrm{sp}(\alpha\mathfrak{a}^{-1})}(\alpha\mathfrak{a}^{-1}) + 1) \log \mathrm{Nm}\,(\mathrm{sp}(\alpha\mathfrak{a}^{-1})).$$

Intermezzo: rewriting the theta-series

Let τ_i be defined by an embedding $\alpha_i : \mathcal{O}_i \to R_q$ for $i = 1, 2$. This yields actions of the $\mathcal{O}_i$ on B_q, and as such, an action of L through

$$\mathcal{O}_L \cong \mathcal{O}_1 \otimes_{\mathbb{Z}} \mathcal{O}_2 : (x \otimes y) * b = \alpha_1(x) b \alpha_2(y).$$

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Proposition

There exists a unique F-linear quadratic form $\det_F : B_q \to F$ with the property that $\mathrm{tr}_{F/\mathbb{Q}}(\det_F(b)) = \mathrm{Nm}(b)$.

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There exists a unique F-linear quadratic form $\det_F : B_q \to F$ with the property that $\mathrm{tr}_{F/\mathbb{Q}}(\det_F(b)) = \mathrm{Nm}(b)$.

It satisfies

$$\frac{\tau_2 - b\tau_1}{\tau_2 - b\tau_1'} \frac{\tau_2' - b\tau_1'}{\tau_2' - b\tau_1} = \frac{\det_F(b)}{\det_F'(b)}.$$

As such,

$$\frac{\Theta(\tau_1, \tau_1'; \tau_2)}{\Theta(\tau_1, \tau_1'; \tau_2')} = \prod_{b \in \Gamma_q^p} \frac{\det_F(b)}{\det_F'(b)}.$$

From quaternions to ideals

Let $\iota : B_q \to L$ be an isomorphism of L-vector spaces. For $b \in B_q$, define the ideal

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Proposition

Ranging over all possible pairs of embeddings α_1, α_2, the association $b \mapsto I_b$ establishes a bijection between

$$\{b \in R_q/\{\pm 1\} \mid \det_F(b) = \nu\}$$

and

$$\{I \subset \mathcal{O}_L \mid \mathrm{Nm}_{L/F}(I) = (\nu)\mathfrak{q}^{-1}\mathcal{D}_F\}.$$

Rewriting the theta series further

Note that we have a correspondence

$$\Gamma_q^p = R_q[1/p]_1^\times \leftrightarrow \lim_{n\to\infty} \left\{ b \in R_q \mid \mathrm{Nm}(b) = p^{2n} \right\}.$$

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Taking the logarithm;

$$\log_p \frac{\Theta(\tau_1, \tau_1'; \tau_2)}{\Theta(\tau_1, \tau_1'; \tau_2')} = \lim_{n\to\infty} \sum_{\mathrm{tr}(\nu)=p^{2n}} \#\{b \in R_q \mid \det_F(b) = \nu\} \log_p(\nu/\nu') = \lim_{n\to\infty} \sum_{\mathrm{tr}(\nu)=p^{2n}} \rho((\nu)\mathfrak{q}^{-1}\mathcal{D}_F) \log_p(\nu/\nu').$$

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- Explicitly compute its Fourier coefficients a_ν for all $\nu \gg 0$;
- The ϵ-part then yields a meaningful derivative.

Deforming $1 \oplus \chi$

Again let $\rho = 1 \oplus \chi$. Write $\tilde{\rho}$ for a deformation of ρ to the ring $GL_2(\mathbb{Q}_p[\epsilon])$ where $\epsilon^2 = 0$.

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Proposition

Let $a, b, c, d : G_F \to \mathbb{Q}_p$ be those functions such that

$$\tilde{\rho}(\tau) = \left(1 + \epsilon \begin{pmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{pmatrix}\right) \cdot \rho(\tau)$$

for all $\tau \in G_F$. Then these functions must respectively satisfy

$$a, d \in \mathrm{Hom}(G_F, \mathbb{Q}_p), \quad \text{and} \quad b, c \in H^1(G_F, \mathbb{Q}_p(\chi)).$$

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Note that $\dim \mathrm{Hom}(G_F, \mathbb{Q}_p) = 1$ spanned by the p-adic cyclotomic character:

$$\phi_p^{cyc} : G_F \to \mathrm{Gal}(F(\zeta_p^\infty)/F) \cong \mathbb{Z}_p^\times \xrightarrow{\log_p} \mathbb{Q}_p.$$

