The fppf and fpqc sites

Mike Daas

4th of March, 2021

Universiteit Leiden

Memories from a distant past

- Recall that a *site* on a category $\mathcal{C}$ is a set $\mathrm{Cov}(\mathcal{C})$ consisting of families of morphisms with *fixed target* $\{U_i \to U\}$ in $\mathcal{C}$ satisfying the following three axioms:
  - if $V \to U$ is an isomorphism, $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$;
  - if $\{U_i \to U\} \in \mathrm{Cov}(\mathcal{C})$ and for all i, $\{U_{ij} \to U_i\} \in \mathrm{Cov}(\mathcal{C})$, then also $\{U_{ij} \to U\} \in \mathrm{Cov}(\mathcal{C})$;
  - if $\{U_i \to U\} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$, then $\{U_i \times_U V \to V\} \in \mathrm{Cov}(\mathcal{C})$.

Memories from a distant past

- Recall that a *site* on a category $\mathcal{C}$ is a set $\mathrm{Cov}(\mathcal{C})$ consisting of families of morphisms with *fixed target* $\{U_i \to U\}$ in $\mathcal{C}$ satisfying the following three axioms:
  - if $V \to U$ is an isomorphism, $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$;
  - if $\{U_i \to U\} \in \mathrm{Cov}(\mathcal{C})$ and for all i, $\{U_{ij} \to U_i\} \in \mathrm{Cov}(\mathcal{C})$, then also $\{U_{ij} \to U\} \in \mathrm{Cov}(\mathcal{C})$;
  - if $\{U_i \to U\} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$, then $\{U_i \times_U V \to V\} \in \mathrm{Cov}(\mathcal{C})$.
- If P is a property[1] of morphisms of schemes such that isomorphisms have P and P is stable under composition and base change, we may define the *big P-site* as the site defined by coverings $\{U_i \to U\}$ for which every morphism has P and their images cover all of U.

[1] We chose this letter for the sake of comedy.

Memories from a distant past

- Recall that a *site* on a category $\mathcal{C}$ is a set $\mathrm{Cov}(\mathcal{C})$ consisting of families of morphisms with *fixed target* $\{U_i \to U\}$ in $\mathcal{C}$ satisfying the following three axioms:
  - if $V \to U$ is an isomorphism, $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$;
  - if $\{U_i \to U\} \in \mathrm{Cov}(\mathcal{C})$ and for all i, $\{U_{ij} \to U_i\} \in \mathrm{Cov}(\mathcal{C})$, then also $\{U_{ij} \to U\} \in \mathrm{Cov}(\mathcal{C})$;
  - if $\{U_i \to U\} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$, then $\{U_i \times_U V \to V\} \in \mathrm{Cov}(\mathcal{C})$.
- If P is a property[1] of morphisms of schemes such that isomorphisms have P and P is stable under composition and base change, we may define the *big P-site* as the site defined by coverings $\{U_i \to U\}$ for which every morphism has P and their images cover all of U.
- A *presheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. A presheaf $\mathcal{F}$ on a site is called a *sheaf* if for all $\{U_i \to U\} \in \mathrm{Cov}(\mathcal{C})$ the following diagram is an equaliser:

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j).$$

[1] We chose this letter for the sake of comedy.

Why do we sometimes call a site a topology?

The set of coverings $\text{Cov}(\mathcal{C})$[2] is called a Grothendieck *pre*-topology on $\mathcal{C}$.
But what does this have to do with a topology in the classical sense?
Sure, we can define sheaves on sites, but can we also define a *topology*?

[2] Never call your category "id" in this context.

Why do we sometimes call a site a topology?

The set of coverings $\mathrm{Cov}(\mathcal{C})$[2] is called a Grothendieck *pre*-topology on $\mathcal{C}$. But what does this have to do with a topology in the classical sense? Sure, we can define sheaves on sites, but can we also define a *topology*?

Definition

Let $\mathcal{C}$ be a category and $U \in \mathrm{Ob}(\mathcal{C})$. A *sieve* S on U comprises for every $T \in \mathrm{Ob}(\mathcal{C})$ a subset $S(T)$ of all morphisms $T \to U$ satisfying that

$$T \to U \in S(T) \quad \text{and} \quad T' \to T \text{ in } \mathcal{C} \implies T' \to T \to U \in S(T').$$

[2] Never call your category "id" in this context.

Why do we sometimes call a site a topology?

The set of coverings $\mathrm{Cov}(\mathcal{C})$[2] is called a Grothendieck *pre*-topology on $\mathcal{C}$. But what does this have to do with a topology in the classical sense? Sure, we can define sheaves on sites, but can we also define a *topology*?

Definition

Let $\mathcal{C}$ be a category and $U \in \mathrm{Ob}(\mathcal{C})$. A *sieve* S on U comprises for every $T \in \mathrm{Ob}(\mathcal{C})$ a subset $S(T)$ of all morphisms $T \to U$ satisfying that

$$T \to U \in S(T) \quad \text{and} \quad T' \to T \text{ in } \mathcal{C} \implies T' \to T \to U \in S(T').$$

Given a sieve S on U and a morphism $f : V \to U$, we may define the *pullback* sieve $S \times_U V$ by

$$T \to V \in (S \times_U V)(T) \iff T \to V \to U \in S(T).$$

[2] Never call your category "id" in this context.

