The fppf and fpqc sites

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Memories from a distant past

Recall that a site on a category $\mathcal{C}$ is a set $\text{Cov}(\mathcal{C})$ consisting of families of morphisms with fixed target $\{U_i \to U\}$ in $\mathcal{C}$ satisfying the following three axioms:

- if $V \to U$ is an isomorphism, $\{V \to U\} \in \text{Cov}(\mathcal{C})$;
- if $\{U_i \to U\} \in \text{Cov}(\mathcal{C})$ and for all $i$, $\{U_{ij} \to U_i\} \in \text{Cov}(\mathcal{C})$, then also $\{U_{ij} \to U\} \in \text{Cov}(\mathcal{C})$;
- if $\{U_i \to U\} \in \text{Cov}(\mathcal{C})$ and $V \to U$, then $\{U_i \times_U V \to V\} \in \text{Cov}(\mathcal{C})$. 

If $P$ is a property of morphisms of schemes such that isomorphisms have $P$ and $P$ is stable under composition and base change, we may define the big $P$-site as the site defined by coverings $\{U_i \to U\}$ for which every morphism has $P$ and their images cover all of $U$. 

A presheaf on $\mathcal{C}$ is a functor $\mathcal{C}^{\text{op}} \to \text{Set}$. A presheaf $F$ on a site is called a sheaf if for all $\{U_i \to U\} \in \text{Cov}(\mathcal{C})$ the following diagram is an equaliser:

$$F(U) \to \prod_i F(U_i) \twoheadrightarrow \prod_{i, j} F(U_i \times U_j).$$
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- if $\{U_i \to U\} \in \text{Cov}(\mathcal{C})$ and $V \to U$, then $\{U_i \times_U V \to V\} \in \text{Cov}(\mathcal{C})$.

If $P$ is a property\footnote{We chose this letter for the sake of comedy.} of morphisms of schemes such that isomorphisms have $P$ and $P$ is stable under composition and base change, we may define the big $P$-site as the site defined by coverings $\{U_i \to U\}$ for which every morphism has $P$ and their images cover all of $U$. 

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If $P$ is a property\(^1\) of morphisms of schemes such that isomorphisms have $P$ and $P$ is stable under composition and base change, we may define the big $P$-site as the site defined by coverings $\{U_i \to U\}$ for which every morphism has $P$ and their images cover all of $U$.

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\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j).
$$

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Why do we sometimes call a site a topology?
The set of coverings $\text{Cov}(C)^2$ is called a Grothendieck pre-topology on $C$. But what does this have to do with a topology in the classical sense? Sure, we can define sheaves on sites, but can we also define a topology?  

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Definition
Let $C$ be a category and $U \in \text{Ob}(C)$. A sieve $S$ on $U$ comprises for every $T \in \text{Ob}(C)$ a subset $S(T)$ of all morphisms $T \to U$ satisfying that

$$T \to U \in S(T) \quad \text{and} \quad T' \to T \in C \implies T' \to T \to U \in S(T').$$

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T \to U \in S(T) \quad \text{and} \quad T' \to T \text{ in } \mathcal{C} \implies T' \to T \to U \in S(T').
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Given a sieve \( S \) on \( U \) and a morphism \( f : V \to U \), we may define the pullback sieve \( S \times_U V \) by
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T \to V \in (S \times_U V)(T) \iff T \to V \to U \in S(T).
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A topology $J$ on $C$ comprises for every $U \in \text{Ob}(C)$ a subset $J(U)$ of the set of all sieves on $U$, satisfying that $\text{Hom}(-, U) \in J(U)$ for all $U$, that $S \times_U V \in J(V)$ for all $S \in J(U)$ and $f : V \to U$, and that

$$S \in J(U) \text{ and } \forall f \in S(V), \ S' \times_U V \in J(V) \implies S' \in J(U).$$

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More abstract nonsense

Another definition
Given \( \{ f_i : U_i \to U \} \) there exists a smallest sieve on \( U \) containing all these morphisms, given by

\[
S(U) = \bigcup_i \{ f_i \circ g | g \text{ has target } U_i \}.
\]

It is called the sieve generated by these morphisms.

