The projective model structure on chain complexes of modules

Mike Daas

13th of May, 2020

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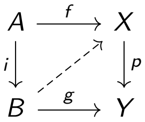
4. A commutative square admits a lift if either (i) i is a cofibration and p is an acyclic fibration, or (ii) i is an acyclic cofibration and p is a fibration.



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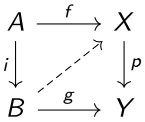

5. Any morphism f can be factored as $f = pi$ where (i) i is a cofibration and p is an acyclic fibration, or (ii) i is an acyclic cofibration and p is a fibration.

The main result

Let R be a ring. We write $\mathbf{Ch}_R$ for the category of complexes of left R-modules

$$M : \ldots \to M_2 \to M_1 \to M_0$$

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Define a morphism $f : M \to N$ to be

- a *weak equivalence* if the induced maps $H_k M \to H_k N$ on the homology are all isomorphisms;
- a *cofibration* if for each $k \geq 0$ the map $f_k : M_k \to N_k$ is injective and has a projective R-module as cokernel;
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Theorem

These choices make $\mathbf{Ch}_R$ into a model category.

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The first statement is trivial, and so are compositions of weak equivalences and fibrations. For the composition of cofibrations it suffices to show that B/A and C/B projective implies C/A projective. Indeed,

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$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{r'} & X' \end{array}$$

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$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{r} & X \\ f\downarrow & & \downarrow g & & \downarrow f \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{r'} & X' \end{array}$$

The bottom row induces maps $X'/X \to Y'/Y \to X'/X$ making X'/X a retract of Y'/Y. Since the latter is projective, so is the former.

The next condition

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If f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f.

If g is surjective, then so is every retract f of g. Indeed, if $x' \in X'$, then $i'(x') \in Y'$ and so $g(y) = i'(x')$ for some $y \in Y$. Then $f(r(y)) = r'(g(y)) = r'(i'(x')) = x'$. Hence the result also follows for fibrations.

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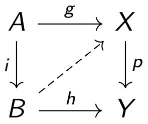
Lastly, for weak equivalences we note that if g is an isomorphism in any category, then so is every retract f of g. To see this, observe that $i'g^{-1}r$ is an inverse of f. Hence if g is an isomorphism on homology, then so is f. The result for weak equivalences follows. $\square$

Starting the fourth condition

Condition 4(i)

If i is a cofibration and p is an acyclic fibration, then the following diagram admits a lift.



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$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \xrightarrow{h} & Y \end{array}$$

Proof: We first claim that $p_0 : X_0 \to Y_0$ is surjective. To see this, since p_0 is an isomorphism on homology, for any $y_0 \in Y_0$ we can find some $x_0 \in X_0$ such that $p_0(x_0) = y_0 + \partial'(y_1)$ for some $y_1 \in Y_1$. But if $p_1(x_1) = y_1$, it follows that $p_0(x_0 - \partial(x_1)) = y_0$, showing surjectivity.

$$\begin{array}{ccccc} X_1 & \xrightarrow{\partial} & X_0 & \longrightarrow & X_0/\mathrm{im}(\partial) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_0} & & \downarrow\wr \\ Y_1 & \xrightarrow{\partial'} & Y_0 & \longrightarrow & Y_0/\mathrm{im}(\partial') \end{array}$$

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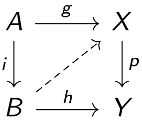


If we let $K_i = \ker(X_i \to Y_i)$ we can construct a short exact sequence of complexes

$$0 \to K \to X \to Y \to 0,$$

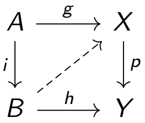
which has long exact sequence looking like

$$\ldots \to H_n K \to H_n X \xrightarrow{\sim} H_n Y \to H_{n-1} K \to \ldots$$

and so $H_n K = 0$ for all $n \geq 0$. We will now construct the map $B \to X$.