Why do we sometimes call a site a topology?

The set of coverings $\mathrm{Cov}(\mathcal{C})$[2] is called a Grothendieck *pre*-topology on $\mathcal{C}$. But what does this have to do with a topology in the classical sense? Sure, we can define sheaves on sites, but can we also define a *topology*?

Definition

Let $\mathcal{C}$ be a category and $U \in \mathrm{Ob}(\mathcal{C})$. A *sieve* S on U comprises for every $T \in \mathrm{Ob}(\mathcal{C})$ a subset $S(T)$ of all morphisms $T \to U$ satisfying that

$$T \to U \in S(T) \quad \text{and} \quad T' \to T \text{ in } \mathcal{C} \implies T' \to T \to U \in S(T').$$

Given a sieve S on U and a morphism $f : V \to U$, we may define the *pullback* sieve $S \times_U V$ by

$$T \to V \in (S \times_U V)(T) \iff T \to V \to U \in S(T).$$

A *topology* J on $\mathcal{C}$ comprises for every $U \in \mathrm{Ob}(\mathcal{C})$ a subset $J(U)$ of the set of all sieves on U, satisfying that $\mathrm{Hom}(-, U) \in J(U)$ for all U, that $S \times_U V \in J(V)$ for all $S \in J(U)$ and $f : V \to U$, and that

$$S \in J(U) \text{ and } \forall f \in S(V),\ S' \times_U V \in J(V) \implies S' \in J(U).$$

[2] Never call your category "id" in this context.

More abstract nonsense

Another definition

Given $\{f_i : U_i \to U\}$ there exists a smallest sieve on U containing all these morphisms, given by

More abstract nonsense

Another definition

Given $\{f_i : U_i \to U\}$ there exists a smallest sieve on U containing all these morphisms, given by

$$S(U) = \bigcup_i \{f_i \circ g \ \mid \ g \text{ has target } U_i \}.$$

It is called the sieve *generated by* these morphisms.

More abstract nonsense

Another definition

Given $\{f_i : U_i \to U\}$ there exists a smallest sieve on U containing all these morphisms, given by

$$S(U) = \bigcup_i \{ f_i \circ g \mid g \text{ has target } U_i \}.$$

It is called the sieve *generated by* these morphisms.

The legend of sheaves

Let J be a topology on a category $\mathcal{C}$ and let $\mathcal{F}$ be a presheaf of sets. We say that $\mathcal{F}$ is a *sheaf* on C if for every $U \in \mathrm{Ob}(\mathcal{C})$ and for every $S \in J(U)$, the canonical map

$$\mathcal{F}(U) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_U, \mathcal{F}) \to \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(S, \mathcal{F})$$

is a bijection.

All of this can be found in detail on Stacks 00YW.

Connecting the dots

Theorem

Let $\mathcal{C}$ be a site. We can define a topology J on $\mathcal{C}$ by writing $J(U)$ for the set of sieves S on U with the property that there exists a covering $\{U_i \to U\}$ so that the sieve generated by this covering is contained in S. Furthermore, a presheaf is a sheaf for this topology if and only if it is a sheaf on the site.

Connecting the dots

Theorem

Let $\mathcal{C}$ be a site. We can define a topology J on $\mathcal{C}$ by writing $J(U)$ for the set of sieves S on U with the property that there exists a covering $\{U_i \to U\}$ so that the sieve generated by this covering is contained in S. Furthermore, a presheaf is a sheaf for this topology if and only if it is a sheaf on the site.

The proof is by pure abstract nonsense and can be found on Stacks 00ZB. We see that if two different sites generate the same topology, they must have the same categories of sheaves, but more is true (00VS).

Connecting the dots

Theorem

Let $\mathcal{C}$ be a site. We can define a topology J on $\mathcal{C}$ by writing $J(U)$ for the set of sieves S on U with the property that there exists a covering $\{U_i \to U\}$ so that the sieve generated by this covering is contained in S. Furthermore, a presheaf is a sheaf for this topology if and only if it is a sheaf on the site.

The proof is by pure abstract nonsense and can be found on Stacks 00ZB. We see that if two different sites generate the same topology, they must have the same categories of sheaves, but more is true (00VS).

Refinement: the last definition

Let $\mathcal{U} = \{U_i \to U\}$ and $\mathcal{V} = \{V_j \to U\}$ be two families of morphisms with fixed target. If for every i there is a j and a morphism $U_i \to V_j$ over U, we say that $\mathcal{U}$ *refines* $\mathcal{V}$.

Connecting the dots

Theorem

Let $\mathcal{C}$ be a site. We can define a topology J on $\mathcal{C}$ by writing $J(U)$ for the set of sieves S on U with the property that there exists a covering $\{U_i \to U\}$ so that the sieve generated by this covering is contained in S. Furthermore, a presheaf is a sheaf for this topology if and only if it is a sheaf on the site.

