CM values of p-adic theta-functions

Mike Daas

Universiteit Leiden

3rd of April, 2023
Let $D_1, D_2 < 0$ be coprime discriminants and write $D = D_1 D_2$. Set

$$K_1 = \mathbb{Q}(\sqrt{D_1}), \quad K_2 = \mathbb{Q}(\sqrt{D_2}),$$

$$F = \mathbb{Q}(\sqrt{D}), \quad L = \mathbb{Q}(\sqrt{D_1}, \sqrt{D_2}).$$
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Let $\chi$ be the genus character of $L/F$: if $p \subset \mathfrak{o}_F$ is prime, then

$$\chi(p) = \begin{cases} 
1 & \text{if } p \text{ splits in } L/F; \\
-1 & \text{if } p \text{ is inert in } L/F.
\end{cases}$$
Let \( I \subset \mathcal{O}_F \) be an ideal. Define
\[
\rho(I) = \#\{J \subset \mathcal{O}_L \mid \text{Nm}_{F/F}(J) = I\};
\]
\[
\text{sp}(I) = \begin{cases} 
\nu & \text{if } \nu \text{ is unique with } \chi(\nu) = -1 \text{ and } \nu_p(I) \text{ odd}; \\
1 & \text{otherwise}.
\end{cases}
\]

**Important:** \( \rho(I) = 0 \) if and only if \( I \) has at least one special prime.
Let $I \subset \mathcal{O}_F$ be an ideal. Define
\[
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\[
sp(I) = \begin{cases} 
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**Important:** $\rho(I) = 0$ if and only if $I$ has at least one special prime.

Let $E_1$ be an elliptic curve with CM by $\mathcal{O}_1$ and $E_2$ an elliptic curve with CM by $\mathcal{O}_2$. Then by CM theory, $j(E_i) \in H_i$ for $i = 1, 2$, where $H_i$ is the Hilbert class field of $K_i$. For simplicity, assume $D_i \neq -3, -4$. 
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**Theorem (Gross-Zagier, 1984)**

Setting $\alpha = \nu \sqrt{D}$ and $D_F = (\sqrt{D})$, the following equality holds:

$$\log \text{Nm}_{Q}^{H_1H_2}(j(E_1) - j(E_2)) = \sum_{\nu \in D_F^{-1,+}, \ \text{tr}(\nu) = 1} \rho(\text{sp}(\alpha)\alpha)(\nu_{\text{sp}(\alpha)}(\alpha) + 1) \log(\text{sp}(\alpha)).$$
Example

Let $D_1 = -7$ and $D_2 = -19$. Then

$E_1 : y^2 + xy = x^3 - x^2 - 2x - 1, \quad j(E_1) = -3^35^3$;

$E_2 : y^2 + y = x^3 - 38x + 90, \quad j(E_2) = -2^{15}3^3$. 
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If $\nu \in \mathcal{D}_F^{-1,+}$ and $\text{tr}(\nu) = 1$, then

$$\alpha = \nu \sqrt{D} = \frac{x + \sqrt{D}}{2},$$

where $x^2 < D = 133$ and $x$ is odd.
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<table>
<thead>
<tr>
<th>$x$</th>
<th>$\pm 1$</th>
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<tbody>
<tr>
<td>$(D - x^2)/4$</td>
<td>$3 \cdot 11$</td>
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<td>$(\nu_{\text{sp}(\alpha)}(\alpha) + 1)/2$</td>
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Let’s check:

$$j(E_1) - j(E_2) = -3^3 5^3 + 2^{15} 3^3 = 881361$$
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j(E_1) - j(E_2) = -3^35^3 + 2^{15}3^3 = 881361 = 3^7 \cdot 13 \cdot 31.
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Zagier’s proof

