1 The conjecture

Let K/k be an abelian extension of number fields and let S be a finite set of places of k containing the archimedean places and all the places that ramify in K. Set $G = \mathrm{Gal}(K/k)$ and let $\chi \in \widehat{G}$ be a character. By class field theory, every unramified prime ideal $\mathfrak{p}$ of k induces a Frobenius $\mathrm{Frob}_\mathfrak{p} \in G$.

Definition 1. The *Artin L-series* $L_S(s, \chi)$ is defined for $\Re(s) > 1$ by

$$L_S(s,\chi) = \prod_{\mathfrak{p}\notin S} (1 - \chi(\mathrm{Frob}_\mathfrak{p}) \mathrm{Nm}\, \mathfrak{p}^{-s})^{-1}.$$

We write $\chi = 1$ for the trivial character of G.

Lemma 2. *The function $L_S(s, 1)$ has a zero of order $\#S - 1$ at $s = 0$.*

Proof. By definition, $L_S(s, 1)$ is related to the usual *Dedekind ζ-function*

$$\zeta_k(s,\chi) = \sum_{I\subset \mathcal{O}_k} \mathrm{Nm}\, I^{-s} = \prod_{\mathfrak{p}} (1 - \mathrm{Nm}\, \mathfrak{p}^{-s})^{-1}$$

through

$$L_S(s,1) = \zeta_k(s,\chi) \cdot \prod_{\mathfrak{p}\in S} (1 - \mathrm{Nm}\, \mathfrak{p}^{-s}).$$

Each of the added factors has a simple zero at $s = 0$, whereas the Dedekind ζ-function has a zero of order $\mathrm{rk}\, \mathcal{O}_k^\times$. As such, $L_S(s, 1)$ has a zero of order

$$\#S_{\text{finite}} + \mathrm{rk}\, \mathcal{O}_k^\times = \#S - 1.$$

Note that we used that

$$\mathrm{rk}\, \mathcal{O}_k^\times = \#\text{Archimedean places} - 1,$$

which is Dirichlet's unit theorem. □

Let $h(k)$ denote the class number of k and $w(k)$ its number of roots of unity. We recall the definition of the regulator $R(k)$. Let $\{\sigma_1, \ldots, \sigma_{r+1}\}$ be a complete set of pairwise non-conjugate archimedean places of k and let $\{u_1, \ldots, u_r\}$ be a basis for $\mathcal{O}_k^\times$. Then

$$R(k) = |\det(n_i \log|\sigma_i(u_j)|_{i,j=1}^{r})|,$$

where $n_i = 1$ if σ_i is real and $n_i = 2$ otherwise.

Corollary 3. *The special value at $s = 0$ of $L_S(s, 1)$ is given by*

$$\lim_{s\to 0} \frac{L_S(s,1)}{s^{\#S-1}} = -\frac{h(k)R(k)}{w(k)} \cdot \prod_{\mathfrak{p}\in S} \log(\mathrm{Nm}\, \mathfrak{p}).$$

Proof. This follows from the identity

$$\frac{L_S(s,1)}{s^{\#S-1}} = \frac{\zeta_k(s,\chi)}{s^{\#\mathrm{Arch}-1}} \cdot \prod_{\mathfrak{p}\in S} \frac{1 - \mathrm{Nm}\, \mathfrak{p}^{-s}}{s}.$$

Namely, the Dedekind ζ-function has a residue as in the corollary and each of the factors clearly has residue $\log(\mathrm{Nm}\, \mathfrak{p})$. □

Proposition 4. *In general, $L_S(s,\chi)$ has a zero at $s = 0$ of order the number of primes $\mathfrak{q} \in S$ such that χ restricted to the decomposition group of $\mathfrak{q}$ is trivial.*

We opt to omit the proof of this statement. One may wonder what the special value of $L_S(s,\chi)$ is in general. Let us consider the following situation: S is a set of places containing all archimedean places, all places that ramify in K/k and *one distinguished prime* $\mathfrak{p}$ that splits completely in K/k that may itself be archimedian or ramified. The effect of the splitting condition on $\mathfrak{p}$ is that its decomposition group in G will be trivial and as such, by the proposition above, $L_S(s,\chi)$ will certainly vanish at $s = 0$. Let $|-|_{\mathfrak{P}}$ denote the standard norm on the $\mathfrak{P}$-adic completion. Stark conjectures the following.

Conjecture 5. *Let $\mathfrak{P}$ be any prime of K above $\mathfrak{p}$. Then there is some $\epsilon \in K$ that is a unit at all places of K not above S, such that*

- *for all characters χ of G,*

$$L_S'(0,\chi) = -\frac{1}{w(K)} \sum_{\gamma \in G} \chi(\gamma) \log |\epsilon^{\gamma}|_{\mathfrak{P}};$$

- $K(\epsilon^{1/w(K)})$ *is an abelian extension of k.*

This conjecture is known for $k = \mathbb{Q}$ or k an imaginary quadratic number field, but it is still open in this generality. One may next wonder what happens in case we choose not only one distinguished prime, but instead two. This leads to the following question.

Question 6. *Let S contain two distinguished primes and let $\mathfrak{P}_1$ and $\mathfrak{P}_2$ be two primes of K lying over them. Do there exist $\epsilon_1, \epsilon_2 \in K$, units at all places of K not above S, such that*

$$\frac{1}{2} L_S''(0,\chi) = \det \left[\sum_{\gamma \in G} \chi(\gamma) \begin{pmatrix} -\frac{1}{w(K)} \log |\epsilon_1^{\gamma}|_{\mathfrak{P}_1} & -\frac{1}{w(K)} \log |\epsilon_1^{\gamma}|_{\mathfrak{P}_2} \\ -\frac{1}{w(K)} \log |\epsilon_2^{\gamma}|_{\mathfrak{P}_1} & -\frac{1}{w(K)} \log |\epsilon_2^{\gamma}|_{\mathfrak{P}_2} \end{pmatrix} \right],$$

and such that $K(\epsilon_1^{1/w(K)})$ and $K(\epsilon_2^{1/w(K)})$ are abelian over k? Further, can one require these fields to coincide, and that $\epsilon_i^{\gamma} \mathcal{O}_K = \epsilon_i \mathcal{O}_K$ for all $\gamma \in G$ and $i \in \{1, 2\}$?

There is reason to doubt that the first equality can always be made to hold. However, we will work out an example that shows that it can at least sometimes hold. Before ending this section, we will record one more fact about Artin L-series.

Proposition 7. *The following equality holds:*

$$\prod_{\chi \in \widehat{G}} L_S(s,\chi) = \zeta_K(s,\chi) \cdot \prod_{\mathfrak{P} | \mathfrak{p} \in S_{\text{finite}}} (1 - \text{Nm}\, \mathfrak{P}^{-s}).$$

Proof. We may rewrite the left hand side like

$$\prod_{\chi \in \widehat{G}} L_S(s,\chi) = \prod_{\chi \in \widehat{G}} \prod_{\mathfrak{p} \notin S} (1 - \chi(\text{Frob}_{\mathfrak{p}}) \text{Nm}\, \mathfrak{p}^{-s})^{-1} = \prod_{\mathfrak{p} \notin S} \prod_{\chi \in \widehat{G}} (1 - \chi(\text{Frob}_{\mathfrak{p}}) \text{Nm}\, \mathfrak{p}^{-s})^{-1}.$$

We treat this prime by prime. Suppose first that $\mathfrak{p} \notin S$. It now suffices to prove the equality

$$\prod_{\chi \in \widehat{G}} (1 - \chi(\mathrm{Frob}_{\mathfrak{p}}) \mathrm{Nm}\, \mathfrak{p}^{-s}) = \prod_{\mathfrak{P} | \mathfrak{p}} (1 - \mathrm{Nm}\, \mathfrak{P}^{-s}).$$