The proof is by pure abstract nonsense and can be found on Stacks 00ZB. We see that if two different sites generate the same topology, they must have the same categories of sheaves, but more is true (00VS).

Refinement: the last definition

Let $\mathcal{U} = \{U_i \to U\}$ and $\mathcal{V} = \{V_j \to U\}$ be two families of morphisms with fixed target. If for every i there is a j and a morphism $U_i \to V_j$ over U, we say that $\mathcal{U}$ *refines* $\mathcal{V}$.

Theorem

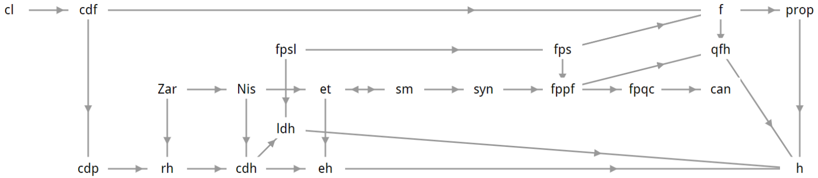
Let Cov_1 and Cov_2 define two sites on $\mathcal{C}$. If for each $\mathcal{U}$ in Cov_1 there exists some $\mathcal{V} \in \mathrm{Cov}_2$ such that $\mathcal{V}$ refines $\mathcal{U}$, and vice versa, the categories of sheaves of these sites are equal.

A bestiary of topologies on Sch

How many topologies are there really? E. A. Berengoltz BSc. introduced two already, and I will introduce some more. Do we know them all then?

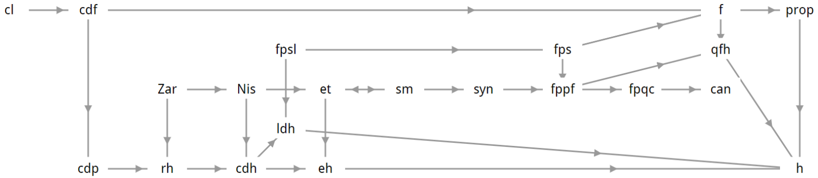
A bestiary of topologies on Sch

How many topologies are there really? E. A. Berengoltz BSc. introduced two already, and I will introduce some more. Do we know them all then?



A bestiary of topologies on Sch

How many topologies are there really? E. A. Berengoltz BSc. introduced two already, and I will introduce some more. Do we know them all then?

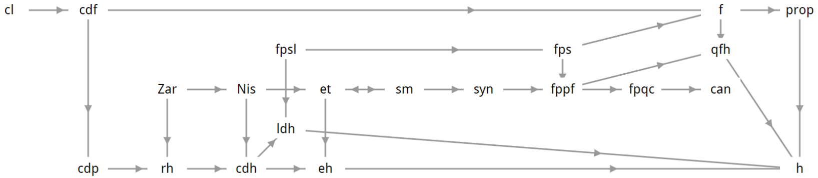


A short game: dead Russian guy (R), French stuff (F) or English (E)?

- cdf =

A bestiary of topologies on Sch

How many topologies are there really? E. A. Berengoltz BSc. introduced two already, and I will introduce some more. Do we know them all then?

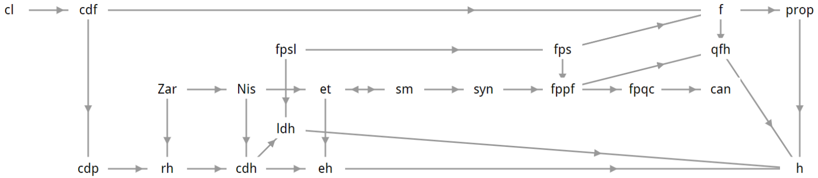


A short game: dead Russian guy (R), French stuff (F) or English (E)?

- cdf = Completely decomposed finite topology. (E)
- nis =

A bestiary of topologies on Sch

How many topologies are there really? E. A. Berengoltz BSc. introduced two already, and I will introduce some more. Do we know them all then?

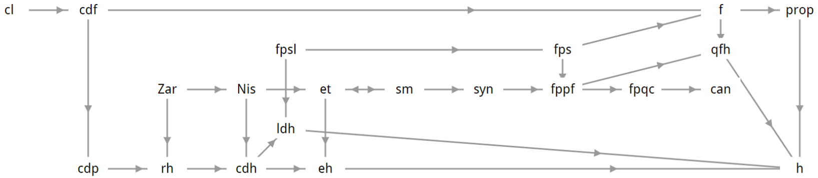


A short game: dead Russian guy (R), French stuff (F) or English (E)?

- cdf = Completely decomposed finite topology. (E)
- nis = Nisnevich topology. (R)
- can =

A bestiary of topologies on Sch

How many topologies are there really? E. A. Berengoltz BSc. introduced two already, and I will introduce some more. Do we know them all then?