Hida’s Hecke Algebra

For simplicity, choose

$$\tilde{\rho}(\tau) = \begin{pmatrix} 1 + \phi_p^{\mathrm{cyc}} \epsilon & 0 \\ 0 & \chi - \chi \phi_p^{\mathrm{cyc}} \epsilon \end{pmatrix}.$$

Suppose that this deformation is modular. That would yield a morphism $\varphi : \mathbb{T} \to \mathbb{Q}_p[\epsilon]$, where $\mathbb{T}$ is Hida’s p-adic Hecke algebra, generated by:

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Let $\mathbb{K}$ be the ring of fractions of Hida's *nearly ordinary cuspidal Hecke algebra*. There exists a semisimple Galois representation π: $G_F \to GL_2(\mathbb{K})$ with the following properties:

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- π is continuous, odd and unramified outside p;
- For each prime $\mathfrak{l} \nmid p$, it holds that

$$\det\left(1 - \pi(\mathrm{Frob}_\mathfrak{l})X\right) = 1 - T_\mathfrak{l}X + \langle \mathfrak{l} \rangle \mathrm{Nm}(\mathfrak{l})X^2.$$

Solving the recursion

We recover φ from

$$\varphi(T_l) = \mathrm{tr}(\tilde{\rho}(\mathrm{Frob}_l)) = \begin{cases} 2 & \text{if } \chi(l) = 1; \\ 2\log_p(\mathrm{Nm}(l))\epsilon & \text{if } \chi(l) = -1. \end{cases}$$

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We can solve this in each case explicitly:

$$\varphi(T_{\mathfrak{l}^n}) = \begin{cases} n+1 & \text{if } \chi(\mathfrak{l}) = 1; \\ (n+1)\log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon & \text{if } \chi(\mathfrak{l}) = -1 \text{ and } n \text{ is odd}; \\ 1 & \text{if } \chi(\mathfrak{l}) = -1 \text{ and } n \text{ is even}. \end{cases}$$

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The integral parts are precisely $\rho(\mathfrak{l}^n)$. We can thus write

$$\varphi(T_{\mathfrak{l}^n}) = \rho(\mathfrak{l}^n) + \frac{1}{2}(n+1)\big(1 - \chi(\mathfrak{l}^n)\big)\log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon.$$

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Let $J \subset \mathcal{O}_F$ be any ideal coprime to p. Then

$$\varphi(T_J) = \rho(J) + \frac{1}{2}\sum_{\mathfrak{l}^n \| J}\Big((n+1)\big(1-\chi(\mathfrak{l}^n)\big)\rho(J/\mathfrak{l}^n)\Big)\log_p(\mathrm{Nm}(\mathfrak{l}))\epsilon.$$

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Proposition

If J is a primitive ideal coprime to p, then the quantity

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Indeed, the factor $1-\chi(\mathfrak{l}^n) = 0$ unless $\mathfrak{l}$ is a special prime of J, and if $J/\mathfrak{l}^n$ still has another special prime, $\rho(J/\mathfrak{l}^n) = 0$. It can thus only be non-zero when $\mathfrak{l}$ is the unique special prime; the rest matches up.

Fourier coefficients

For convenience, let us denote

$$\log \mathcal{F}(J) = \rho(\mathrm{sp}(J)J)(\nu_{\mathrm{sp}(J)}(J) + 1)\log(\mathrm{sp}(J)),$$

so that very concisely, for J coprime to p,

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Theorem

For any $\nu \in (\mathcal{D}_F^{-1}\mathfrak{q})^+$, let $J_\nu = (\nu)\mathcal{D}_F\mathfrak{q}^{-1}$. Then it holds that

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The term $\log(\nu/\nu')$ comes from ν at the two places above p, as

$$\varphi(U_\pi) = -1 + \log_p(\pi)\epsilon; \quad \varphi(U_{\pi'}) = 1 + \log_p(\pi')\epsilon.$$

Ordinary projection

We take the diagonal restriction:

$$\mathrm{diag}(f_{\mathfrak{q}}) = \sum_{n=1}^{\infty} \Big(\sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{q})^+ \\ \mathrm{tr}(\nu)=n}} a_\nu \Big) q^n.$$