First step is to rewrite the task at hand to proving

$$\log Nm_Q^{H_1 H_2}(j(E_1) - j(E_2)) = \sum_{\nu \in D_F^{-1,+}} \sum_{I | (\nu) \mathcal{D}_F \text{ and } \text{tr}(\nu) = 1} \chi(I) \log Nm(I).$$
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This reminds one of a diagonal restriction of a weight \(k\) Hilbert Eisenstein series:

\[
E_{k,\chi}(z, z) = \text{const} + \sum_{\nu \in \mathbb{D}_F^{-1,+}} \left( \sum_{I|\nu \mathbb{D}_F, \text{tr}(\nu)=n} \chi(I) Nm(I)^{k-1} \right) q^n.
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- Consider a family parametrised by a “weight” \( s \in \mathbb{C} \);
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- Apply a so-called holomorphic projection.

This must be in \( M_2(\text{SL}_2(\mathbb{Z})) = 0 \).
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This must be in $M_{2}(SL_{2}(\mathbb{Z})) = 0$. The explicit formula for its Fourier coefficients involves two terms, one for each side $\implies$ equal. Hard.
What is the \( j \)-function really?

Consider \( M_2(\mathbb{Q}) \); this is a quaternion algebra with norm det. Here, a maximal order is given by

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M_2(\mathbb{Z}) \subset M_2(\mathbb{Q}).
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Its units of norm 1 are precisely

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Since $M_2(\mathbb{Q})$ acts on $\mathbb{C}$, we may consider the quotient

$$Y_1(\mathbb{C}) = \text{SL}_2(\mathbb{Z}) \setminus \mathcal{H}.$$

Its function field is generated by the j-function.
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**Question**

What happens if we change $M_2(\mathbb{Q})$ to a different quaternion algebra?
Choose two primes $p \neq q$ and let $N = pq$. Let $B_N$ denote the quaternion algebra ramified at $p$ and $q$. Let $R_N$ be a maximal order and let $R_{N,1}^\times$ denote the subgroup of units of norm 1. We may choose an embedding $R_{N,1}^\times \to M_2(\mathbb{R})$ to form the quotient

$$X_N(\mathbb{C}) = R_{N,1}^\times \setminus \mathcal{H};$$

this is known as a Shimura curve, which is an algebraic curve over $\mathbb{Q}$. 

Proposition

The Shimura curve $X_N$ is of genus 0 if and only if $N \in \{6, 10, 22\}$. Suppose henceforth that we are in one of these cases. Then there exists a generator $j_N$ of the function field. Note this choice is not unique. Let $\tau_1, \tau_2 \in \mathcal{H}$ be CM points: fixed points in $\mathbb{C}$ of embeddings $\mathcal{O}_i \to R_N$. These exist when $p$ and $q$ are inert in both $K_i$. We want to study $N m_{j_N}(\tau_1) - j_N(\tau_2)$. They are algebraic by Shimura reciprocity.
Shimura curves

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$$\text{Nm}(j_N(\tau_1) - j_N(\tau_2)).$$

They are algebraic by Shimura reciprocity.
Let $B_q$ denote the quaternion algebra ramified at $q$ and $\infty$. Let $R_q$ be a maximal order. Now $B_q$ is definite, so consider the group

$$\Gamma_q^p = R_q [1/p]^\times$$

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Cerednik-Drinfeld

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where $\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$ is the $p$-adic upper half plane.
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**Theorem (Cerednik-Drinfeld)**

The quotient $\Gamma_q^p \backslash \mathcal{H}_p$ is as rigid $p$-adic space isomorphic to $X_N(\mathbb{C}_p)$.
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**Theorem (Cerednik-Drinfeld)**

The quotient $\Gamma_p^q \backslash \mathcal{H}_p$ is as rigid $p$-adic space isomorphic to $X_N(C_p)$.

**Question**

Which functions on $\Gamma_p^q \backslash \mathcal{H}_p$ correspond to $j_N$ on the other side?
Let $w_1, w_2 \in \mathcal{H}_p$. Then consider the expression

$$\Theta(w_1, w_2; z) = \prod_{\gamma \in \Gamma_q^p} \frac{z - \gamma w_1}{z - \gamma w_2}.$$ 

If $N \in \{6, 10, 22\}$, this expression descends to a rigid analytic meromorphic function on $\Gamma_q^p \backslash \mathcal{H}_p$ with divisor $[w_1] - [w_2]$. 