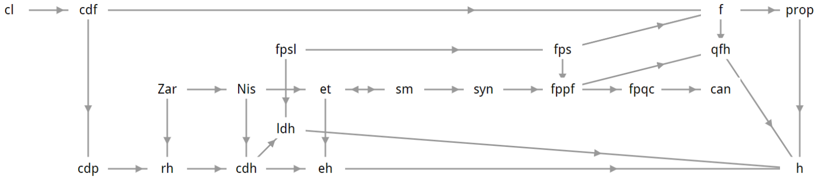


A short game: dead Russian guy (R), French stuff (F) or English (E)?

- cdf = Completely decomposed finite topology. (E)
- nis = Nisnevich topology. (R)
- can = Canonical topology. (E)
- syn =

A bestiary of topologies on Sch

How many topologies are there really? E. A. Berengoltz BSc. introduced two already, and I will introduce some more. Do we know them all then?

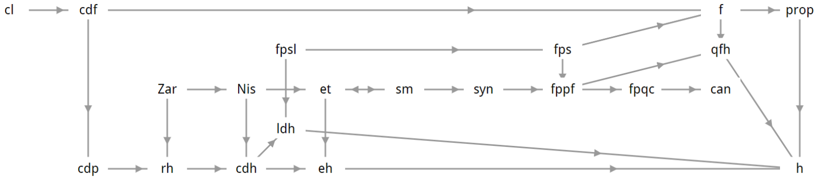


A short game: dead Russian guy (R), French stuff (F) or English (E)?

- cdf = Completely decomposed finite topology. (E)
- nis = Nisnevich topology. (R)
- can = Canonical topology. (E)
- syn = Syntomic / Syntomique topology. (E/F)
- fpqc =

A bestiary of topologies on Sch

How many topologies are there really? E. A. Berengoltz BSc. introduced two already, and I will introduce some more. Do we know them all then?

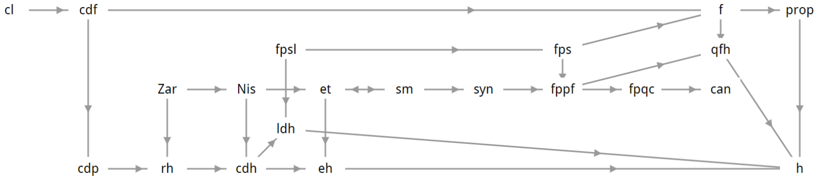


A short game: dead Russian guy (R), French stuff (F) or English (E)?

- cdf = Completely decomposed finite topology. (E)
- nis = Nisnevich topology. (R)
- can = Canonical topology. (E)
- syn = Syntomic / Syntomique topology. (E/F)
- fpqc = Fidèlement plat quasi-compacte. (F)
- f =

A bestiary of topologies on Sch

How many topologies are there really? E. A. Berengoltz BSc. introduced two already, and I will introduce some more. Do we know them all then?

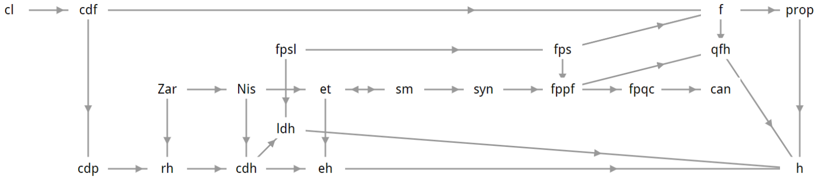


A short game: dead Russian guy (R), French stuff (F) or English (E)?

- cdf = Completely decomposed finite topology. (E)
- nis = Nisnevich topology. (R)
- can = Canonical topology. (E)
- syn = Syntomic / Syntomique topology. (E/F)
- fpqc = Fidèlement plat quasi-compacte. (F)
- f = Finite topology. (E)
- h =

A bestiary of topologies on Sch

How many topologies are there really? E. A. Berengoltz BSc. introduced two already, and I will introduce some more. Do we know them all then?



A short game: dead Russian guy (R), French stuff (F) or English (E)?

- cdf = Completely decomposed finite topology. (E)
- nis = Nisnevich topology. (R)
- can = Canonical topology. (E)
- syn = Syntomic / Syntomique topology. (E/F)
- fpqc = Fidèlement plat quasi-compacte. (F)
- f = Finite topology. (E)
- h = ???

Smooth vs. étale

Recall that a morphism is étale if it is smooth of relative dimension 0. It turns out that the smooth site and the étale site are very similar. Their topologies are *not* the same, but we do have the following lemma (055S):

Lemma

Let $X \to S$ be a smooth morphism of schemes and let $s \in S$ be in its image. Then there exists an étale neighbourhood $S' \to S$ of s and an S-morphism $S' \to X$.

$$\begin{array}{ccc} S' & \longrightarrow & X \\ & {\scriptstyle \text{ét}} \searrow & \downarrow {\scriptstyle \text{sm}} \\ & & S \end{array}$$

Smooth vs. étale

Recall that a morphism is étale if it is smooth of relative dimension 0. It turns out that the smooth site and the étale site are very similar. Their topologies are *not* the same, but we do have the following lemma (055S):

Lemma

Let $X \to S$ be a smooth morphism of schemes and let $s \in S$ be in its image. Then there exists an étale neighbourhood $S' \to S$ of s and an S-morphism $S' \to X$.

$$\begin{array}{ccc} S' & \longrightarrow & X \\ & \underset{\text{ét}}{\searrow} & \downarrow{\scriptstyle \text{sm}} \\ & & S \end{array}$$

Corollary

Let $\mathcal{U}$ be a smooth covering of a scheme S. Then there exists an étale covering $\mathcal{V}$ of S that refines $\mathcal{U}$. Consequently, the categories of sheaves defined by the smooth site and the étale site coincide[3].