Let us write e for the number of primes above in K. Then the right hand side is simply equal to

$$(1 - \mathrm{Nm}\, \mathfrak{p}^{-fs})^{e},$$

where $\mathfrak{P}$ has residue field degree f with respect to k. Also, $\mathrm{Frob}_{\mathfrak{p}}$ will be of order f in G and as such the values $\chi(\mathrm{Frob}_{\mathfrak{p}})$ will simply range over the set $\{1, \zeta_f, \ldots, \zeta_f^{f-1}\}$ precisely e times. Hence the left hand side becomes

$$\prod_{k=0}^{f-1} (1 - \zeta_f^k \mathrm{Nm}\, \mathfrak{p}^{-s})^{e}.$$

The desired conclusion now follows from the obvious identity

$$\prod_{k=0}^{f-1} (1 - \zeta_f^k X) = 1 - X^f.$$

Finally, the finite primes $\mathfrak{p} \in S$ are completely omitted from the left hand side, so we are simply to cancel their contributions from the right hand side as well. □

Disclaimer: This note is completely based on the 1997/1998 paper by David Grant, titled *Units from 5-torsion on the Jacobian of* $y^2 = x^5 + 1/4$ *and the Conjectures of Stark and Rubin*, [2], which is in turn heavily supported by [1]. Almost no original work has been done by yours truly in establishing this exposition. We provide many details for innocent looking steps that Grant understood to be well known, but we will often omit parts of Grant's impressive calculations, as copying those will serve neither the enjoyment of the author of this note, nor that of the reader. Suggestions to improve this note are most welcome and can be sent to m.a.daas@math.leidenuniv.nl.

2 The fields k and K

This section is mainly devoted to introduction our notation, together with some amusing calculations in the typical style of an introductory course in algebraic number theory.

- Let ζ denote a primitive fifth root of unity.
- Let $k = \mathbb{Q}(\zeta)$ be the corresponding quatric cyclotomic extension.
- Set $\epsilon = -\zeta^2 - \zeta^3$.
- Set $\lambda = 1 - \zeta$.

We will start off with some very elementary properties.

Lemma 8. *The following statements are true.*

- *The ring of integers of* k *is given by* $\mathcal{O}_k = \mathbb{Z}[\zeta]$
- *The number* ϵ *is equal to the golden ratio. As such,* $\epsilon + \epsilon^{-1} = 2\epsilon - 1 = \sqrt{5}$*.*
- *It is also a fundamental unit of* k*, and as such,* $R(k) = |2\log|\epsilon||$*.*
- *The class number of* k *is* $h(k) = 1$ *and* k *contains 10 roots of unity, so* $w(k) = 10$*.*
- *The discriminant of* k *is* $D(k) = 5^3$ *and* λ *generates the only prime above 5 in* k*.*

Proof. Since k is generated by $\Phi_5(X) = X^4 + X^3 + X^2 + X + 1$, which is quickly computed to have discriminant 125, we need only check the prime 5 to use the Kummer-Dedekind theorem to compute $\mathcal{O}_k$. However, since $25 \nmid 5 = \Phi_5(1)$, using that $\Phi_5(X) = (X-1)^4$ mod 5, it follows that $\mathcal{O}_k = \mathbb{Z}[\zeta]$. This also shows that λ generates the only prime above 5 in k. We next compute that

$$(2\epsilon - 1)^2 = (2\zeta^{-3} + 2\zeta^{-2} + 1)^2 = 4\zeta^{-1} + 4\zeta^{-4} + 1 + 4\zeta^{-3} + 4\zeta^{-2} + 8 = -4 + 1 + 8 = 5,$$

from which the claims about ϵ follow. Since $(1,1)$ is trivially the smallest solution to the unit equation $x^2 + xy - y^2 = 1$, ϵ is a fundamental unit in $\mathbb{Q}(\sqrt{5})$. Now if ϵ were not fundamental in k as well, then it would be the nth power of some $\eta \in k \setminus \mathbb{Q}(\sqrt{5})$. As such, we would have $k = \mathbb{Q}(\sqrt{5}, \eta)$. Since this extension is Galois, it would contain all nth roots of ϵ; in particular at least one real one, so we may choose $\eta \in \mathbb{R}$. However, this would yield a real embedding of k; a contradiction. For the class number of k, we compute its Minkowski bound to be

$$M_k = \left(\frac{4}{\pi}\right)^2 \cdot \frac{4!}{4^4} \cdot \sqrt{125} < 2,$$

immediately yielding the claim. For its roots of unity, we observe μ_k is always cyclic and generated by some ζ_n with $\varphi(n) \leqslant [k : \mathbb{Q}]$. Since $-\zeta \in k$, we know that $10 \mid n$ and $\varphi(n) \leqslant 4$. This forces $n = 10$. □

Now we introduce the second player of our story. Define $K = k(\epsilon^{1/5})$.

Lemma 9. *The following statements are true.*

- k *is its own ray class field of conductor* λ^3 *and* K *is the ray class field of* k *of conductor* λ^4*.*
- *The discriminant of* K *is* $D(K) = 5^{31}$*, the class number of* K *is* $h(K) = 1$ *and* K *contains precisely 10 roots of unity.*

We choose to omit the proof and leave it as an exercise to the reader who is willing to take a deep dive into the computational side of class field theory and algebraic number theory, as degree 20 extensions are simply too much for naive methods and require some more background, e.g. the Odlyzko bound.

For the rest of this note, we set $G = \mathrm{Gal}(K/k)$ unless specified otherwise and we fix the element $\sigma \in G$ that satisfies $\sigma(\epsilon^{1/5}) = \zeta^2\epsilon^{1/5}$.

3 The units of K

This section will contain all the geometry. Let C be the curve of genus 2 defined by

$$C : y^2 = x^5 + 1/4$$

over $\mathbb{Q}$ and let J be its Jacobian. The projective closure of C is given by the equation

$$\overline{C} : y^2 z^3 = x^5 + z^5/4$$

and from this it can be seen that C contains only one point at infinity, which we will denote by $\infty = [0 : 1 : 0]$. There is an obvious automorphism $[\zeta] : C \to C$ given by $(x, y) \mapsto (\zeta x, y)$. This map induces an automorphism of J as well and as such, we obtain a map

$$[-] : \mathcal{O}_k = \mathbb{Z}[\zeta] \to \mathrm{End}(J).$$

We will denote the kernel of the endomorphism $[\alpha]$ by $J[\alpha]$.

The idea is to use this Jacobian to construct units in K. As it is a totally imaginary number field of degree 20 over $\mathbb{Q}$, by Dirichlet's unit theorem, its unit group will be of rank $20/2 - 1 = 9$ over $\mathbb{Z}$. Computing a basis for such a huge group is generally a very difficult problem, but using the geometry, we can actually manage.

Lemma 10. *We have that* $K = k(J[\lambda^4])$.

Proof. Recall that as ideals, $(\lambda)^5 = (5)$. From Theorem 4 in [3] it follows that $K = k(J[5])$. However, we will see shortly that λ^4-torsion points are sufficient to generate a field of degree 5 over k, ensuring equality. □

From the above result, it is clear that our approach will be to produce λ^4-torsion points on J to obtain elements from K that will turn out to be units. For this, we will define a few more special points on J. We set $P = (0, 1/2) - \infty$ and $Q = (1, \sqrt{5}/2) - \infty$. Finally, we let R be any point such that $[\lambda]R = Q$.

