Mysterious factorisations and Knopp's cocycle

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Setting up

Recall the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$:

$$\text{if} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{then} \quad \gamma \cdot x = \frac{ax+b}{cx+d}.$$

On $\mathbb{R}(z)$ we have the *weight* 2 action:

$$f(z)|_2\gamma = (cz+d)^{-2} f(\gamma \cdot z).$$

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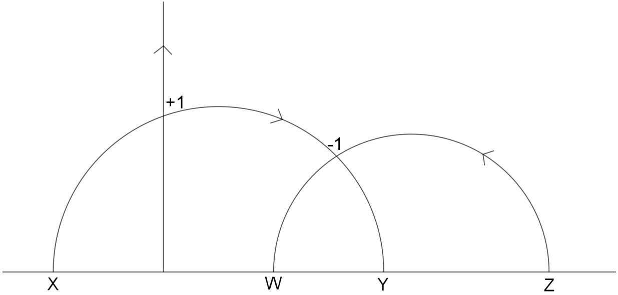
Let $\tau \in \mathbb{R}$ be of degree 2 over $\mathbb{Q}$. Its Galois conjugate is denoted τ'.

Lemma

The stabiliser of $\tau \in \mathbb{R}$ is infinite cyclic: $\Gamma_\tau = \langle \gamma_\tau \rangle \subset \mathrm{PSL}_2(\mathbb{Z})$.

Intersecting geodesics

Any pair $(x, y) \subset \widehat{\mathbb{R}}^2$ defines a unique geodesic in $\mathcal{H}$ by drawing a half circle:



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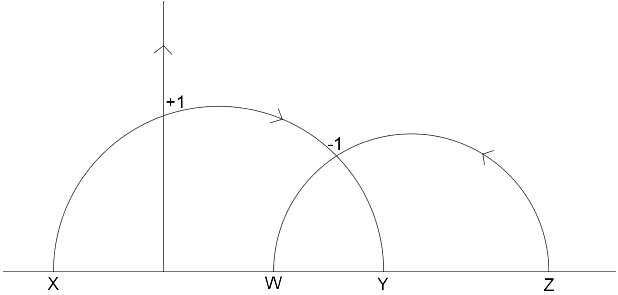

Two geodesics can intersect either positively or negatively, depending on their orientations. We denote this intersection by

$$(x \to y) \cap (z \to w) \in \{-1, 0, 1\}.$$

Knopp's cocycle

Definition

The *Knopp cocycle* f_τ associated to τ is defined by

$$f_\tau(\gamma) = \sum_{\delta \in \mathrm{SL}_2(\mathbb{Z})/\Gamma_\tau} \frac{(\delta\tau' \to \delta\tau) \cap (\gamma\infty \to \infty)}{z - \delta\tau}.$$

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- So $\delta\tau' \to \delta\tau$ contributes to the sum if and only if $\gamma\infty \in \mathbb{Q}$ is in between $\delta\tau$ and $\delta\tau'$. If $\delta\tau > \gamma\infty$ it contributes 1, otherwise -1.

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- Every element in $\mathrm{SL}_2(\mathbb{Z}) \cdot \tau$ has the same discriminant as τ. So, the set $\mathrm{SL}_2(\mathbb{Z})/\Gamma_\tau$ bijects with the set of real quadratic irrationalities σ satisfying $\mathrm{disc}(\sigma) = \mathrm{disc}(\tau) = D$.

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$$f_\tau(\gamma_1\gamma_2) = f_\tau(\gamma_1) + f_\tau(\gamma_2)|_2\gamma_1^{-1}.$$

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Remark

This sum is finite. Consider e.g. $\gamma\infty = 0$. We look for zeroes of $px^2 + qx + r = 0$ with $q^2 - 4pr = D$ and $\sigma' < 0 < \sigma$. The latter means that $pr < 0$. However, this bounds $q^2 = D + 4pr < D$, yielding only finitely many possibilities.