The legend of sheaves
Let \( J \) be a topology on a category \( C \) and let \( F \) be a presheaf of sets. We say that \( F \) is a sheaf on \( C \) if for every \( U \in \text{Ob}(C) \) and for every \( S \in J(U) \), the canonical map

\[
F(U) \cong \text{Hom}_{PSh(C)}(h_U, F) \to \text{Hom}_{PSh(C)}(S, F)
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is a bijection.

All of this can be found in detail on Stacks 00YW.
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Connecting the dots

**Theorem**

Let $\mathcal{C}$ be a site. We can define a topology $J$ on $\mathcal{C}$ by writing $J(U)$ for the set of sieves $S$ on $U$ with the property that there exists a covering $\{U_i \to U\}$ so that the sieve generated by this covering is contained in $S$. Furthermore, a presheaf is a sheaf for this topology if and only if it is a sheaf on the site.
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The proof is by pure abstract nonsense and can be found on Stacks 00ZB. We see that if two different sites generate the same topology, they must have the same categories of sheaves, but more is true (00VS).

Refinement: the last definition
Let $U = \{U_i \to U\}$ and $V = \{V_j \to U\}$ be two families of morphisms with fixed target. If for every $i$ there is a $j$ and a morphism $U_i \to V_j$ over $U$, we say that $U$ refines $V$.

Theorem
Let Cov$_1$ and Cov$_2$ define two sites on $C$. If for each $U$ in Cov$_1$ there exists some $V \in$ Cov$_2$ such that $V$ refines $U$, and vice versa, the categories of sheaves of these sites are equal.
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**Theorem**

Let \( C \) be a site. We can define a topology \( J \) on \( C \) by writing \( J(U) \) for the set of sieves \( S \) on \( U \) with the property that there exists a covering \( \{U_i \to U\} \) so that the sieve generated by this covering is contained in \( S \). Furthermore, a presheaf is a sheaf for this topology if and only if it is a sheaf on the site.

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**Refinement: the last definition**

Let \( \mathcal{U} = \{U_i \to U\} \) and \( \mathcal{V} = \{V_j \to U\} \) be two families of morphisms with fixed target. If for every \( i \) there is a \( j \) and a morphism \( U_i \to V_j \) over \( U \), we say that \( \mathcal{U} \) refines \( \mathcal{V} \).
Connecting the dots

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**Theorem**
Let $\text{Cov}_1$ and $\text{Cov}_2$ define two sites on $C$. If for each $\mathcal{U}$ in $\text{Cov}_1$ there exists some $\mathcal{V} \in \text{Cov}_2$ such that $\mathcal{V}$ refines $\mathcal{U}$, and vice versa, the categories of sheaves of these sites are equal.
A bestiary of topologies on Sch

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![Diagram](image-url)
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\[
\begin{array}{ccc}
\text{cl} & \rightarrow & \text{cdf} \\
\downarrow & & \downarrow \\
\text{Zar} & \rightarrow & \text{Nis} \\
\downarrow & & \downarrow \\
\text{cdp} & \rightarrow & \text{rh} \\
\downarrow & & \downarrow \\
\text{cdh} & \rightarrow & \text{eh} \\
\end{array}
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- $\text{h} = ???$
Smooth vs. étale

Recall that a morphism is étale if it is smooth of relative dimension 0. It turns out that the smooth site and the étale site are very similar. Their topologies are not the same, but we do have the following lemma (055S):

**Lemma**

Let \( X \to S \) be a smooth morphism of schemes and let \( s \in S \) be in its image. Then there exists an étale neighbourhood \( S' \to S \) of \( s \) and an \( S \)-morphism \( S' \to X \).

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S & & \\
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\]

**Corollary**
Let $\mathcal{U}$ be a smooth covering of a scheme $S$. Then there exists an étale covering $\mathcal{V}$ of $S$ that refines $\mathcal{U}$. Consequently, the categories of sheaves defined by the smooth site and the étale site coincide$^3$.

**Proof:** Apply the proposition above to every $s \in S$. 

$^3$Perhaps coindsite would have been more appropriate.
The long awaited fppf-site

Definition
An \textit{fppf} covering of a scheme $T$ is a family of morphisms $\{f_i : T_i \to T\}$ that are all flat and locally of finite presentation, and with the property that $T = \bigcup_i f_i(T_i)$. 