Lemma 11. *It holds that* $P \in J[\lambda]$ *and* $[\lambda^2]Q = P$. *As such,* $R \in J[\lambda^4]$.

Proof. Since its x-coordinate is zero, it is clear that $[\zeta]P = P$ and as such, $[\lambda]P = [1 - \zeta]P = 0$; this proves the first claim. For the second, we leave it to the reader to check that the function

$$f = y - (\zeta + 1)x^2 - \zeta^4 x + 1/2$$

satisfies the property that $[1 - \zeta]^2 Q = P + \mathrm{div}(f)$, yielding equality on J. Finally, we see that

$$[\lambda^4]R = [\lambda^3][\lambda]R = [\lambda][\lambda^2]Q = [\lambda]P = 0,$$

completing the proof of the lemma. □

Given the equality $K = k(J[\lambda^4])$, one may wonder what Galois elements, for instance $\sigma \in \mathrm{Gal}(K/k)$, look like on the λ^4-torsion of J. The following lemma establishes this.

Lemma 12. *On* $J[\lambda^4]$, *the automorphism* σ *acts as translation by* P, *i.e.* $\sigma(R) = R + P$.

Proof. There seems to be no trick here; in [1], Grant basically produces an unfathomably nasty function that after many arduous calculations comes up with the correct divisor to relate $\sigma(R)$ and $R + P$. We will not copy these calcuations. □

To produce units, we will define two functions $\nu, \xi \in k(J)$ that evaluated at R turn out to produce units. To understand the language of Grant's calculations, we will prove the following results first.

Lemma 13. *On* C *the canonical divisor is given by* $K = 2[\infty]$.

Proof. Consider the differential $\omega = dy/x^4$. We compute its divisor. Indeed, as long as $x \neq 0$, the function y is a uniformiser on C and as such the divisor of ω is not supported here. Only the points $(0, 1/2)$, $(0, -1/2)$ and ∞ remain. For the first two points, we must compute that

$$2y\,dy = 5x^4 dx, \quad \text{and so} \quad \omega = \frac{5}{2y} dx.$$

In these points, x is a uniformiser and as such, we find no zeroes or poles here either. Only the point at infinity remains. For this, we look at the affine patch $D(y)$ of $\overline{C}$ to find the affine curve

$$z^3 = x^5 + z^5/4.$$

At the origin, z has a 5-fold zero whereas x only has a triple zero, so $w = x^2/z$ is a uniformiser. We rewrite the equation to

$$xw = w^4 + x^5/4, \quad \text{on which} \quad (x - 4w^3)dw = (w - 5x^4/4)dx.$$

Now since $x - 4w^3$ has a triple zero and $w - 5x^4/4$ only has a simple zero, their quotient will have a double zero. Hence so will ω. □

Lemma 14. *The Jacobian* J *is birationally equivalent to the symmetric product* $C^{(2)} = (C \times C)/S_2$.

Proof. We apply the Riemann-Roch theorem to C, which states that for any divisor D on C, we have that

$$\ell(D) - \ell(K - D) = \deg(D) - 1,$$

where $\ell(D)$ denotes the dimension of the space of functions f satisfying $\mathrm{div}(f) + D \geqslant 0$. Now let D be any degree 0 divisor on C and apply the above theorem to $D + 2[\infty]$. Then the theorem says that

$$\ell(D + 2[\infty]) \geqslant 1.$$

In other words, there exists some function $f \in k(C)$ such that $\mathrm{div}(f) + D + 2[\infty] \geqslant 0$. Since this divisor is of degree 2, we find $P_1, P_2 \in C$ such that

$$\mathrm{div}(f) + D + 2[\infty] = P_1 + P_2, \quad \text{i.e.} \quad D + 2[\infty] = P_1 + P_2 \quad \text{on } J.$$

This shows that every point of J is of the form $P_1 + P_2 - 2[\infty]$. We now show that, away from the origin, the choice of P_1, P_2 is unique up to interchanging these points. For this, it suffices to show that the choice of f in the argument above was unique; i.e. that equality holds in the inequality. So, we reduce to showing that $\ell(-D) = \ell(K - D - 2[\infty]) = 0$, where we used Lemma 13. Any function g in this space would satisfy $\operatorname{div}(g) - D \geqslant 0$. However, this divisor is of degree 0, so equality must hold. In other words, $D = \operatorname{div}(g)$, so $D = 0$ on J. So unicity holds on all of $J \setminus \{0\}$. This yields an isomorphism between this open subvariety and $C^{(2)}$, yielding the birational equivalence. □

Corollary 15. *The function field of* J *is isomorphic to the field of symmetric functions in pairs of points* $P_i = (x_i, y_i)$, $i \in \{1, 2\}$, *on* C.

Proof. Birationally equivalent varieties have isomorphic function fields, and indeed the functions on $C^{(2)}$ are as written in the corollary. □

Now we can write down the expressions for ν and ξ. We set

$$\nu = -\zeta x_1 x_2, \qquad \xi = \frac{(x_1 + x_2)(x_1 x_2)^2 + 1/2 - 2y_1 y_2}{(2\zeta^2 + \zeta + 2)(x_1 - x_2)^2}.$$

According to Grant, it is nothing but painful calculations to verify that $\nu(R)$ is a root of

$$X^5 + 5X^4 + 5X^2 + 1,$$

and that $\xi(R)$ is a root of

$$X^5 + \ldots + \epsilon^3,$$

where the dots omit four terms with particularly nasty coefficients. Regardless, the results are two units $\nu(R)$ and $\xi(R)$ in K. Their Galois conjugates are of course again units, giving us 8 in total if we omit one because it is dependent on the others due to the norm relation. Adding $\epsilon^{1/5}$ itself, we have found 9 units, as desired. One may wonder if they are a basis for $\mathcal{O}_K^\times$. A numerical calculation shows that the regulator in this case is

$$R\left(\nu(R), \nu(R)^\sigma, \nu(R)^{\sigma^2}, \nu(R)^{\sigma^3}, \xi(R), \xi(R)^\sigma, \xi(R)^{\sigma^2}, \xi(R)^{\sigma^3}, \epsilon^{1/5}\right) \approx 7715.$$

Sadly, this turns out to be not quite minimal yet. Namely, another laborious calculation shows that

$$\zeta^{-1}\nu(R)^{(1-\sigma^2)(1-\sigma^3)(1-\sigma^4)} = \mu^5$$

for some $\mu \in \mathcal{O}_K^\times$. This brings the value of the regulator to

$$R\left(\mu, \nu(R), \nu(R)^\sigma, \nu(R)^{\sigma^2}, \xi(R), \xi(R)^\sigma, \xi(R)^{\sigma^2}, \xi(R)^{\sigma^3}, \epsilon^{1/5}\right) \approx 7715/5 = 1543.$$

We will see in the next section that this is actually the correct regulator and as such, that these units generate $\mathcal{O}_K^\times$.

4 Computing L-series

Let S be the set containing the two archimedean places of k and the place λ above 5, so $\#S = 3$. Then S consists of precisely those places that ramify on K/k. We let the two archimedian places play the role of the *distinguished primes*. By Corollary 3 and Lemma 8, we have for the trivial character $\chi = 1$ that

$$\lim_{s\to 0} \frac{L_S(s,1)}{s^2} = -\frac{1\cdot|2\log|\epsilon||}{10}\cdot\log(5) = -\frac{|\log|\epsilon||\log(5)}{5}.$$

We also wish to analyse this value for the nontrivial characters χ of the group $\mathrm{Gal}(K/k) \cong \mathbb{Z}/5\mathbb{Z}$. The key here is to observe that all non-trivial characters induce the same L-function.