Proof: Apply the proposition above to every $s \in S$. □

[3] Perhaps coin*site* would have been more appropriate.

The long awaited fppf-site

Definition

An *fppf* covering of a scheme T is a family of morphisms $\{f_i : T_i \to T\}$ that are all flat and locally of finite presentation, and with the property that $T = \bigcup_i f_i(T_i)$.

The long awaited fppf-site

Definition

An *fppf* covering of a scheme T is a family of morphisms $\{f_i : T_i \to T\}$ that are all flat and locally of finite presentation, and with the property that $T = \bigcup_i f_i(T_i)$.

This defines a site. Note that a smooth covering is certainly an fppf-covering, because smooth morphisms satisfy the above and additionally that all fibres are smooth. We thus have

$$\text{Zariski} < \text{étale} < \text{smooth} < \text{fppf}.$$

Lemma

Flat morphisms that are locally of finite presentation are (universally) open. (Stacks 01UA)

The long awaited fppf-site

Definition

An *fppf* covering of a scheme T is a family of morphisms $\{f_i : T_i \to T\}$ that are all flat and locally of finite presentation, and with the property that $T = \bigcup_i f_i(T_i)$.

This defines a site. Note that a smooth covering is certainly an fppf-covering, because smooth morphisms satisfy the above and additionally that all fibres are smooth. We thus have

$$\text{Zariski} < \text{étale} < \text{smooth} < \text{fppf}.$$

Lemma

Flat morphisms that are locally of finite presentation are (universally) open. (Stacks 01UA)

Examples

- Recall that $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ is étale if and only if K/k is finite and separable. For fppf we merely require finiteness.

The long awaited fppf-site

Definition

An *fppf* covering of a scheme T is a family of morphisms $\{f_i : T_i \to T\}$ that are all flat and locally of finite presentation, and with the property that $T = \bigcup_i f_i(T_i)$.

This defines a site. Note that a smooth covering is certainly an fppf-covering, because smooth morphisms satisfy the above and additionally that all fibres are smooth. We thus have

$$\text{Zariski} < \text{étale} < \text{smooth} < \text{fppf}.$$

Lemma

Flat morphisms that are locally of finite presentation are (universally) open. (Stacks 01UA)

Examples

- Recall that $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ is étale if and only if K/k is finite and separable. For fppf we merely require finiteness.
- The map $\mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k$ defined by $(x:1) \mapsto (x^2:1)$ and $(1:0) \mapsto (1:0)$ is an fppf-cover of $\mathbb{P}^1_k$. Also just $\mathbb{P}^1_k \to \mathbb{P}^1_k$.

Splitting up the étale site

It can happen that you want to prove something for all covers in a given site. Sometimes it is possible to reduce the problem to simpler coverings.

Proposition

Let $\{U_i \to S\}$ be an étale covering of S. Then there exist:

- a Zariski open covering $\{V_j \to S\}$;
- for each j a surjective, finite, locally free morphism $W_j \to V_j$;
- ...and a Zariski open covering $\{W_{j,k} \to W_j\}$,

such that $\{W_{j,k} \to S\}$ is a refinement of the covering $\{U_i \to S\}$.

Splitting up the étale site

It can happen that you want to prove something for all covers in a given site. Sometimes it is possible to reduce the problem to simpler coverings.

Proposition

Let $\{U_i \to S\}$ be an étale covering of S. Then there exist:

- a Zariski open covering $\{V_j \to S\}$;
- for each j a surjective, finite, locally free morphism $W_j \to V_j$;
- ...and a Zariski open covering $\{W_{j,k} \to W_j\}$,

such that $\{W_{j,k} \to S\}$ is a refinement of the covering $\{U_i \to S\}$.

In other words, we can split up an étale covering into two Zariski coverings and a surjecture finite locally free covering. This means that in order to prove something about all étale coverings, it can be enough to prove it for Zariski coverings and surj. fin. loc. free coverings[4]. (Stacks 04HE)

This has an application to the fppf-site.

[4] Stacks actually gives an example of someone who used this abstract trick to prove something meaningful, which quite surprised me. Part of the reason for this footnote is to fill up the slide some more, for it would have been a little empty otherwise.

Splitting up the fppf site

Proposition

Let $\{S_i \to S\}$ be an fppf covering. Then there exist

- an étale covering $\{S'_a \to S\}$;
- for each a a surjective, finite, locally free morphism $V_a \to S'_a$,

such that $\{V_a \to S\}$ is an fppf-covering that refines $\{S_i \to S\}$.