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$$a_n(\partial \mathrm{diag}(f_{\mathfrak{q}})) = \sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{q})^+ \\ \mathrm{tr}(\nu)=n}} (-1)^{\nu_{\mathfrak{p}}(\nu)} \Big(\log_p(\mathcal{F}(\widetilde{J_\nu})) - \rho(\widetilde{J_\nu}) \log_p(\nu/\nu') \Big).$$

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Now we take the *ordinary projection* e^{ord}:

$$\begin{aligned} a_1(e^{\mathrm{ord}}(\partial \mathrm{diag}(f_{\mathfrak{q}}))) &= \lim_{n\to\infty} a_{p^{2n}}(\partial \mathrm{diag}(f_{\mathfrak{q}})) \\ &= \lim_{n\to\infty} \sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{q})^+ \\ \mathrm{tr}(\nu)=p^{2n}}} (-1)^{\nu_{\mathfrak{p}}(\nu)} \Big(\log_p(\mathcal{F}(\widetilde{J_\nu})) - \rho(\widetilde{J_\nu}) \log_p(\nu/\nu') \Big). \end{aligned}$$

The crux!

One can show that the result must be a classical cusp form of weight 2 and level N, but one can check that

$$S_2(\Gamma_0(6)) = S_2(\Gamma_0(10)) = 0 \quad \text{and} \quad S_2(\Gamma_0(22)) \approx 0.$$

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In other words, if

$$A := \lim_{n\to\infty} \sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{q})^+ \\ \operatorname{tr}(\nu)=p^{2n}}} (-1)^{\nu_{\mathfrak{p}}(\nu)} \rho(\widetilde{J_\nu}) \log_p(\nu/\nu')$$

and

$$B := \lim_{n\to\infty} \sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{q})^+ \\ \operatorname{tr}(\nu)=p^{2n}}} (-1)^{\nu_{\mathfrak{p}}(\nu)} \log_p(\mathcal{F}(\widetilde{J_\nu})),$$

then $A = B$.

The crux!

One can show that the result must be a classical cusp form of weight 2 and level N, but one can check that

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In other words, if

$$A := \lim_{n\to\infty} \sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{q})^+ \\ \mathrm{tr}(\nu) = p^{2n}}} (-1)^{\nu_\mathfrak{p}(\nu)} \rho(\widetilde{J_\nu}) \log_p(\nu/\nu')$$

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then $A = B$. Recall our expression for the theta series

$$\log_p \frac{\Theta(\tau_1, \tau_1'; \tau_2)}{\Theta(\tau_1, \tau_1'; \tau_2')} = \lim_{n\to\infty} \sum_{\mathrm{tr}(\nu) = p^{2n}} \rho(J_\nu) \log_p(\nu/\nu').$$

It easily follows that

$$A = \log \mathrm{Nm}\, J_q^p(\tau_1, \tau_2).$$

Conclusion

One can show that the limit in B equals the first term:

$$B = \sum_{\substack{\nu \in (\mathcal{D}_F^{-1}\mathfrak{q})^+ \\ \mathrm{tr}(\nu)=1}} (-1)^{\nu_{\mathfrak{p}}(\nu)} \log_p(\mathcal{F}(\widetilde{J_\nu}))$$

where

$$\log \mathcal{F}(J) = \rho(\mathrm{sp}(J)J)(\nu_{\mathrm{sp}(J)}(J) + 1) \log_p(\mathrm{sp}(J)).$$

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Now use $A = B$ to complete the proof:

Theorem (D., 2023)

The expression

$$\log \mathrm{Nm}_{\mathbb{Q}}^{H_1 H_2} J_q^p(\tau_1, \tau_2)$$

is up to sign explicitly equal to

$$\sum_{\mathrm{Nm}(\mathfrak{a})=N} \delta(\mathfrak{a}) \sum_{\substack{\nu \in \mathcal{D}_F^{-1,+} \\ \mathrm{tr}(\nu)=1}} \rho(\mathrm{sp}(\alpha\mathfrak{a}^{-1})\alpha\mathfrak{a}^{-1})(\nu_{\mathrm{sp}(\alpha\mathfrak{a}^{-1})}(\alpha\mathfrak{a}^{-1})+1)\log \mathrm{Nm}\,(\mathrm{sp}(\alpha\mathfrak{a}^{-1})).$$

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Preprint is on arXiv: https://arxiv.org/abs/2309.17251