We establish this through a group theoretical construction.

Lemma 16. *Let* G *be an abelian normal subgroup of some finite group* H*. Then there is a homomorphism* $H/G \to \mathrm{Aut}(G)$ *by mapping any coset* xG *to the conjugation by* x *map* $c_x \in \mathrm{Aut}(G)$*.*

Proof. To see that the map $H \to \mathrm{Aut}(G)$ is well defined, it suffices to note that G being normal in H implies that it is mapped to itself under conjugation by any element of H. To see that the homomorphism $H/G \to \mathrm{Aut}(G)$ is also well defined, it suffices to show that the image of the class G is trivial. This is clear, because this image is conjugation by some $g \in G$ and G is abelian. □

Proposition 17. *Suppose that* $G = \mathrm{Gal}(L/M) \cong \mathbb{Z}/p\mathbb{Z}$ *and* $H = \mathrm{Gal}(L/\mathbb{Q})$ *for some tower of Galois fields* $L/M/\mathbb{Q}$ *are such that the map from Lemma 16 above is surjective. Then* $L_S(\chi, s)$ *is independent of the character* $\chi \in \widehat{G} \setminus \{1\}$*.*

Proof. We start with the definition

$$L_S(s,\chi) = \prod_{\mathfrak{p}\notin S}(1-\chi(\mathrm{Frob}_{\mathfrak{p}})\mathrm{Nm}\,\mathfrak{p}^{-s})^{-1}.$$

This time, instead of looking prime by prime, we will consider H/G-orbits of primes $\mathfrak{p}$ of M. So, we consider the polynomials

$$\prod_{(H/G)\cdot\mathfrak{p}}(1-\chi(\mathrm{Frob}_{\mathfrak{p}})X).$$

For this to be independent of χ, it suffices to show that the set $\{\chi(\mathrm{Frob}_{\mathfrak{q}}) \mid \mathfrak{q} \in (H/G)\cdot\mathfrak{p}\}$ is independent of the non-trivial character χ. For this, we observe that all the non-trivial characters of $G \cong \mathbb{Z}/p\mathbb{Z}$ assume the same values, and of course they all assume the value 1 at the identity. So, it suffices to show that $\{\mathrm{Frob}_{\mathfrak{q}} \mid \mathfrak{q} \in (H/G)\cdot\mathfrak{p}\} = G \setminus \{\mathrm{id}\}$. To this end, recall that for any $\sigma \in H/G \cong \mathrm{Gal}(M/\mathbb{Q})$, we have that $\mathrm{Frob}_{\sigma\mathfrak{p}} = \sigma\mathrm{Frob}_{\mathfrak{p}}\sigma^{-1}$. In other words, $\mathrm{Frob}_{\mathfrak{q}}$ for $\mathfrak{q} \in (H/G)\cdot\mathfrak{p}$ ranges over the H/G-orbit of $\mathrm{Frob}_{\mathfrak{p}}$ under the conjugation action of H/G on G from above. By assumption, this is equal to the $\mathrm{Aut}(G)$-orbit, and because $G \cong \mathbb{Z}/p\mathbb{Z}$, this is equal to $G \setminus \{\mathrm{id}\}$. □

We now return to our setting from before.

Lemma 18. *Let* $G = \mathrm{Gal}(K/k) \cong \mathbb{Z}/5\mathbb{Z}$ *and* $H = \mathrm{Gal}(K/\mathbb{Q})$*. Then the map* $H/G \to \mathrm{Aut}(G)$ *is well defined and surjective.*

Proof. The map $H/G \to \mathrm{Aut}(G)$ is well defined because G is abelian and normal, because the extension $k/\mathbb{Q}$ is Galois. We then consider the automorphism ρ of k in $H/G \cong \mathrm{Gal}(k/\mathbb{Q})$ that maps ζ to ζ^2, extended to K leaving $\epsilon^{1/5}$ invariant. We may conjugate the element $\sigma \in \mathrm{Gal}(K/k)$ sending $\epsilon^{1/5}$ to $\zeta^2\epsilon^{1/5}$ with ρ to compute that

$$\rho \circ \sigma \circ \rho^{-1}(\epsilon^{1/5}) = \zeta^4\epsilon^{1/5} = \sigma^2(\epsilon^{1/5}).$$

Since $\langle\sigma\rangle = G \cong \mathbb{Z}/5\mathbb{Z}$, it follows that the automorphism sending $\sigma \mapsto \sigma^2$ generates the automorphism group, as claimed. □

Corollary 19. *The L-function* $L_S(s, \chi)$ *is independent of the non-trivial character* χ*.*

Proof. This follows directly from Proposition 17 and Lemma 18 above. □

Proposition 20. *Let* χ *be any non-trivial character of* G*. Then*

$$\left(\lim_{s\to 0} \frac{L_S(s,\chi)}{s^2}\right)^4 = \frac{R(K)}{2|\log|\epsilon||}.$$

Proof. Since all L-functions coming from non-trivial characters coincide, we find from Proposition 7 that

$$L(s,\chi)^4 \cdot L(s,1) = \zeta_K(s) \cdot (1 - 5{-}s).$$

Hence we see that

$$\begin{aligned}\left(\lim_{s\to 0} \frac{L_S(s,\chi)}{s^2}\right)^4 &= \frac{\lim_{s\to 0} \zeta_K(s)/s^9 \cdot \lim_{s\to 0}(1 - 5{-}s)/s}{\lim_{s\to 0} L(s,1)/s^2} \\ &= \frac{-h(K)R(K)/w(K) \cdot \log(5)}{-|\log|\epsilon|| \log 5/5} \\ &= \frac{1 \cdot R(K)/10}{|\log|\epsilon||/5} \\ &= \frac{R(K)}{2|\log|\epsilon||}.\end{aligned}$$

Note that we used the residue of $L_S(s,1)$ that we computed at the start of this section. □

One may now numerically approximate the limiting value $L''(0,\chi)$ with one's favourite method to obtain information about the value of $R(K)$. Computing the value of the L-function at $s = 0$ forces us to remind ourselves how this value is defined in the first place. Indeed, our infinite product expansion will never converge near $s = 0$, but it will always converge near $s = 1$. We must make use of a functional equation that relates the values at these two points; according to Grant, general theory provides us with the statement that the slightly scaled completed L-function

$$\Lambda(s,\chi) = (2\pi)^{-2s}\Gamma(s)^{-2}A^{s/2}L(s,\chi),$$

where Γ denotes the Gamma-function and $A = D_k \cdot N(f(\chi)) = 5^3 \cdot 5^4 = 5^7$, satisfies

$$\Lambda(s,\chi) = w(\chi)\Lambda(1-s,\overline{\chi}),$$

where $\overline{\chi}$ is the complex conjugate of χ. Here $f(\chi)$ denoted the conductor of the character χ, which by Lemma 9 must necessarily be λ^4. From the above, it follows that in our case $L(s,\chi) = L(s,\overline{\chi})$ and as such, $\Lambda(s,\chi) = \Lambda(s,\overline{\chi})$. The *root number* $w(\chi)$ can be computed locally, but this would take us too far afield at present. The result turns out to be $w(\chi) = 1$, so that

$$\Lambda(s,\chi) = \Lambda(1-s,\chi)$$

is simply symmetric in the line $\Re(s) = 1/2$. Grant describes a way to evaluate $\Lambda(0,\chi)$ directly using integrals of some Bessel function that we choose to not discuss here. We content ourselves with the naive approach. Writing out the symmetry formula yields

$$(2\pi)^{-2s}\Gamma(s)^2 A^{s/2} L(s,\chi) = (\pi/2)^{2s-2}\Gamma(1-s)^2 A^{(1-s)/2} L(1-s,\chi),$$

and as such,

$$L(s,\chi) = (2\pi)^{4s-2}\Gamma(s)^{-2}\Gamma(1-s)^2 A^{(1-2s)/2} L(1-s,\chi).$$