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**Theta functions**

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If $N \in \{6, 10, 22\}$, this expression descends to a rigid analytic meromorphic function on $\Gamma_q^p \backslash \mathcal{H}_p$ with divisor $[w_1] - [w_2]$. We obtain

$$\Theta(w_1, w_2; z) = c(w_1, w_2) \cdot \frac{j_N(z) - j_N(w_1)}{j_N(z) - j_N(w_2)},$$

for some $c(w_1, w_2) \in \mathbb{C}_p$. 

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$$\Theta(w_1, w_2; z) = c(w_1, w_2) \cdot \frac{j_N(z) - j_N(w_1)}{j_N(z) - j_N(w_2)}, \text{ for some } c(w_1, w_2) \in \mathbb{C}_p.$$ 

Now choose $w_1 = \tau_1$ and $w_2 = \tau'_1$; its Galois conjugate. Because we don’t know $c(\tau_1, \tau'_1)$, we opt to study instead

$$\frac{j_N(\tau_2) - j_N(\tau_1) j_N(\tau'_2) - j_N(\tau'_1)}{j_N(\tau_2) - j_N(\tau'_1) j_N(\tau'_2) - j_N(\tau_1)} = \prod_{\gamma \in \Gamma^p_q} \frac{\tau_2 - \gamma \tau_1}{\tau_2 - \gamma \tau'_1} \frac{\tau'_2 - \gamma \tau_1}{\tau'_2 - \gamma \tau'_1}.$$
The conjecture

One can $p$-adically approximate the quantity

$$J_p^{q}(\tau_1, \tau_2) := \prod_{\gamma \in \Gamma_p^q} \frac{\tau_2 - \gamma \tau_1}{\tau_2 - \gamma \tau'_1} \frac{\tau'_2 - \gamma \tau_1}{\tau'_2 - \gamma \tau'_1}$$

and recognise it as an algebraic number.
The conjecture

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and recognise it as an algebraic number.

There are four ideals $a$ of norm $N = pq$ in $\mathcal{O}_F$; they come in two $\text{Gal}(F/Q)$ orbits. Assign one orbit $\delta(a) = +1$, the other $\delta(a) = -1$. 
The conjecture

One can $p$-adically approximate the quantity

$$J_p^q(\tau_1, \tau_2) := \prod_{\gamma \in \Gamma_p^q} \frac{\tau_2 - \gamma \tau_1}{\tau_2 - \gamma \tau_1'} \frac{\tau_2' - \gamma \tau_1}{\tau_2' - \gamma \tau_1'}$$

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Conjecture (Giampietro, Darmon)

The expression

$$\log Nm_Q^{H_1 H_2} J_p^q(\tau_1, \tau_2)$$

is up to sign explicitly equal to

$$\sum_{Nm(a) = N} \delta(a) \sum_{\nu \in D_F^{-1, +}} \rho(sp(\alpha a^{-1}) \alpha a^{-1})(\nu_{sp(\alpha a^{-1})}(\alpha a^{-1}) + 1) \log(sp(\alpha a^{-1})).$$
Intermezzo: rewriting the theta-series

Let \( \tau_i \) be defined by an embedding \( \alpha_i : \mathcal{O}_i \rightarrow \mathbb{R}_q \) for \( i = 1, 2 \). This yields actions of the \( \mathcal{O}_i \) on \( B_q \), and as such, an action of \( L \) through

\[
\mathcal{O}_L \cong \mathcal{O}_1 \otimes \mathbb{Z} \mathcal{O}_2 : (x \otimes y) * b = \alpha_1(x)b\alpha_2(y).
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**Proposition**

There exists a unique $F$-linear quadratic form $\det_F : B_q \to F$ with the property that $\text{tr}_{F/\mathbb{Q}}(\det_F(b)) = \text{Nm}(b)$. 