Splitting up the fppf site

Proposition

Let $\{S_i \to S\}$ be an fppf covering. Then there exist

- an étale covering $\{S'_a \to S\}$;
- for each a a surjective, finite, locally free morphism $V_a \to S'_a$,

such that $\{V_a \to S\}$ is an fppf-covering that refines $\{S_i \to S\}$.

Corollary[5]

Let $\{S_i \to S\}$ be an fppf covering of S. Then there exist:

- a Zariski open covering $\{U_j \to S\}$;
- for each j a surjective, finite, locally free morphism $W_j \to U_j$;
- ...and a Zariski open covering $\{W_{j,k} \to W_j\}$;
- for each k a surjective, finite, locally free morphism $T_{j,k} \to W_{j,k}$,

such that $\{T_{j,k} \to S\}$ is an fppf cover that refines $\{S_i \to S\}$.

Proof: Apply the result of the previous slide to the étale covering from the above proposition. More details at Stacks 05WM. □

[5] Algebraic geometry used double team! Student hurt itself in confusion.

An example of this splitting process

Let k be algebraically closed. We consider the map

$$f : \mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k, \quad (x:1) \mapsto (x^2:1) \quad \text{and} \quad (1:0) \mapsto (1:0)$$

that we discussed before. How can we refine this by two Zariski coverings and two sets of surjective, finite, locally free morphisms?

An example of this splitting process

Let k be algebraically closed. We consider the map

$$f : \mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k, \quad (x:1) \mapsto (x^2:1) \quad \text{and} \quad (1:0) \mapsto (1:0)$$

that we discussed before. How can we refine this by two Zariski coverings and two sets of surjective, finite, locally free morphisms?

- Step 1: clearly $\mathrm{id} : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is a Zariski covering.

An example of this splitting process

Let k be algebraically closed. We consider the map

$$f : \mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k, \quad (x:1) \mapsto (x^2:1) \quad \text{and} \quad (1:0) \mapsto (1:0)$$

that we discussed before. How can we refine this by two Zariski coverings and two sets of surjective, finite, locally free morphisms?

- Step 1: clearly $\mathrm{id} : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is a Zariski covering.
- Step 2: the map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ also defined by $x \mapsto x^2$ is surjective, finite and locally free.

An example of this splitting process

Let k be algebraically closed. We consider the map

$$f : \mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k, \quad (x:1) \mapsto (x^2:1) \quad \text{and} \quad (1:0) \mapsto (1:0)$$

that we discussed before. How can we refine this by two Zariski coverings and two sets of surjective, finite, locally free morphisms?

- Step 1: clearly $\mathrm{id} : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is a Zariski covering.
- Step 2: the map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ also defined by $x \mapsto x^2$ is surjective, finite and locally free.
- Step 3: the maps

  $$\mathrm{id}\colon \mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k,$$
  $$\mathrm{id}\colon \mathbb{P}^1_k \setminus \{(1:-1)\} \to \mathbb{P}^1_k$$

  form a Zariski covering of $\mathbb{P}^1_k$.

An example of this splitting process

Let k be algebraically closed. We consider the map

$$f : \mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k, \quad (x:1) \mapsto (x^2:1) \quad \text{and} \quad (1:0) \mapsto (1:0)$$

that we discussed before. How can we refine this by two Zariski coverings and two sets of surjective, finite, locally free morphisms?

- Step 1: clearly $\mathrm{id} : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is a Zariski covering.
- Step 2: the map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ also defined by $x \mapsto x^2$ is surjective, finite and locally free.
- Step 3: the maps
$$\mathrm{id}\colon \mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k,$$
$$\mathrm{id}\colon \mathbb{P}^1_k \setminus \{(1:-1)\} \to \mathbb{P}^1_k$$
form a Zariski covering of $\mathbb{P}^1_k$.
- Step 4: simply take the identity again. This refines f by taking $\mathrm{id} : \mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k \setminus \{(1:1)\}$ and the obvious map $-\mathrm{id} : \mathbb{P}^1_k \setminus \{(1:-1)\} \to \mathbb{P}^1_k \setminus \{(1:1)\}$. These are maps over $\mathbb{P}^1_k$.

An example of this splitting process

Let k be algebraically closed. We consider the map

$$f : \mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k, \quad (x:1) \mapsto (x^2:1) \quad \text{and} \quad (1:0) \mapsto (1:0)$$

that we discussed before. How can we refine this by two Zariski coverings and two sets of surjective, finite, locally free morphisms?

- Step 1: clearly $\mathrm{id} : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is a Zariski covering.
- Step 2: the map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ also defined by $x \mapsto x^2$ is surjective, finite and locally free.
- Step 3: the maps
$$\mathrm{id}\colon \mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k,$$
$$\mathrm{id}\colon \mathbb{P}^1_k \setminus \{(1:-1)\} \to \mathbb{P}^1_k$$
form a Zariski covering of $\mathbb{P}^1_k$.
- Step 4: simply take the identity again. This refines f by taking $\mathrm{id} : \mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k \setminus \{(1:1)\}$ and the obvious map $-\mathrm{id} : \mathbb{P}^1_k \setminus \{(1:-1)\} \to \mathbb{P}^1_k \setminus \{(1:1)\}$. These are maps over $\mathbb{P}^1_k$.