Hence

$$\begin{aligned}\lim_{s\to 0}\frac{L(s,\chi)}{s^2} &= (2\pi)^{-2}\big(\lim_{s\to 0} s\Gamma(s)\big)^{-2}\Gamma(1)^{-2}A^{1/2}L(1,\chi)\\ &= (2\pi)^{-2}\cdot 1\cdot 1\cdot 5^{7/2}\cdot L(1,\chi).\end{aligned}$$

Using the infinite product or series expansion of $L(s,\chi)$, one may numerically determine that $L(1,\chi) \approx 0.895$. Hence $L(1,\chi)^4 \approx 0.64$ and as such,

$$\left(\lim_{s\to 0}\frac{L(s,\chi)}{s^2}\right)^4 \approx 5^{14}\cdot 0.64/(2\pi)^8 \approx 1608.$$

Finally, we compute using Proposition 20 that

$$R(K) \approx 1608 \cdot 2|\log|\epsilon|| \approx 1548.$$

Indeed, this matches our claimed regulator from the previous section, showing its correctness up to numerical error analysis arguments that neither we nor the author of the paper [2] in question could be bothered to provide.

5 The key equality

Now that we know exactly what the units in K are, in addition to having an expression for both $L_S(s,1)$ and $L_S(s,\chi)$ from Proposition 20, we are in a good position to attempt to verify Stark's Question 6. During this section, all our considerations are motivated by our direct desire to carry out this verification. The reader should keep this in mind should any concept feel arbitrary or unnatural; additionally, the author has attempted to stress precisely when and where we are lead directly by this goal.

Recall first that we chose $S = \{\lambda, \mathfrak{p}_1, \mathfrak{p}_2\}$, where the $\mathfrak{p}_i$ are the two archimedian primes of k. We may lift this set to K and call it Σ, which thus contains all the archimedian primes and the single prime above λ of K.

Recall further Stark's question concerns some choice of primes $\mathfrak{P}_1$, $\mathfrak{P}_2$ of K above the two distinguished primes in S of k, that we took to be the two archimedian primes. To be completely explicit, we let $\mathfrak{p}_1$ send ζ to $e^{2\pi i/5}$ and $\mathfrak{p}_2$ sends it to $e^{4\pi i/5}$. For now, we will fix the primes $\mathfrak{P}_i$ by requiring them to send the unit $\nu(R)$, which had minimal polynomial $X^5 + 5X^4 + 5X^2 + 1$, to its unique real root near $X \approx -5.2$. We will see in the next section that making an explicit choice here will not harm our generality. Similarly, we take $\epsilon^{1/5}$ to be the fifth root of ϵ that is real under both $\mathfrak{P}_i$.

Our main challenge is to relate the expression for $L_S(s,\chi)$ from Proposition 20, which features the regulator, the determininant of some 9×9 matrix of units, to something more innocent; the determinant of a sum of 2×2-matrices. To this end, we will have to define a few helpful quantities to ease the notation. Let us start with the following one that should remind the reader strongly of Stark's Question 6, and let us explore some of its properties.

Definition 21. Let $u_1, u_2 \in K^\times$ and $\chi \in \widehat{G}$. We then set

$$R_\chi(u_1 \wedge u_2) := \det \left[\sum_{\gamma \in G} \chi(\gamma) \begin{pmatrix} \log |u_1^\gamma|_{\mathfrak{P}_1} & \log |u_1^\gamma|_{\mathfrak{P}_2} \\ \log |u_2^\gamma|_{\mathfrak{P}_1} & \log |u_2^\gamma|_{\mathfrak{P}_2} \end{pmatrix} \right].$$

Even though there is meaning to the usage of the wedge product in this definition, for this note the reader can just think of it as a notational quirk, as we will not use wedge products in any meaningful way. That said, with some basic knowledge of the properties of wedge products, the following property should not come as a surprise.

Lemma 22. *Let* $u_1, u_2, u_3 \in K^\times$. *Then* $R_\chi(u_1 \wedge u_2 u_3) = R_\chi(u_1 \wedge u_2) + R_\chi(u_1 \wedge u_3)$.

Proof. This is a direct consequence of the identity

$$\det \begin{pmatrix} a & b \\ c + c' & d + d' \end{pmatrix} = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \det \begin{pmatrix} a & b \\ c' & d' \end{pmatrix}.$$

Indeed, $a(d + d') - b(c + c') = (ad - bc) + (ad' - bc')$. □

Since $L_S(s,\chi)$ does not depend on the non-trivial character χ by Corollary 19, it will be of interest to work out when $R_\chi(u_1 \wedge u_2)$ has this same property. A simple case is taken care of by the following lemma, for which we recall our distinguished element $\sigma \in G$ given by $\sigma(\epsilon^{1/5}) = \zeta^2 \epsilon^{1/5}$.

Lemma 23. *Let* $u_1, u_2 \in K^\times$. *If* $u_2^{\sigma-1}$ *is a root of unity, then* $R_\chi(u_1 \wedge u_2) = 0$ *for any* $\chi \in \widehat{G} \setminus \{1\}$.

Proof. Explicitly, $u_2^{\sigma-1}$ being a root of unity gives us that $\sigma(u_2)$ and u_2 differ only up to a root of unity. In other words, $|\sigma(u_2)|_{\mathfrak{P}} = |u_2|_{\mathfrak{P}}$ for any archimedian place $\mathfrak{P}$, and since σ generates G, we find $|u_2^\gamma|_{\mathfrak{P}} = |u_2|_{\mathfrak{P}}$ for any $\gamma \in G$. Therefore, the bottom row of the matrix will be constant, so summing them with weights $\chi(\gamma)$, which sum to zero, yields a matrix with vanishing bottom row. Its determinant therefore vanishes. □

There are also more subtle conditions, however. The following makes reference to the Galois element τ that fixes $\nu(R)$ but sends ζ to ζ^2. Note that τ interchanges $\mathfrak{P}_1$ and $\mathfrak{P}_2$ because of its action on ζ.

Lemma 24. *Let* $u_1, u_2 \in K^\times$. *Suppose that* $u_2^{\tau-1}$ *and* $u_1^{\tau+1}$ *are both roots of unity. Then the quantity* $R_\chi(u_1 \wedge u_2)$ *is independent of the choice of* $\chi \in \widehat{G} \setminus \{1\}$.