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It satisfies

$$\frac{\tau_2 - b\tau_1 \tau'_2 - b\tau_1}{\tau_2 - b\tau'_1 \tau'_2 - b\tau'_1} = \frac{\det_F(b)}{\det'_F(b)}.$$  

As such,

$$\frac{\Theta(\tau_1, \tau'_1; \tau_2)}{\Theta(\tau_1, \tau'_1; \tau'_2)} = \prod_{b \in \Gamma_q^p} \frac{\det_F(b)}{\det'_F(b)}.$$
Let $\iota : B \rightarrow L$ be an isomorphism of $L$-vector spaces. For $b \in B_q$, define the ideal

$$I_b = \iota(b)/\iota(R_q).$$
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**Proposition**

Ranging over all possible pairs of embeddings $\alpha_1, \alpha_2$, the association $b \mapsto I_b$ establishes a bijection between

$$\{ b \in R_q/\{\pm 1\} \mid \det_F(b) = \nu \}$$

and

$$\{ I \subset O_L \mid Nm_{L/F}(I) = (\nu)q^{-1}D_F \}. $$
Rewriting the theta series further

Note that we have a correspondence

\[ \Gamma_p^q = R_q[1/p]_1^\times \leftrightarrow \lim_{n \to \infty} \{ b \in R_q \mid Nm(b) = p^{2n} \} . \]
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Taking the logarithm;

$$\log_p \frac{\Theta(\tau_1, \tau'_1; \tau_2)}{\Theta(\tau_1, \tau'_1; \tau'_2)} = \lim_{n \to \infty} \sum_{\text{tr}(\nu) = p^{2n}} \# \left\{ b \in R_q \mid \det_F(b) = \nu \right\} \log_p \left( \frac{\nu}{\nu'} \right)$$

$$= \lim_{n \to \infty} \sum_{\text{tr}(\nu) = p^{2n}} \rho((\nu)q^{-1}D_F) \log_p \left( \frac{\nu}{\nu'} \right).$$
Idea of the proof

We consider a $p$-stabilisation of the Hilbert Eisenstein series $E_{1,\chi}$. We wish to do the following three steps:

1. Find explicit family of Hilbert modular forms around $E_{1,\chi}$;
2. Take its derivative and compute its coefficients explicitly;
3. Apply the ordinary projection; argue why the result must vanish and obtain an equality by equating its coefficients to 0.

But writing down explicit families of modular forms is hard. Idea: Consider its associated Galois representation $1 \oplus \chi$; deform it infinitesimally ($\epsilon^2 = 0$) and explicitly; argue why these deformations are modular; explicitly compute its Fourier coefficients $a_\nu$ for all $\nu \gg 0$; the $\epsilon$-part then yields a meaningful derivative.
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- The \( \epsilon \)-part then yields a meaningful derivative.
Again let $\rho = 1 \oplus \chi$. Write $\tilde{\rho}$ for a deformation of $\rho$ to the ring $GL_2(\mathbb{Q}_p[\epsilon])$ where $\epsilon^2 = 0$. 

Proposition

Let $a, b, c, d : GF \to \mathbb{Q}_p$ be those functions such that $\tilde{\rho}(\tau) = (1 + \epsilon(a(\tau)b(\tau)c(\tau)d(\tau))) \cdot \rho(\tau)$ for all $\tau \in GF$. Then these functions must respectively satisfy $a, d \in \text{Hom}(GF, \mathbb{Q}_p)$, and $b, c \in H^1(GF, \mathbb{Q}_p[\chi])$.

Note that $\dim \text{Hom}(GF, \mathbb{Q}_p) = 1$ spanned by the $p$-adic cyclotomic character: $\phi_{\text{cyc}} : GF \to \text{Gal}(F(\zeta_{\infty}^p)/F) \overset{\sim}{\longrightarrow} \mathbb{Z} \times p \log p \overset{\sim}{\longrightarrow} \mathbb{Q}_p$. 