Examples
\begin{itemize}
\item Recall that $\text{Spec}(K) \to \text{Spec}(k)$ is \'{e}tale if and only if $K/k$ is finite and separable. For fppf we merely require finiteness.
\item The map $P^1_k\{\begin{pmatrix} 1:1 \end{pmatrix}\} \to P^1_k$ defined by $\begin{pmatrix} x:1 \end{pmatrix} \mapsto \begin{pmatrix} x^2:1 \end{pmatrix}$ and $\begin{pmatrix} 1:0 \end{pmatrix} \mapsto \begin{pmatrix} 1:0 \end{pmatrix}$ is an fppf-cover of $P^1_k$. Also just $P^1_k \to P^1_k$.
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This defines a site. Note that a smooth covering is certainly an fppf-covering, because smooth morphisms satisfy the above and additionally that all fibres are smooth. We thus have

$$\text{Zariski} < \text{étale} < \text{smooth} < \text{fppf}.$$ 

Lemma
Flat morphisms that are locally of finite presentation are (universally) open. (Stacks 01UA)
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- Recall that $\text{Spec}(K) \to \text{Spec}(k)$ is étale if and only if $K/k$ is finite and separable. For fppf we merely require finiteness.
- The map $\mathbb{P}^1_k \setminus \{(1 : 1)\} \to \mathbb{P}^1_k$ defined by $(x : 1) \mapsto (x^2 : 1)$ and $(1 : 0) \mapsto (1 : 0)$ is an fppf-cover of $\mathbb{P}^1_k$. Also just $\mathbb{P}^1_k \to \mathbb{P}^1_k$. 
Splitting up the étale site

It can happen that you want to prove something for all covers in a given site. Sometimes it is possible to reduce the problem to simpler coverings.

Proposition

Let \( \{ U_i \to S \} \) be an étale covering of \( S \). Then there exist:

- a Zariski open covering \( \{ V_j \to S \} \);
- for each \( j \) a surjective, finite, locally free morphism \( W_j \to V_j \);
- ...and a Zariski open covering \( \{ W_{j,k} \to W_j \} \),

such that \( \{ W_{j,k} \to S \} \) is a refinement of the covering \( \{ U_i \to S \} \).
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such that \( \{W_{j,k} \rightarrow S\} \) is a refinement of the covering \( \{U_i \rightarrow S\} \).

In other words, we can split up an étale covering into two Zariski coverings and a surjective finite locally free covering. This means that in order to prove something about all étale coverings, it can be enough to prove it for Zariski coverings and surj. fin. loc. free coverings\(^4\). (Stacks 04HE)

This has an application to the fppf-site.

\(^4\)Stacks actually gives an example of someone who used this abstract trick to prove something meaningful, which quite surprised me. Part of the reason for this footnote is to fill up the slide some more, for it would have been a little empty otherwise.
Splitting up the fppf site

Proposition
Let \( \{ S_i \to S \} \) be an fppf covering. Then there exist

- an étale covering \( \{ S'_a \to S \} \);
- for each \( a \) a surjective, finite, locally free morphism \( V_a \to S'_a \), such that \( \{ V_a \to S \} \) is an fppf-covering that refines \( \{ S_i \to S \} \).
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Let \( \{ S_i \rightarrow S \} \) be an fppf covering. Then there exist
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such that \( \{ V_a \rightarrow S \} \) is an fppf-covering that refines \( \{ S_i \rightarrow S \} \).

Corollary\(^5\)
Let \( \{ S_i \rightarrow S \} \) be an fppf covering of \( S \). Then there exist:
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  \item for each \( j \) a surjective, finite, locally free morphism \( W_j \rightarrow U_j \);
  \item ...and a Zariski open covering \( \{ W_{j,k} \rightarrow W_j \} \);
  \item for each \( k \) a surjective, finite, locally free morphism \( T_{j,k} \rightarrow W_{j,k} \),
end{itemize}
such that \( \{ T_{j,k} \rightarrow S \} \) is an fppf cover that refines \( \{ S_i \rightarrow S \} \).