Question: Is $\mathrm{Spec}(k[x,y]/(y^2-x^3)) \to \mathrm{Spec}(k)$ a smooth cover?

An example of this splitting process

Let k be algebraically closed. We consider the map

$$f : \mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k, \quad (x:1) \mapsto (x^2:1) \quad \text{and} \quad (1:0) \mapsto (1:0)$$

that we discussed before. How can we refine this by two Zariski coverings and two sets of surjective, finite, locally free morphisms?

- Step 1: clearly $\mathrm{id} : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is a Zariski covering.
- Step 2: the map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ also defined by $x \mapsto x^2$ is surjective, finite and locally free.
- Step 3: the maps
$$\mathrm{id}\colon \mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k,$$
$$\mathrm{id}\colon \mathbb{P}^1_k \setminus \{(1:-1)\} \to \mathbb{P}^1_k$$
form a Zariski covering of $\mathbb{P}^1_k$.
- Step 4: simply take the identity again. This refines f by taking $\mathrm{id} : \mathbb{P}^1_k \setminus \{(1:1)\} \to \mathbb{P}^1_k \setminus \{(1:1)\}$ and the obvious map $-\mathrm{id} : \mathbb{P}^1_k \setminus \{(1:-1)\} \to \mathbb{P}^1_k \setminus \{(1:1)\}$. These are maps over $\mathbb{P}^1_k$.

Question: Is $\mathrm{Spec}(k[x,y]/(y^2 - x^3)) \to \mathrm{Spec}(k)$ a smooth cover? And is it maybe an fppf cover?

The fabled fpqc-site

Definition

An fpqc covering $\{f_i : T_i \to T\}_{i \in I}$ of a scheme T consists of flat morphisms f_i such that for every affine open $U \subset T$, there exist a *finite* subset $J \subset I$ and affine opens $V_j \subset T_j$ for all $j \in J$, such that $\bigcup f_j(V_j) = U$.

The fabled fpqc-site

Definition

An fpqc covering $\{f_i : T_i \to T\}_{i \in I}$ of a scheme T consists of flat morphisms f_i such that for every affine open $U \subset T$, there exist a *finite* subset $J \subset I$ and affine opens $V_j \subset T_j$ for all $j \in J$, such that $\bigcup f_j(V_j) = U$.

Lemma

An fppf covering is an fpqc covering.

Proof: Indeed all f_i are flat, and by an earlier lemma, they are all open. Let $U \subset T$ be an affine open. Then we can write $f_i^{-1}(U) = \bigcup U_{ij}$ as a union of affine opens for each i. But then $U = \bigcup_{i,j} f_i(U_{ij})$, and since affine schemes are quasi-compact, we can find a finite subcover. □

The fabled fpqc-site

Definition

An fpqc covering $\{f_i : T_i \to T\}_{i \in I}$ of a scheme T consists of flat morphisms f_i such that for every affine open $U \subset T$, there exist a *finite* subset $J \subset I$ and affine opens $V_j \subset T_j$ for all $j \in J$, such that $\bigcup f_j(V_j) = U$.

Lemma

An fppf covering is an fpqc covering.

Proof: Indeed all f_i are flat, and by an earlier lemma, they are all open. Let $U \subset T$ be an affine open. Then we can write $f_i^{-1}(U) = \bigcup U_{ij}$ as a union of affine opens for each i. But then $U = \bigcup_{i,j} f_i(U_{ij})$, and since affine schemes are quasi-compact, we can find a finite subcover. □

Now we have established

$$\text{Zariski} < \text{étale} < \text{smooth} < \text{fppf} < \text{fpqc}.$$

What this means for rings

A map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an fpqc covering if and only if $A \to B$ is faithfully flat.

Examples of fpqc things

A nugatory warning

Thanks to set theory, the fpqc site is a bit poorly behaved. For example, there does not exist a *set* A of fpqc coverings of T such that any fpqc covering of T can be refined to one in A. This leads to pathological presheaves that do not have sheafifications. A solution is to only consider rings with a cardinality bounded by some strongly inaccessible cardinal. A better solution is to simply forget about it. See Stacks 022A.

Examples of fpqc things

A nugatory warning

Thanks to set theory, the fpqc site is a bit poorly behaved. For example, there does not exist a *set* A of fpqc coverings of T such that any fpqc covering of T can be refined to one in A. This leads to pathological presheaves that do not have sheafifications. A solution is to only consider rings with a cardinality bounded by some strongly inaccessible cardinal. A better solution is to simply forget about it. See Stacks 022A.

Two examples

- For any field k, the morphism $\coprod_{x \in \mathbb{A}^1_k} \mathrm{Spec}(\mathcal{O}_{\mathbb{A}^1_k, x}) \to \mathbb{A}^1_k$ may be flat and surjective, but it is not quasi-compact. Hence this is *not* an fpqc covering.