Constructing the lift

We will use induction. For $k = 0$, we note that since $i_0 : A_0 \to B_0$ is injective with projective quotient, we may write $B_0 \cong A_0 \oplus P_0$ with P_0 projective. Then h_0 induces a map $P_0 \to Y_0$ which lifts to a map $P_0 \to X_0$ because $p_0 : X_0 \to Y_0$ is surjective. Combined with $g_0 : A_0 \to X_0$ we obtain a map $f_0 : B_0 \to X_0$ making the diagram commute.



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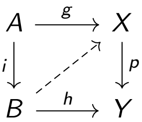

Now suppose that for some $k > 0$ we have constructed maps $f_j : B_j \to X_j$ for all $j < k$ such that $\partial f_j = f_{j-1}\partial$ for all $0 < j < k$ and each f_j makes the corresponding lift diagram commute. Construct $\tilde{f}_k$ using the decomposition $B_k \cong A_k \oplus P_k$ as above. Then the lift diagram commutes, but we need to make adjustments to ensure $\partial f_k = f_{k-1}\partial$.

Completing the proof of 4(i)

Define $\mathcal{E} = \partial \tilde{f}_k - f_{k-1}\partial$. Using the commuting diagrams we see

$$\partial\mathcal{E} = -\partial f_{k-1}\partial = f_{k-2}\partial^2 = 0, \quad p_{k-1}\mathcal{E} = \partial p_k \tilde{f}_k - h_{k-1}\partial = 0$$

and similarly $\mathcal{E} i_k = \partial g_k - f_{k-1} i_{k-1}\partial = 0$. Hence $\mathcal{E}$ induces a map

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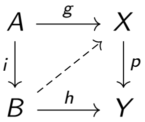

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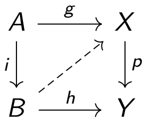

Since $H_{k-1}K = 0$, the map $K_k \to \ker(K_{k-1} \to K_{k-2})$ is surjective. We thus obtain a map $P_k \to K_k$, and so a map

$$\phi : B_k \to P_k \to K_k \to X_k.$$

We leave it as an exercise to check that $f_k = \tilde{f}_k - \phi$ satisfies all the necessary conditions. $\square$

Disks

Definition

For any R-module A and $n \geq 1$, define the *disk* $D_n(A)$ as the complex with $D_n(A)_n = D_n(A)_{n-1} = A$ and $D_n(A)_k = 0$ otherwise, with sole nontrivial boundary map id_A.

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\cdots & \longrightarrow & M_{n+1} & \longrightarrow & M_n & \xrightarrow{\partial} & M_{n-1} & \longrightarrow & M_{n-2} & \longrightarrow & \cdots
\end{array}$$

it is clear that for any morphism we must have $\partial f_n = f_{n-1}$, and so we obtain

$$\mathrm{Hom}_{\mathbf{Ch}_R}(D_n(A), M) \to \mathrm{Hom}_{\mathbf{Mod}_R}(A, M_n)$$

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which is a bijection. It follows that for A projective and epimorphism $M \to N$ in $\mathbf{Ch}_R$, any map $D_n(A) \to N$ lifts to a map $D_n(A) \to M$. It is not hard to see that this property is also satisfied for any direct sum $\oplus_i D_{n_i}(A_i)$. These complexes are called projective.

A useful lemma for 4(ii)

Lemma

Suppose that $P \in \mathbf{Ch}_R$ satisfies that P_k is projective and that $H_k P = 0$ for all $k \geq 0$. Then all of $K_k := \ker(P_k \to P_{k-1})$ are also projective, and

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Proof: For convenience we define

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For $k = 1$ the short exact sequence

$$0 \to P^{(2)} \to P^{(1)} \to P^{(1)}/P^{(2)} \cong D_1(P_0) \to 0$$

splits because the identity on $D_1(P_0)$ lifts to a map $D_1(P_0) \to P^{(1)}$ by the property discussed before.

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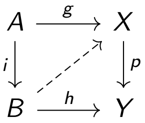
If i is an acyclic cofibration and p is a fibration, then the following diagram admits a lift.