Mike Daas
CM values of $p$-adic theta-functions
3rd of April, 2023
Deforming $1 \oplus \chi$

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Let $a, b, c, d : G_F \rightarrow Q_p$ be those functions such that

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for all $\tau \in G_F$. Then these functions must respectively satisfy

$$a, d \in \text{Hom}(G_F, Q_p), \quad \text{and} \quad b, c \in H^1(G_F, Q_p(\chi)).$$
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$$\phi_p^{\text{cyc}} : G_F \rightarrow \text{Gal}(F(\zeta_p^\infty)/F) \cong \mathbb{Z}_p^\times \xrightarrow{\log_p} Q_p.$$
For simplicity, choose

$$\tilde{\rho}(\tau) = \begin{pmatrix} 1 + \phi_p^{\text{cyc}} \epsilon & 0 \\ 0 & \chi - \chi \phi_p^{\text{cyc}} \epsilon \end{pmatrix}. $$

Suppose that this deformation is modular. That would yield a morphism $\varphi : \mathbb{T} \to \mathbb{Q}_p[\epsilon]$, where $\mathbb{T}$ is Hida’s $p$-adic Hecke algebra, generated by adèles of $F$, but in practice:
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- operators \( T_l, \langle l \rangle \) for all primes \( l \subset \mathcal{O}_F \) prime to \( p \);
For simplicity, choose

\[ \tilde{\rho}(\tau) = \left( \begin{array}{cc} 1 + \phi_p^{\text{cyc}} \epsilon & 0 \\ 0 & \chi - \chi \phi_p^{\text{cyc}} \epsilon \end{array} \right). \]

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Images of Frobenius

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We recover $\varphi$ from

$$\varphi(T_l) = \text{tr}(\tilde{\rho}(\text{Frob}_l)) = \begin{cases} 2 & \text{if } \chi(l) = 1; \\ 2 \log_p(Nm(l)) \epsilon & \text{if } \chi(l) = -1. \end{cases}$$
Images of Frobenius

For simplicity, choose

\[ \bar{\rho}(\tau) = \begin{pmatrix} 1 + \phi_p^{\text{cyc}} \epsilon & 0 \\ 0 & \chi - \chi \phi_p^{\text{cyc}} \epsilon \end{pmatrix}. \]

Suppose that this deformation is modular. That would yield a morphism \( \varphi : \mathbb{T} \to \mathbb{Q}_p[\epsilon] \), where \( \mathbb{T} \) is Hida’s \( p \)-adic Hecke algebra, generated by adèles of \( F \), but in practice:

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We recover \( \varphi \) from

\[ \varphi(T_I) = \text{tr}(\bar{\rho}(\text{Frob}_I)) = \begin{cases} 2 & \text{if } \chi(I) = 1; \\ 2 \log_p(Nm(I)) \epsilon & \text{if } \chi(I) = -1. \end{cases} \]

Further, note that

\[ \varphi(\langle I \rangle Nm(I)) = \det(\bar{\rho}(\text{Frob}_I)) = \chi(I). \]
Solving the recursion

Remember the essential recursion relation

\[ T_{n+1} = T_n T_I - \langle I \rangle Nm(I) T_{n-1}. \]
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$$T_{n+1} = T_n T_I - \langle I \rangle Nm(I) T_{n-1}.$$ 

We can solve this in each case explicitly:

$$\varphi(T_n) = \begin{cases} 
  n + 1 & \text{if } \chi(I) = 1; \\
  (n + 1) \log_p (Nm(I)) \epsilon & \text{if } \chi(I) = -1 \text{ and } n \text{ is odd}; \\
  1 & \text{if } \chi(I) = -1 \text{ and } n \text{ is even}. 
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\end{cases} \]