Examples of fpqc things

A nugatory warning

Thanks to set theory, the fpqc site is a bit poorly behaved. For example, there does not exist a *set* A of fpqc coverings of T such that any fpqc covering of T can be refined to one in A. This leads to pathological presheaves that do not have sheafifications. A solution is to only consider rings with a cardinality bounded by some strongly inaccessible cardinal. A better solution is to simply forget about it. See Stacks 022A.

Two examples

- For any field k, the morphism $\coprod_{x \in \mathbb{A}^1_k} \mathrm{Spec}(\mathcal{O}_{\mathbb{A}^1_k,x}) \to \mathbb{A}^1_k$ may be flat and surjective, but it is not quasi-compact. Hence this is *not* an fpqc covering.
- The cover consisting of $\mathrm{Spec}(k) \to \mathrm{Spec}(k[x])$, so $\{0\} \to \mathbb{A}^1_k$, and $\mathrm{Spec}(k[x, x^{-1}]) \to \mathrm{Spec}(k[x])$, so $\mathbb{A}^1_k \setminus \{0\} \to \mathbb{A}^1_k$, is fpqc.

Examples of fpqc things

A nugatory warning

Thanks to set theory, the fpqc site is a bit poorly behaved. For example, there does not exist a *set* A of fpqc coverings of T such that any fpqc covering of T can be refined to one in A. This leads to pathological presheaves that do not have sheafifications. A solution is to only consider rings with a cardinality bounded by some strongly inaccessible cardinal. A better solution is to simply forget about it. See Stacks 022A.

Two examples

- For any field k, the morphism $\coprod_{x \in \mathbb{A}^1_k} \mathrm{Spec}(\mathcal{O}_{\mathbb{A}^1_k, x}) \to \mathbb{A}^1_k$ may be flat and surjective, but it is not quasi-compact. Hence this is *not* an fpqc covering.
- The cover consisting of $\mathrm{Spec}(k) \to \mathrm{Spec}(k[x])$, so $\{0\} \to \mathbb{A}^1_k$, and $\mathrm{Spec}(k[x, x^{-1}]) \to \mathrm{Spec}(k[x])$, so $\mathbb{A}^1_k \setminus \{0\} \to \mathbb{A}^1_k$, is fpqc.
- The infinite coproduct $\coprod \mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Q})$ is an fpqc covering that is clearly not quasi-compact.

A short note about sheaves

We're almost there

- Consider the presheaf $\mathcal{F} : (\mathrm{Sch}/\mathbb{F}_p) \to \mathrm{Ab}$ that sends a scheme T over $\mathbb{F}_p$ to $\Gamma(T, \Omega_{T/\mathbb{F}_p})$. This is a sheaf on the Zariski site. However, $\mathrm{Spec}(\mathbb{F}_p[x]) \to \mathrm{Spec}(\mathbb{F}_p[y])$ mapping y to x^p is an fpqc covering, but on differentials it induces the zero map as $dy \mapsto d(x^p) = 0$. As a result, the diagram

$$\mathbb{F}_p \cdot dy \xrightarrow{0} \mathbb{F}_p \cdot dx \rightrightarrows \mathcal{F}\Big(\mathrm{Spec}(\mathbb{F}_p[x]) \otimes_{\mathrm{Spec}(\mathbb{F}_p[y])} \mathrm{Spec}(\mathbb{F}_p[x])\Big)$$

  is not an equaliser. Hence this is not a sheaf on the fpqc site.

A short note about sheaves

We're almost there

- Consider the presheaf $\mathcal{F} : (\mathrm{Sch}/\mathbb{F}_p) \to \mathrm{Ab}$ that sends a scheme T over $\mathbb{F}_p$ to $\Gamma(T, \Omega_{T/\mathbb{F}_p})$. This is a sheaf on the Zariski site. However, $\mathrm{Spec}(\mathbb{F}_p[x]) \to \mathrm{Spec}(\mathbb{F}_p[y])$ mapping y to x^p is an fpqc covering, but on differentials it induces the zero map as $dy \mapsto d(x^p) = 0$. As a result, the diagram

$$\mathbb{F}_p \cdot dy \xrightarrow{0} \mathbb{F}_p \cdot dx \rightrightarrows \mathcal{F}\Big(\mathrm{Spec}(\mathbb{F}_p[x]) \otimes_{\mathrm{Spec}(\mathbb{F}_p[y])} \mathrm{Spec}(\mathbb{F}_p[x])\Big)$$

  is not an equaliser. Hence this is not a sheaf on the fpqc site.
- Representable presheaves are always sheaves on the fpqc site. In particular, $\mathcal{F} : \mathrm{Sch} \to \mathrm{Ab}$ that sends a scheme T to $\Gamma(T, \mathcal{O}_T)$ is a sheaf on the fpqc site.

Some more information on the fpqc site can be found at Stacks 03NV.

Fin

Thanks for listening!