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Defining $P_k = \mathrm{coker}(A_k \to B_k)$ and observing that i_k is injective, we obtain a short exact sequence of complexes

$$0 \to A \to B \to P \to 0,$$

Now P contains only projectives and the long exact sequence gives

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and so $H_k P = 0$ for all $k \geq 0$. So the previous lemma applies, and thus P is a direct sum of disks. As before, this means that the short exact sequence splits, and we obtain $B \cong A \oplus P$. Then a map $B \to X$ is obtained by taking $g : A \to X$ and combining it with any lift $P \to X$ of the map $P \to Y$, using that $p_k : X_k \to Y_k$ is surjective for $k \geq 1$.

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Let $\mathbf{C}$ be any category, $A \in \mathbf{C}$ be any object and let $B : \mathbf{Z}^+ \to \mathbf{C}$ be any functor. Then there are natural maps $B_n \to \text{colim } B$ and so natural maps $\text{Hom}(A, B_n) \to \text{Hom}(A, \text{colim } B)$. These combine to form a map

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If this is a bijection for every B, we say that $A \in \mathbf{C}$ is *sequentially small*.

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Let $\mathbf{Z}^+$ denote the category

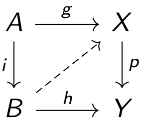
$$\star_1 \to \star_2 \to \star_3 \to \dots$$

Let $\mathbf{C}$ be any category, $A \in \mathbf{C}$ be any object and let $B : \mathbf{Z}^+ \to \mathbf{C}$ be any functor. Then there are natural maps $B_n \to \operatorname{colim} B$ and so natural maps $\operatorname{Hom}(A, B_n) \to \operatorname{Hom}(A, \operatorname{colim} B)$. These combine to form a map

$$\operatorname{colim} \operatorname{Hom}(A, B_n) \to \operatorname{Hom}(A, \operatorname{colim} B).$$

If this is a bijection for every B, we say that $A \in \mathbf{C}$ is *sequentially small*.

Recall that a map $p : X \to Y$ is said to have the RLP (*right lifting property*) with respect to $i : A \to B$ if every diagram of the following form admits a lift.



Some helpful results

Lemma 1

A map $f : X \to Y$ in $\mathbf{Ch}_R$ is a fibration if and only if it has the RLP with respect to the maps $0 \to D_n(R)$ for all $n \geq 1$.

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Proof: Recall that a map $D_n(R) \to Y$ is given by a map $R \to Y_n$, which can have any image of $1 \in R$. So if a lift always exists, f_n must be surjective. Similarly, if f_n is surjective for $n > 0$, then we can always choose a lift. $\square$

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Lemma 2

A map $f : X \to Y$ in $\mathbf{Ch}_R$ is an acyclic fibration if and only if it has the RLP with respect to the inclusions $K_{n-1}(R) \to D_n(R)$ for all $n \geq 1$, where $K_n(A)$ denotes the complex consisting of A in degree n and zeroes elsewhere.

Again some diagram chasing, which we leave as an exercise.

The gluing construction

Let $\mathcal{F} = \{f_i : A_i \to B_i\}_{i \in I}$ be a set of morphisms in $\mathbf{C}$. Let $p : X \to Y$ be any morphism. Let S_i be the set of pairs of morphisms (g, h) such that

$$\begin{array}{ccc} A_i & \xrightarrow{g} & X \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle p} \\ B_i & \xrightarrow{h} & Y \end{array}$$

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commutes for all i. Now let $G^1(\mathcal{F}, p)$ be the pushout of the diagram

$$\begin{array}{ccc} \coprod_{i \in I} \coprod_{(g,h) \in S_i} A_i & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle i_1} \\ \coprod_{i \in I} \coprod_{(g,h) \in S_i} B_i & \longrightarrow & G^1(\mathcal{F}, p) \end{array}$$

It is important that this pushout gives us a natural map $p_1 : G^1(\mathcal{F}, p) \to Y$ such that $p = p_1 i_1$.