Proof. Since mapping χ to χ^2 is a transitive action on $\widehat{G} \setminus \{1\}$, it suffices to show that $R_{\chi^2}(u_1 \wedge u_2) = R_\chi(u_1 \wedge u_2)$ under the assumptions from the lemma. It is easy to check that $\tau\sigma\tau^{-1} = \sigma^3$, so this holds for any $\gamma \in G$. Starting with the definition of $R_{\chi^2}(u_1 \wedge u_2)$, we first replace γ by γ^3 to change $\chi(\gamma^2)$ to $\chi(\gamma)$. Subsequently, we replace γ^3 by $\tau\gamma\tau^{-1}$, and finally note that

$$|u_i^{\tau\gamma\tau^{-1}}|_{\mathfrak{P}_j} = |u_i^{\tau\gamma}|_{\mathfrak{P}_{3-j}} = \begin{cases} |u_1^{-\gamma}|_{\mathfrak{P}_{3-j}} & \text{if } i = 1; \\ |u_2^{\gamma}|_{\mathfrak{P}_{3-j}} & \text{if } i = 2, \end{cases}$$

where first used that τ interchanges the $\mathfrak{P}_j$ and subsequently the assumptions on the u_i in the same way as in the proof of the lemma above. After taking the logarithm, the minus signs in the exponents will appear in front, after which multiplying the top row by -1 and swapping the two columns yield two cancelling signs in the determinant and transform our expression precisely into $R_\chi(u_1 \wedge u_2)$, completing the proof. The author apologises for the verbal treatment of this calculation, but the above should suffice for anyone who wants to carry out the proof effortlessly by hand. □

Now let $\phi = \epsilon^{1/5} + \epsilon^{-1/5}$, so that after a brief computation $\mathrm{Nm}^{K/k}(\phi) = \sqrt{5}$. In particular, we see that ϕ is a Σ-unit of K. We set

$$\phi_1 = \phi^{(1+\sigma+\sigma^4)}/\lambda,$$

which is also a Σ-unit of K. Further, we set

$$\omega = \nu(R)^{2-\sigma-\sigma^4} \quad \text{and} \quad \phi_2 = \epsilon^{1/5}\omega^2.$$

All calculations from this point onward serve to show that ϕ_1 and ϕ_2 will eventually turn out to be the units we are after to positively answer Stark's question. Indeed, these units have the following Stark-y property with respect to the trivial character.

Proposition 25. *With all notation from above, we have the equality*

$$\frac{1}{2}L_S''(0,1) = \frac{1}{w(K)} \det \left[\sum_{\gamma \in G} \begin{pmatrix} \log|\phi_1^\gamma|_{\mathfrak{P}_1} & \log|\phi_1^\gamma|_{\mathfrak{P}_2} \\ \log|\phi_2^\gamma|_{\mathfrak{P}_1} & \log|\phi_2^\gamma|_{\mathfrak{P}_2} \end{pmatrix} \right].$$

Proof. We found at the beginning of Section 4 that

$$\frac{1}{2}L_S''(0,1) = \lim_{s\to 0} \frac{L_S(s,1)}{s^2} = -\frac{2|\log|\epsilon|| \log(5)}{10}.$$

It thus suffices to show that

$$\det \left[\sum_{\gamma \in G} \begin{pmatrix} \log|\phi_1^\gamma|_{\mathfrak{P}_1} & \log|\phi_1^\gamma|_{\mathfrak{P}_2} \\ \log|\phi_2^\gamma|_{\mathfrak{P}_1} & \log|\phi_2^\gamma|_{\mathfrak{P}_2} \end{pmatrix} \right] = -2|\log|\epsilon|| \log(5).$$

To this end, we note that for any $x \in K$,

$$\sum_{\gamma \in G} \log|x^\gamma|_{\mathfrak{P}_i} = \log \left| \prod_{\gamma \in G} x^\gamma \right|_{\mathfrak{P}_i} = \log \left|\mathrm{Nm}_k^K(x)\right|_{\mathfrak{p}_i},$$

because $\mathfrak{P}_i$ and $\mathfrak{p}_i$ coincide on k. Using the above already eluded to computation

$$\mathrm{Nm}_k^K(\phi) = \mathrm{Nm}_k^K(\epsilon^{1/5} + \epsilon^{-1/5}) = \epsilon + \epsilon^{-1} = \sqrt{5},$$

we may then compute that

$$\begin{aligned} R_1(\phi_1 \wedge \phi_2) &= \det \begin{pmatrix} \log\left|\mathrm{Nm}\,_k^K(\phi_1)\right|_{\mathfrak{p}_1} & \log\left|\mathrm{Nm}\,_k^K(\phi_1)\right|_{\mathfrak{p}_2} \\ \log\left|\mathrm{Nm}\,_k^K(\phi_2)\right|_{\mathfrak{p}_1} & \log\left|\mathrm{Nm}\,_k^K(\phi_2)\right|_{\mathfrak{p}_2} \end{pmatrix} \\ &= \det \begin{pmatrix} \log|\sqrt{5}^3/\lambda^5|_{\mathfrak{p}_1} & \log|\sqrt{5}^3/\lambda^5|_{\mathfrak{p}_2} \\ \log|\epsilon|_{\mathfrak{p}_1} & \log|\epsilon|_{\mathfrak{p}_1} \end{pmatrix}. \end{aligned}$$

Clearly we have $|\sqrt{5}|_{\mathfrak{p}_i} = 5$. To work out the bottom entries, we must recall that we chose ϵ to satisfy $2\epsilon - 1 = \sqrt{5}$ and as such, using the definitions of the embeddings $\mathfrak{p}_1$ and $\mathfrak{p}_2$, once may check that $|\epsilon|_{\mathfrak{p}_1} = |\epsilon|^2$, whereas $|\epsilon|_{\mathfrak{p}_2} = 1/|\epsilon|^2$. Plugging everything in and noting that both diagonals yield very similar contributions to the determinant, we find that

$$R_1(\phi_1 \wedge \phi_2) = -2\log|\epsilon|\big(6\log(5) - 5(\log|\lambda|_{\mathfrak{p}_1} + \log|\lambda|_{\mathfrak{p}_2})\big).$$

Finally,

$$\log(|\lambda|_{\mathfrak{p}_1} + \log(|\lambda|_{\mathfrak{p}_2} = \log(|\lambda|_{\mathfrak{p}_1}|\lambda|_{\mathfrak{p}_2}) = \log(\mathrm{Nm}\,|\lambda|) = \log(5).$$

So indeed, $R_1(\phi_1 \wedge \phi_2) = -2\log|\epsilon|\log(5)$, as claimed. □

We also record here a property of the unit ϕ_2 that will be useful later.

Lemma 26. *The field extension* $K(\phi_2^{1/10})/K$ *is abelian.*

Proof. The Galois group of the compositum of the fields $K(\phi_2^{1/2})$ and $K(\phi_2^{1/5})$, which is precisely $K(\phi_2^{1/10})$, must be abelian if both of these fields are. Indeed, we have an injective map $\mathrm{Gal}(K(\phi_2^{1/10})/k) \to \mathrm{Gal}(K(\phi_2^{1/5})/k) \times \mathrm{Gal}(K(\phi_2^{1/2})/k)$ given by restriction, showing the claim. Hence it suffices to show that these two groups are abelian. For the first, we note that $\phi_2^{1/2} = \epsilon^{1/10}\omega$ so that $K(\phi_2^{1/2}) = K(\epsilon^{1/10})$. In Grant [2] it is shown that $\omega \in k(\epsilon^{1/10})$, so that even $K(\phi_2^{1/2}) = k(\epsilon^{1/10})$, completing the proof of this part because this is a Kummer extension of k and as such abelian.

For the other, we note that $[K(\phi_2^{1/5}) : k] \mid [K(\phi_2^{1/5}) : K][K : k] = 5^2$ and since all groups dividing this order are abelian, it suffices to show normality. This in turn comes down to proving that $\sigma(\phi_2)/\phi_2$ is a fifth power in K. Grant does this explicitly using his unit μ from a while back that appeared in the regulator of K. □

Now the L-functions $L_S(s,\chi)$ displayed a striking independence from the choice of $\chi \in \widehat{G} \setminus \{1\}$. For a statement similar to the above for $\chi = 1$ to generalise to all characters, as Stark's question asks, we would certainly require such an invariance for $R_\chi(\phi_1 \wedge \phi_2)$ as well. Fortunately, this turns out to be the case.