The multiplicative Knopp cocycle

Recall the dlog map on polynomials: if $P(z) = \prod (z - \alpha_i)^{n_i}$, then

$$\operatorname{dlog}(P)(z) = \frac{P'(z)}{P(z)} = \sum \frac{n_i}{z - \alpha_i}.$$

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For some appropriate choices of $x_\gamma \in \mathbb{Q}(\tau)$, we have a cocycle defined by

$$F_\tau(\gamma) = x_\gamma \prod_{\delta \in \mathrm{SL}_2(\mathbb{Z})/\Gamma_\tau} (z - \delta\tau)^{(\delta\tau \cap \delta\tau') \cap (\gamma\infty \to \infty)}.$$

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Definition

For τ_1, τ_2 both real quadratic irrationalities, we define

$$F_{\tau_1}[\tau_2] := F_{\tau_1}(\gamma_{\tau_2})(\tau_2).$$

This is independent on the choice of τ_2 in its $\mathrm{SL}_2(\mathbb{Z})$-orbit. However, choosing $\gamma_{\tau_2}^{-1}$ instead will invert the outcome.

Context

Why is this cocycle interesting?
Recent work by H. Darmon and J. Vonk uses a similar construction but then for the group $\mathrm{SL}_2(\mathbb{Z}[1/p])$ on the p-adic upper half plane.

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This Knopp cocycle is the $\mathbb{R}$-baby case. Still interesting to study!

Example

Set $\tau_1 = \frac{1+\sqrt{5}}{2}$ and $\tau_2 = \sqrt{3}$. Then

$$\gamma_{\tau_2} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \quad \text{so} \quad \gamma_{\tau_2}\infty = 2.$$

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As a result,

$$F_{\tau_1}(\gamma_{\tau_2}) = \frac{\left(z - \frac{3+\sqrt{5}}{2}\right)\left(z - \frac{5+\sqrt{5}}{2}\right)}{\left(z - \frac{3-\sqrt{5}}{2}\right)\left(z - \frac{5-\sqrt{5}}{2}\right)}.$$

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Hence

$$F_{\tau_1}[\tau_2] = \frac{\left(-5 + 2\sqrt{3} - \sqrt{5}\right)\left(-3 + 2\sqrt{3} - \sqrt{5}\right)}{\left(-5 + 2\sqrt{3} + \sqrt{5}\right)\left(-3 + 2\sqrt{3} + \sqrt{5}\right)}$$

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Hence

$$F_{\tau_1}[\tau_2] = \frac{\left(-5+2\sqrt{3}-\sqrt{5}\right)\left(-3+2\sqrt{3}-\sqrt{5}\right)}{\left(-5+2\sqrt{3}+\sqrt{5}\right)\left(-3+2\sqrt{3}+\sqrt{5}\right)} = \frac{21-8\sqrt{5}}{11}.$$

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$$\left(-5+2\sqrt{3}-\sqrt{5}\right)\left(-3+2\sqrt{3}-\sqrt{5}\right) = -4(\sqrt{3}-2)(\sqrt{5}+4)$$
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Coincidence? Let's try $\tau_1 = \sqrt{2}$ and $\tau_2 = \sqrt{3}$. Then $F_{\tau_1}[\tau_2]$ equals

$$\frac{\left(4+\sqrt{2}-2\sqrt{3}\right)\left(1+\sqrt{2}-\sqrt{3}\right)\left(2+\sqrt{2}-\sqrt{3}\right)\left(3+\sqrt{2}-\sqrt{3}\right)}{\left(-3+\sqrt{2}+\sqrt{3}\right)\left(-2+\sqrt{2}+\sqrt{3}\right)\left(-1+\sqrt{2}+\sqrt{3}\right)\left(-4+\sqrt{2}+2\sqrt{3}\right)}.$$

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So $F_{\tau_1}[\tau_2]$ equals