The infinite gluing construction

Continuing, we can inductively define $G^k(\mathcal{F}, p) = G^1(\mathcal{F}, p_{k-1})$ and $p_k = (p_{k-1})_1$. We obtain a commutative diagram

$$\begin{array}{ccccccccc}
X & \xrightarrow{i_1} & G^1(\mathcal{F}, p) & \xrightarrow{i_2} & G^2(\mathcal{F}, p_1) & \xrightarrow{i_3} & G^3(\mathcal{F}, p_2) & \xrightarrow{i_4} & \dots \\
\downarrow{\scriptstyle p} & & {\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_3} & & \\
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Define $G^\infty(\mathcal{F}, p)$ to be the colimit of the top row. This comes naturally with a map $i_\infty : X \to G^\infty(\mathcal{F}, p)$ and a map $p_\infty : G^\infty(\mathcal{F}, p) \to Y$. These still satisfy $p = p_\infty i_\infty$.

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Proposition

Let $\mathcal{F} = \{f_i : A_i \to B_i\}_{i \in I}$ be such that each A_i is sequentially small. Then the map $p_\infty : G^\infty(\mathcal{F}, p) \to Y$ has the RLP with respect to *every* map in $\mathcal{F}$.

Proving the proposition

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Proof: Consider any diagram of the form

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Since A_i is sequentially small, we have a bijection

$$\operatorname{colim} \operatorname{Hom}(A, G^k(\mathcal{F}, p)) \to \operatorname{Hom}(A, G^\infty(\mathcal{F}, p)),$$

so g comes from a map $A \to G^k(\mathcal{F}, p)$ for some k. Therefore it factors

$$g : A_i \to G^k(\mathcal{F}, p) \to G^{k+1}(\mathcal{F}, p) \to G^\infty(\mathcal{F}, p).$$

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Hence we obtain a commutative diagram

$$\begin{array}{ccccccc} A_i & \longrightarrow & G^k(\mathcal{F}, p) & \xrightarrow{i_{k+1}} & G^{k+1}(\mathcal{F}, p) & \longrightarrow & G^\infty(\mathcal{F}, p) \\ \downarrow{\scriptstyle f_i} & & {\scriptstyle p_k}\downarrow & & {\scriptstyle p_{k+1}}\downarrow & & \downarrow{\scriptstyle p_\infty} \\ B_i & \xrightarrow{h} & Y & \xrightarrow{\mathrm{id}} & Y & \xrightarrow{\mathrm{id}} & Y \end{array}$$

But now the leftmost square contributes the pair (g, h) to the construction of $G^{k+1}(\mathcal{F}, p)$, so from the pushout diagram we have a natural map $B_i \to G^{k+1}(\mathcal{F}, p)$. Hence we obtain a natural map $B_i \to G^\infty(\mathcal{F}, p)$, which is the desired lift.

Proving condition 5

To show:

Any morphism f can be factored as $f = pi$ where (i) i is a cofibration and p is an acyclic fibration, or (ii) i is an acyclic cofibration and p is a fibration.

Proof: For (i), let $\mathcal{F}$ be the set of inclusions $K_n(R) \to D_n(R)$ and consider

$$X \xrightarrow{i_\infty} G^\infty(\mathcal{F}, f) \xrightarrow{p_\infty} Y.$$

Since R is sequentially small, we conclude that p_∞ has the RLP with respect to every map $K_{n-1}(R) \to D_n(R)$. Hence by the lemma, it is an acyclic fibration. The map i_∞ is a cofibration because each $G^k(\mathcal{F}, f)_n$ and so also $G^\infty(\mathcal{F}, f)_n$ is formed by taking the direct sum of X_n with many copies of R. Hence i_∞ is injective and its quotient is projective.

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Bridge to next week

Now that $\mathbf{Ch}_R$ is a model category, it enjoys all the general results about model categories.

Proposition

For any R-modules A and B and integers $m, n \geq 0$, we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Ch}_R)}(K_m(A), K_n(B)) \cong \mathrm{Ext}_R^{n-m}(A, B).$$

Here $\mathrm{Ho}(\mathbf{Ch}_R)$ is the *homotopy category* associated to the model catagory $\mathbf{Ch}_R$. This will be introduced next week.

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That was it, thanks for listening. Any questions?