Proposition 27. *The quantity* $R_\chi(\phi_1 \wedge \phi_2)$ *is independent of the choice of* $\chi \in \widehat{G} \setminus \{1\}$.

Proof. We claim that

$$R_\chi(\phi_1 \wedge \phi_2) = R_\chi(\phi_1 \wedge \epsilon^{1/5}) + R_\chi(\phi^{\sigma-\sigma^2-\sigma^3+\sigma^4} \wedge \nu(R)^2).$$

With this claim, the proof is easy. Namely, by definition $\sigma(\epsilon^{1/5})$ and $\epsilon^{1/5}$ differ only up to a root of unity, and as such, Lemma 23 tells us that the first term vanishes for all $\chi \neq 1$. For the second, a direct calculation reveals that

$$\tau(\phi^{\sigma-\sigma^2-\sigma^3+\sigma^4}) = \phi^{-\sigma+\sigma^2+\sigma^3-\sigma^4},$$

so that we may apply Lemma 24 to obtain the required independence after noting that, since τ fixes $\nu(R)$ by definition, we have $\nu(R)^{\tau-1} = 1$. It remains to prove the claim. Using Lemma 22 and $\phi_2 = \epsilon^{1/5}\omega^2$, after cancelling a factor of 2 we immediately reduce to showing that

$$R_\chi(\phi_1 \wedge \omega) = R_\chi(\phi^{\sigma-\sigma^2-\sigma^3+\sigma^4} \wedge \nu(R)).$$

Because they are the only quintic subextensions of $\mathbb{Q}$ that are real under both $\mathfrak{P}_1$ and $\mathfrak{P}_2$, we must have that $\mathbb{Q}(\nu(R)) = \mathbb{Q}(\phi^2)$. Therefore the above equality can be checked with a finite amount of effort inside this quintic field. We leave the details to the reader, as unfortunately neither we nor Grant seem willing to provide them. □

All we need now is a way to relate regulators to sums of determinants of smaller matrices. We will do this by first relating the usual regulator of the field K to a slight generalisation of it.

Definition 28. Let Σ be any set of places of K containing all the archimedian ones. Let $\{u_1, \dots, u_{t-1}\}$ be any set of Σ-units, where $t = \#\Sigma$. Then we define

$$R_{\Sigma,K}(u_1, \dots, u_{t-1}) = |\det([\log |u_i|_{\sigma_j}]_{i,j})|,$$

where the σ_j range over all but one place of Σ. If we took a set of generators instead, we obtain the Σ-*regulator* of K with respect to Σ, denoted $R_{\Sigma,K}$.

It is not hard to see that in our case, where Σ consists of precisely all archimedian primes and just one additional prime over λ in k with absolute norm 5, we must have

$$R_{\Sigma,K} = \log(5)R(K).$$

Even though we have seemingly only increased the complexity, by going from a 9×9-matrix to a 10×10, there actually is a trick that allows us to compute this bigger determinant more easily if we pick our units well. Namely, suppose we would be interested to compute

$$R := R_{\Sigma,K}\left(\phi_1, \phi_1^{\sigma}, \phi_1^{\sigma^2}, \phi_1^{\sigma^3}, \phi_1^{\sigma^4}, \phi_2, \phi_2^{\sigma}, \phi_2^{\sigma^2}, \phi_2^{\sigma^3}, \phi_2^{\sigma^4}\right),$$

and we write, omitting the last finite prime when computing the regulator,

$$\Sigma = \left\{\mathfrak{P}_1, \sigma(\mathfrak{P}_1), \sigma^2(\mathfrak{P}_1), \sigma^3(\mathfrak{P}_1), \sigma^4(\mathfrak{P}_1), \mathfrak{P}_2, \sigma(\mathfrak{P}_2), \sigma^2(\mathfrak{P}_2), \sigma^3(\mathfrak{P}_2), \sigma^4(\mathfrak{P}_4)\right\}.$$

With this, we see that we can write

$$R = \det\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where

$$M_{ij} = (\log |\phi_i^{\sigma^k}|_{\sigma^\ell(\mathfrak{P}_j)})_{k,\ell} = (\log |\phi_i^{\sigma^{k-\ell}}|_{\mathfrak{P}_j})_{k,\ell} = (\varphi_{ij}(k-\ell))_{k,\ell},$$

where we set $\varphi_{ij}(a) = \log |\phi_i^{\sigma^a}|_{\mathfrak{P}_j}$ as a complex valued function $G \to \mathbb{C}$. It is this precise form that we relate to sums of 2×2-matrices.

Proposition 29. *With all notation from above, we have that*

$$R = \prod_{\chi \in \widehat{G}} \det\left(\sum_{a \in G} \chi(a)[\varphi_{ij}(a)]_{i,j}\right).$$

Proof. Consider the 5-dimensional $\mathbb{C}$-vector space Ω of complex valued functions $G \to \mathbb{C}$. This vector space comes with two natural bases; the indicator functions $\{\delta_b \mid b \in G\}$ form one, and the set $\widehat{G}$ forms the other. The idea of the proof is as follows: we define a linear operator on Ω^2 whose matrix in the indicator basis is the matrix whose determinant defines R and who at the same time leaves the spaces $(\mathbb{C}\chi \times 0) \oplus (0 \times \mathbb{C}\chi)$ invariant for any $\chi \in \hat{G}$. This means that in the character basis, M will have 5 blocks of size 2×2 on its diagonal, so that its determinant will be the product of the smaller determinants, as desired.

To carry out this plan, for any $\varphi \in \Omega$, we define $\varphi_{(1)}$ to be φ regarded as an element of $\Omega \times 0 \subset \Omega^2$, and $\varphi_{(2)}$ as φ as an element of $0 \times \Omega$. For any $a \in G$, we have a natural linear map on Ω given by

$$(\mathcal{T}_a\varphi)(b) = \varphi(a+b).$$

It is easy to see that

$$\mathcal{T}_a \delta_b = \delta_{b-a} \quad \text{and} \quad \mathcal{T}_a \chi = \chi(a)\chi.$$

Using our distinguished functions φ_{ij} from above, this allows us to define a linear transformation on Ω^2 by extending the formula

$$\mathcal{T}(\varphi_{(i)}) = \sum_{a \in G} \sum_{j=1,2} \varphi_{ij}(a)(\mathcal{T}_a \varphi)_{(j)}$$

to all of Ω^2 by linearity. To see what this map looks like on the basis $\{\delta_b \mid b \in G\}$, we write that

$$\mathcal{T}((\delta_b)_{(i)}) = \sum_{a \in G} \sum_{j=1,2} \varphi_{ij}(a)(\delta_{b-a})_{(j)}.$$

So indeed the $(\delta_c)_{(j)}$ component of $\mathcal{T}((\delta_b)_{(i)})$ is equal to $\varphi_{ij}(b-c)$. Hence the matrix of $\mathcal{T}$ will in this basis coincide with the matrix whose determinant defines R.

On the other hand, we may compute that

$$\mathcal{T}(\chi_{(i)}) = \sum_{a \in G} \sum_{j=1,2} \varphi_{ij}(a)\chi(a)\chi_{(j)}.$$

We see that the space $\mathbb{C}\chi_{(1)} \oplus \mathbb{C}\chi_{(2)}$ is left invariant for any character χ, and it is clear that its matrix entries are as described in the statement. □

Indeed, the above argument is completely general and is in no way, shape or form confined to our particular choices of G and matrix sizes. The author preferred to treat the argument in a pragmatic way as presented above, and to leave the formulation of the precise general statement about determinants to the reader; they should find the same as in [2]. Further, the reader will have noticed that the expression that occurs on the right hand side of the above theorem is precisely a product over all $R_\chi(\phi_1 \wedge \phi_2)$. In other words, we have shown that

$$R = \prod_{\chi \in \widehat{G}} R_\chi(\phi_1 \wedge \phi_2).$$

This is the key point. We are now merely left to precisely relate both sides of the equation to what we are after. Let us start with R. As it is the Σ-regulator of *some* independent set of Σ-units, it must be an integral multiple of

$$R_\Sigma = \log(5) R(K).$$

A quick numerical calculation shows this multiple to be precisely 10^4. Thus, we have

$$R = 10^4 \log(5) R(K).$$

We can now finally state and prove what Grant calls the *key equality*.