Proof: Apply the result of the previous slide to the étale covering from the above proposition. More details at Stacks 05WM. \( \square \)

\(^5\)Algebraic geometry used double team! Student hurt itself in confusion.
An example of this splitting process

Let $k$ be algebraically closed. We consider the map

$$f : \mathbb{P}^1_k \setminus \{(1 : 1)\} \to \mathbb{P}^1_k, \quad (x : 1) \mapsto (x^2 : 1) \quad \text{and} \quad (1 : 0) \mapsto (1 : 0)$$

that we discussed before. How can we refine this by two Zariski coverings and two sets of surjective, finite, locally free morphisms?

---

Step 1: clearly $\text{id} : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is a Zariski covering.

Step 2: the map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ also defined by $x \mapsto x^2$ is surjective, finite.

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Question: Is $\text{Spec}(k[x, y]/(y^2 - x^3)) \to \text{Spec}(k)$ a smooth cover? And is it maybe an fppf cover?
An example of this splitting process

Let $k$ be algebraically closed. We consider the map

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The fabled fpqc-site

Definition
An fpqc covering \( \{ f_i : T_i \to T \}_{i \in I} \) of a scheme \( T \) consists of flat morphisms \( f_i \) such that for every affine open \( U \subset T \), there exist a finite subset \( J \subset I \) and affine opens \( V_j \subset T_j \) for all \( j \in J \), such that \( \bigcup f_j(V_j) = U \).

Lemma
An fppf covering is an fpqc covering.

Proof:
Indeed all \( f_i \) are flat, and by an earlier lemma, they are all open. Let \( U \subset T \) be an affine open. Then we can write \( f_i^{-1}(U) = \bigcup_i U_{ij} \) as a union of affine opens for each \( i \). But then \( U = \bigcup_i \bigcup_j f_i(U_{ij}) \), and since affine schemes are quasi-compact, we can find a finite subcover.

Zariski < étale < smooth < fppf < fpqc.

What this means for rings
A map \( \text{Spec}(B) \to \text{Spec}(A) \) is an fpqc covering if and only if \( A \to B \) is faithfully flat.
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Now we have established

\[ \text{Zariski} < \text{étale} < \text{smooth} < \text{fppf} < \text{fpqc}. \]

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Examples of fpqc things

A nugatory warning

Thanks to set theory, the fpqc site is a bit poorly behaved. For example, there does not exist a *set* $A$ of fpqc coverings of $T$ such that any fpqc covering of $T$ can be refined to one in $A$. This leads to pathological presheaves that do not have sheafifications. A solution is to only consider rings with a cardinality bounded by some strongly inaccessible cardinal. A better solution is to simply forget about it. See Stacks 022A.
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Two examples

- For any field $k$, the morphism $\coprod_{x \in \mathbb{A}^1_k} \text{Spec}(\mathcal{O}_{\mathbb{A}^1_k, x}) \to \mathbb{A}^1_k$ may be flat and surjective, but it is not quasi-compact. Hence this is not an fpqc covering.
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- The infinite coproduct $\coprod \text{Spec}(\mathbb{Q}) \to \text{Spec}(\mathbb{Q})$ is an fpqc covering that is clearly not quasi-compact.
A short note about sheaves

We’re almost there

Consider the presheaf $\mathcal{F} : (\text{Sch}/\mathbb{F}_p) \to \text{Ab}$ that sends a scheme $T$ over $\mathbb{F}_p$ to $\Gamma(T, \Omega_{T/\mathbb{F}_p})$. This is a sheaf on the Zariski site. However, $\text{Spec}(\mathbb{F}_p[x]) \to \text{Spec}(\mathbb{F}_p[y])$ mapping $y$ to $x^p$ is an fpqc covering, but on differentials it induces the zero map as $dy \mapsto d(x^p) = 0$. As a result, the diagram

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is not an equaliser. Hence this is not a sheaf on the fpqc site.

Representable presheaves are always sheaves on the fpqc site. In particular, $\mathcal{F} : \text{Sch} \to \text{Ab}$ that sends a scheme $T$ to $\Gamma(T, \mathcal{O}_T)$ is a sheaf on the fpqc site.

Some more information on the fpqc site can be found at Stacks 03NV.
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Thanks for listening!