Compare this to

\[ \sum_{I | l^n} \chi(I) = \rho(l^n) = \begin{cases} 
  n + 1 & \text{if } \chi(I) = 1; \\
  0 & \text{if } \chi(I) = -1 \text{ and } n \text{ is odd}; \\
  1 & \text{if } \chi(I) = -1 \text{ and } n \text{ is even.} 
\end{cases} \]
Unifying expressions

So we have

\[
\varphi(T_l^n) = \begin{cases} 
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\]

The integral parts are precisely \( \rho(l^n) \). We can thus write

\[
\varphi(T_l^n) = \rho(l^n) + \frac{1}{2} (n + 1)(1 - \chi(l^n)) \log_p(Nm(l)) \epsilon.
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The integral parts are precisely \(\rho(l^n)\). We can thus write

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\]

Let \(J \subset O_F\) be any ideal coprime to \(p\). Then

\[
\varphi(T_J) = \rho(J) + \frac{1}{2} \sum_{nm||J} \left( (n + 1)(1 - \chi(l^n)) \rho(J/l^n) \right) \log_p(Nm(l))\epsilon.
\]
\[ \varphi(T_J) = \rho(J) + \frac{1}{2} \sum_{m \mid J} \left( (n + 1) \left( 1 - \chi(l^n) \right) \rho(J/l^n) \right) \log_p(Nm(l)) \varepsilon. \]
The Magic Moment

\[ \varphi(T_J) = \rho(J) + \frac{1}{2} \sum_{m \parallel J} \left( (n + 1)(1 - \chi(I^n)) \rho(J/I^n) \right) \log_p(Nm(I)) \epsilon. \]

**Proposition**

If \( J \) is a primitive ideal coprime to \( p \), then the quantity

\[ \frac{1}{2} \sum_{m \parallel J} \left( (n + 1)(1 - \chi(I^n)) \rho(J/I^n) \right) \log_p(Nm(I)) \]

is equal to

\[ \rho(sp(J)J)(\nu_{sp(J)}(J) + 1) \log_p(sp(J)). \]
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is equal to

\[ \rho(sp(J)J)(\nu_{sp(J)}(J) + 1) \log_p (sp(J)). \]

Indeed, the factor \( 1 - \chi(l^n) = 0 \) unless \( l \) is a special prime of \( J \), and if \( J/l^n \) still has another special prime, \( \rho(J/l^n) = 0 \). It can thus only be non-zero when \( l \) is the unique special prime; the rest matches up.
Fourier coefficients

For convenience, let us denote

$$\log \mathcal{F}(J) = \rho(\text{sp}(J)J)(\nu_{\text{sp}(J)}(J) + 1) \log(\text{sp}(J)),$$

so that very concisely, for $J$ coprime to $p$,

$$\varphi(T_J) = \rho(J) + \log \mathcal{F}(J)\epsilon.$$

Let $\widetilde{J}$ denote the ideal $J$ without its prime factors dividing $p$. 
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Theorem

For any $\nu \in (D_{F}^{-1}q)^+$, let $J_{\nu} = (\nu)D_{F}q^{-1}$. Then it holds that

$$a_{\nu}(f_{q}) = (-1)^{\nu_{\pi}(\nu)} \left( \rho(\widetilde{J}_{\nu}) + \log_{p}(F(\widetilde{J}_{\nu})) \epsilon - \rho(\widetilde{J}_{\nu}) \log_{p}(\nu/\nu') \epsilon \right).$$
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$$a_\nu(f_q) = (-1)^{\nu\pi(\nu)}(\rho(\tilde{J}_\nu) + \log_p (F(\tilde{J}_\nu))\epsilon - \rho(\tilde{J}_\nu) \log_p (\nu/\nu')\epsilon).$$