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This equals

$$\frac{219+53\sqrt{2}}{23}.$$

Mysterious factorisations II

So $F_{\tau_1}[\tau_2]$ equals

$$\frac{(4+\sqrt{2}-2\sqrt{3})\,(1+\sqrt{2}-\sqrt{3})\,(2+\sqrt{2}-\sqrt{3})\,(3+\sqrt{2}-\sqrt{3})}{(-3+\sqrt{2}+\sqrt{3})\,(-2+\sqrt{2}+\sqrt{3})\,(-1+\sqrt{2}+\sqrt{3})\,(-4+\sqrt{2}+2\sqrt{3})}.$$

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Again, we have factorisations

$$\begin{aligned}
\left(4+\sqrt{2}-2\sqrt{3}\right)\left(2+\sqrt{2}-\sqrt{3}\right) &= -(\sqrt{3}-2)(8+3\sqrt{2})\\
\left(1+\sqrt{2}-\sqrt{3}\right)\left(3+\sqrt{2}-\sqrt{3}\right) &= -(\sqrt{3}-2)(4+2\sqrt{2})\\
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\end{aligned}$$

What's going on here?

Key Lemma

Lemma

Let $M \in \mathrm{GL}_2(\mathbb{Q})$ interchange τ and τ'. Then M^2 acts trivially.

Proof: We claim that $\mathrm{tr}(M) = 0$, so that $M^2 = -\det(M)$ acts trivially.

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$$a\tau + b = c\tau\tau' + d\tau' \iff (a+d)\tau = c\tau\tau' - b + d(\tau + \tau') \in \mathbb{Q}.$$

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Proof: Just write everything out. The equation will be true when

$$pb = qa + rc.$$

Rewrite this as

$$b = -a(\tau + \tau') + c\tau\tau' \iff M\tau = \tau'$$

by the previous proof. □

Idea of the proof

Set $x = \tau_1$ and $\tau = \tau_2$. Then the lemma reads:

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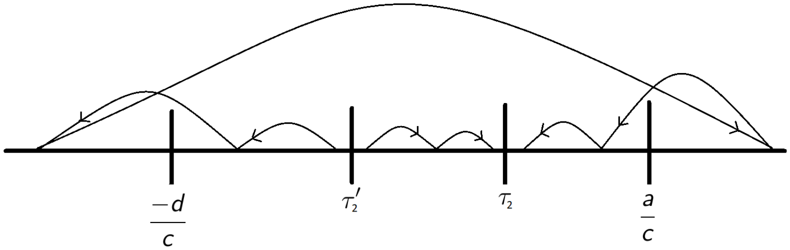
Concern: How do we choose M?
If M interchanges τ_2 and τ_2', then so does $\gamma_{\tau_2}^k M$ for any k.
We need that M preserves being $> \gamma_{\tau_2}\infty$ and $< \gamma_{\tau_2}\infty$.

Choosing the right involution

Note that γ_{τ_2} acts on the number line as follows:



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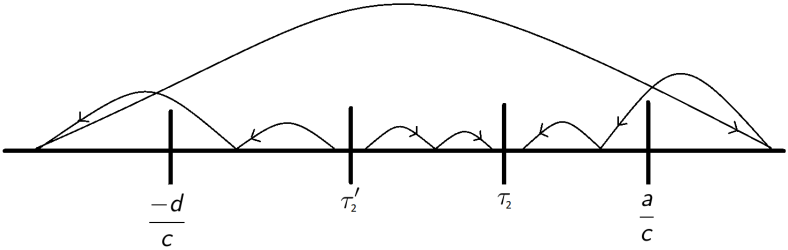

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Then there is a unique $k \in \mathbb{Z}$ such that $\gamma_{\tau_2}^k M\infty > \gamma_{\tau_2}\infty = a/c$.