Theorem 30. *With all notation from above, we have for all characters* χ *on* G *the equality*

$$\frac{1}{2}L_S''(0,\chi) = \frac{1}{w(K)} \det \left[\sum_{\gamma \in G} \chi(\gamma) \begin{pmatrix} \log|\phi_1^\gamma|_{\mathfrak{P}_1} & \log|\phi_1^\gamma|_{\mathfrak{P}_2} \\ \log|\phi_2^\gamma|_{\mathfrak{P}_1} & \log|\phi_2^\gamma|_{\mathfrak{P}_2} \end{pmatrix} \right].$$

Proof. The trivial character has already been dealt with in Proposition 25, so we may assume that $\chi \neq 1$. In this case, we get from Proposition 20 that

$$\left(\frac{1}{2}L_S''(0,\chi)\right)^4 = \left(\lim_{s\to 0} \frac{L_S(s,\chi)}{s^2}\right)^4 = \frac{R(K)}{2|\log|\epsilon||} = \frac{1}{10^4}\frac{R}{2|\log|\epsilon||\log(5)}.$$

Making use of Proposition 27, we find that

$$R = R_1(\phi_1 \wedge \phi_2) \prod_{\chi \neq 1} R_\chi(\phi_1 \wedge \phi_2) = R_1(\phi_1 \wedge \phi_2) \cdot R_\chi(\phi_1 \wedge \phi_2)^4.$$

Again from Proposition 25 it follows that

$$R_1(\phi_1 \wedge \phi_2) = -2|\log|\epsilon||\log(5),$$

and as such,

$$\left(\frac{1}{2}L_S''(0,\chi)\right)^4 = \left(\frac{R_\chi(\phi_1 \wedge \phi_2)}{10}\right)^4.$$

Hence the numbers between the brackets must be equal up to a power of i. A brief numerical calculation shows that equality in fact holds. □

6 Verification of Stark's question

With the *key equality* in hand, we may readily verify Stark's question. Recall that we made a choice for $\mathfrak{P}_i$ at the start of the previous section. Fortunately, it turns out that we need not worry about this all that much, as is provided by the following lemma.

Lemma 31. *If Stark's question has a positive answer for some choice of* $\mathfrak{P}_1, \mathfrak{P}_2$*, then it has a positive answer for any such choice.*

Proof. Suppose that the Σ-units ϵ_1, ϵ_2 positively answer Stark's question for $\mathfrak{P}_1, \mathfrak{P}_2$. Note that any other choice of primes can be written as $\rho_1(\mathfrak{P}_1), \rho_2(\mathfrak{P}_2)$ for some $\rho_1, \rho_2 \in G$, because the Galois group $G = \mathrm{Gal}(K/k)$ acts transitively on the primes above a given prime. Further note that

$$|x|_{\rho(\mathfrak{P})} = |(\rho(\mathfrak{P}))(x)| = |\mathfrak{P}(x^{\rho^{-1}})| = |x^{\rho^{-1}}|_{\mathfrak{P}},$$

where the inverse only appears because we switch from a left- to a right-action. Taking $x = \epsilon^\rho$ shows that the choice of Σ-units $\epsilon_1^{\rho_1}, \epsilon_2^{\rho_1}$ will work for our new choice of archimedian primes, because the resulting matrices are the same. The follow-up questions in Question 6 are also still valid. □

We thus reduce to verifying Stark's question for our particular choice of the $\mathfrak{P}_i$. Fortunately, the *key equality* from Theorem 30 does most of the work.

Theorem 32. *Stark's Question 6 can be answered positively for the field extension* K/k.

Proof. Indeed, by Theorem 30, the Σ-units ϕ_1^{10} and ϕ_2 answer the first equality positively. We only need to include this tenth power because of the appearance of an extra factor of $w(K)$ in the question compared to the key equality. In fact, because

$$\begin{pmatrix} \log|\epsilon_1^a\epsilon_2^b|_{\mathfrak{p}_1} & \log|\epsilon_1^a\epsilon_2^b|_{\mathfrak{p}_2} \\ \log|\epsilon_1^c\epsilon_2^d|_{\mathfrak{p}_1} & \log|\epsilon_1^c\epsilon_2^d|_{\mathfrak{p}_2} \end{pmatrix} = \begin{pmatrix} a\log|\epsilon_1|_{\mathfrak{p}_1} + b\log|\epsilon_2|_{\mathfrak{p}_1} & a\log|\epsilon_1|_{\mathfrak{p}_2} + b\log|\epsilon_2|_{\mathfrak{p}_2} \\ c\log|\epsilon_1|_{\mathfrak{p}_1} + d\log|\epsilon_2|_{\mathfrak{p}_1} & c\log|\epsilon_1|_{\mathfrak{p}_2} + d\log|\epsilon_2|_{\mathfrak{p}_2} \end{pmatrix}$$
$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \log|\epsilon_1|_{\mathfrak{p}_1} & \log|\epsilon_1|_{\mathfrak{p}_2} \\ \log|\epsilon_2|_{\mathfrak{p}_1} & \log|\epsilon_2|_{\mathfrak{p}_2} \end{pmatrix},$$

it follows that for any choice of a, b, c, d such that $ad - bc = 1$, the units $\phi_1^{10a}\phi_2^b$ and $\phi_1^{10c}\phi_2^d$ will also satisfy the first equation from Stark's question. This is useful, because it allows us to positively answer the follow-up questions as well.

Namely, by Lemma 26, the field extensions $K(\phi_1^a\phi_2^{b/10})/k$ and $K(\phi_1^c\phi_2^{d/10})/k$ will always be abelian as they are subfields of $K(\phi_2^{1/10})$. To ensure equality here, it suffices to choose c, d coprime with 10. One such choice is established with $(a, b, c, d) = (1, 3, 2, 7)$. Explicity, we thus choose $\epsilon_1 = \phi_1^{10}\phi_2^3$ and $\epsilon_2 = \phi_1^{20}\phi_2^7$.

Finally, we must show that we may ensure that $\epsilon_i^\sigma\mathcal{O}_K = \epsilon_i\mathcal{O}_K$ for all $\sigma \in G$. In other words, we want that $\epsilon_i^\sigma/\epsilon \in \mathcal{O}_K^\times$. Fortunately, this is easy; ϕ_2 is a unit so we may ignore it completely. Also, ϕ_1^σ/ϕ_1 is a unit because it ϕ_1 only had nonzero λ-adic valuation but this prime, being the only one above 5, is fixed by σ. $\square$

7 References

[1] David Grant. A proof of quintic reciprocity using the arithmetic of $y^2 = x^5 + 1/4$. *Acta Arithmetica*, 75(4):321–337, 1996.

[2] David Grant. Units from 5-torsion on the jacobian of $y^2 = x^5 + 1/4$ and the conjectures of Stark and Rubin. *Journal of Number Theory*, 77(2):227–251, 1999.

[3] Ralph Greenberg. On the jacobian variety of some algebraic curves. *Compositio Mathematica*, 42(3):345–359, 1980.