The term \( \log(\nu/\nu') \) comes from \( \nu \) at the two places above p, as

$$\varphi(U_\pi) = -1 + \log_p (\pi)\epsilon; \quad \varphi(U_{\pi'}) = 1 + \log_p (\pi')\epsilon.$$
We take the diagonal restriction:

$$\text{diag}(f_q) = \sum_{n=1}^{\infty} \left( \sum_{\nu \in (D_F^{-1}q)^+, \text{tr}(\nu)=n} a_{\nu} \right) q^n.$$
Ordinary projection

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Taking its derivative amounts to considering only the $\epsilon$-part:

$$a_n(\partial \text{diag}(f_q)) = \sum_{\nu \in (D_F^{-1}q)^+ \atop \text{tr}(\nu) = n} (-1)^{\nu, \pi(\nu)} \left( \log_p (F(\tilde{J}_\nu)) - \rho(\tilde{J}_\nu) \log_p (\nu/\nu') \right).$$
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Now we take the ordinary projection $e^{\text{ord}}$:

$$a_1(e^{\text{ord}}(\partial \text{diag}(f_q))) = \lim_{n \to \infty} a_{p^{2n}}(\partial \text{diag}(f_q))$$

$$= \lim_{n \to \infty} \sum_{\nu \in (D_F^{-1}q)^+ \atop \text{tr}(\nu) = p^{2n}} (-1)^{\nu \pi(\nu)} \left( \log_p (\mathcal{F}(\tilde{J}_\nu)) - \rho(\tilde{J}_\nu) \log_p (\nu/\nu') \right).$$
One can show that the result must be a classical cusp form of weight 2 and level $N$, but one can check that

$$S_2(\Gamma_0(6)) = S_2(\Gamma_0(10)) = S_2(\Gamma_0(22)) = 0.$$
The crux!

One can show that the result must be a classical cusp form of weight 2 and level \(N\), but one can check that

\[
S_2(\Gamma_0(6)) = S_2(\Gamma_0(10)) = S_2(\Gamma_0(22)) = 0.
\]

In other words, if

\[
A := \lim_{n \to \infty} \sum_{\nu \in (D_F^{-1}q)^+, \text{tr}(\nu) = p^{2n}} (-1)^{\nu \pi(\nu)} \rho(\tilde{J}_\nu) \log_p (\nu/\nu')
\]

and

\[
B := \lim_{n \to \infty} \sum_{\nu \in (D_F^{-1}q)^+, \text{tr}(\nu) = p^{2n}} (-1)^{\nu \pi(\nu)} \log_p (\mathcal{F}(\tilde{J}_\nu)),
\]

then \(A = B\).
Conclusion

One can show that the limit in $B$ equals the first term:

\[ B = \sum_{\nu \in (D_F^{-1}q)^+} (-1)^{\nu \pi(\nu)} \log_p(\mathcal{F}(\tilde{J}_\nu)) \]

where

\[ \log \mathcal{F}(J) = \rho(\text{sp}(J)J)(\nu_{\text{sp}(J)}(J) + 1) \log_p(\text{sp}(J)). \]
One can show that the limit in $B$ equals the first term:

$$B = \sum_{\nu \in (\mathcal{D}^{-1}Fq)^+ \atop \text{tr}(\nu) = 1} (-1)^{\nu \pi(\nu)} \log_p (\mathcal{F}(\tilde{J}_\nu))$$

where

$$\log \mathcal{F}(J) = \rho(\text{sp}(J)J)(\nu_{\text{sp}(J)}(J) + 1) \log_p (\text{sp}(J)).$$

Recall our expression for the theta series

$$\log_p \frac{\Theta(\tau_1, \tau'_1; \tau_2)}{\Theta(\tau_1, \tau'_1; \tau'_2)} = \sum_{\text{tr}(\nu) = p^{2n}} \rho((\nu)q^{-1}\mathcal{D}_F) \log_p (\nu/\nu').$$

This shows that

$$A \approx \log_p \frac{\Theta(\tau_1, \tau'_1; \tau_2)}{\Theta(\tau_1, \tau'_1; \tau'_2)}.$$

This pretty much proves the conjecture.