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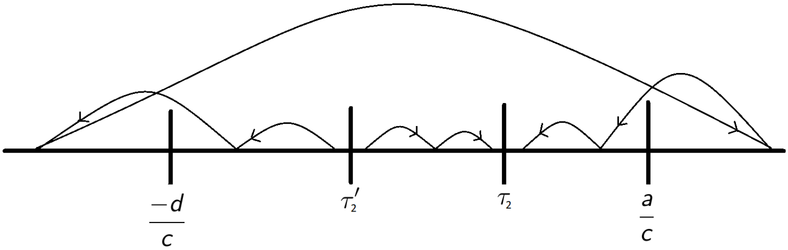
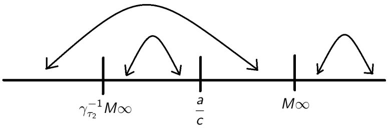
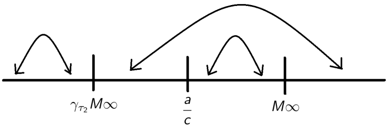
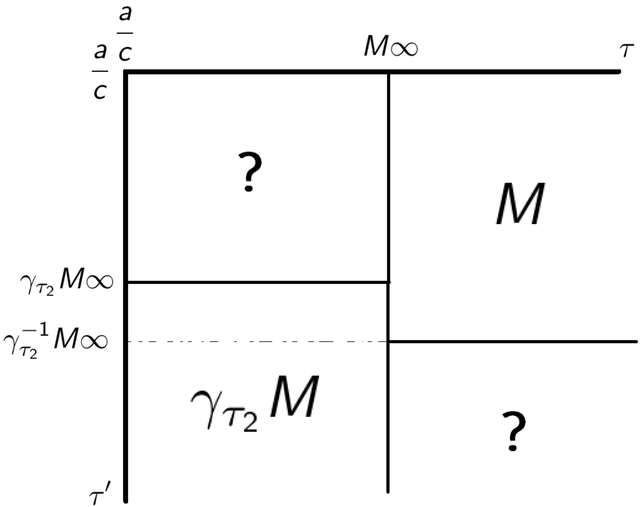


We see: let M interchange τ_2 and τ_2' and suppose that $\det(M) = -1$. Then there is a unique $k \in \mathbb{Z}$ such that $\gamma_{\tau_2}^k M\infty > \gamma_{\tau_2}\infty = a/c$. Choose this involution to be M. Then M and γM act like:





More MS Paint pictures



The final steps

What about the questionmarks? We use the associations

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One can check these interchange the regions appropriately. Why do they induce factorisations?

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$$p_1 b = q_1 a + r_1 c \quad \text{and} \quad p_2 b = q_2 a + r_2 c.$$

These equations are independent over $\mathbb{Q}$ so give a unique solution up to scalars. □

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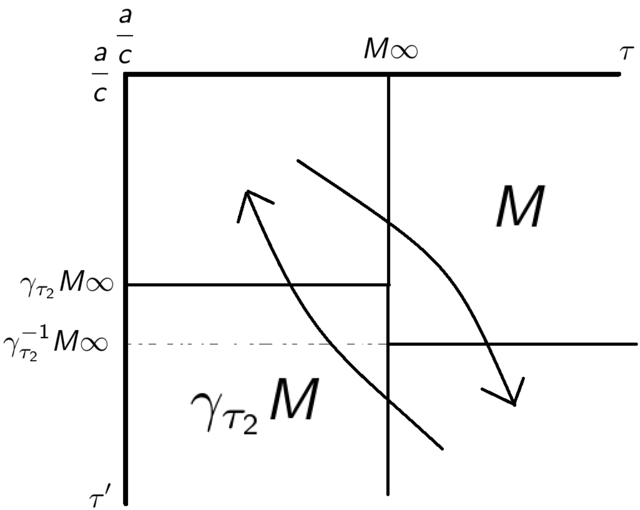
Consider $\gamma_{\tau_2} N$. It still interchanges τ_2, τ_2', but now maps τ_1 to $\gamma_{\tau_2}\tau_1'$. So our associations above are secretly acting by involutions and thus give factorisations.

Fin



$\frac{a}{c}$
$\frac{a}{c}$
$M\infty$
τ
M
$\gamma_{\tau_2} M\infty$
$\gamma_{\tau_2}^{-1} M\infty$
$\gamma_{\tau_2} M$
τ'