

Blowup in the Complex Ginzburg-Landau Equation

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Chapter 1

Introduction

1.1 Finite-time blowup

In this thesis, we study the stability of a finite-time blowup solution of a partial differential equation (PDE). Partial differential equations can be used to model phenomena in a wide range of applications. Examples of well known partial differential equations are: the heat equation which models heat conduction in a medium; the Navier-Stokes equation which describes the motion of fluids; and (a system of coupled nonlinear) reaction-diffusion equations which model(s) the density of for example chemical substances that can undergo a reaction. In initial value problems (also called Cauchy problems), an initial state at $t = 0$ and boundary conditions are specified. And, if local existence and uniqueness of solutions is established, the aim of initial value problems is to solve these equations and thereby determine the state, depending on space, for $t > 0$. A priori, it is, however, not clear that through solving the equation it is possible to determine the state for all $t > 0$.

We first give an introductory example of this by considering an ordinary differential equation, i.e. an equation for a function that only depends on time and not on a space. Consider the initial value problem

$$\begin{cases} \dot{u} = u^2, \\ u(0) = u_0, \end{cases} \quad (1.1a)$$

$$(1.1b)$$

where $t \geq 0$ and we denote differentiation with respect to time t by (over)dot. As is taught in basic courses on differential equations, the unique solution to this equation is given by

$$u(t; u_0) = \begin{cases} \frac{1}{\frac{1}{u_0} - t} & \text{if } u_0 \neq 0, \\ 0 & \text{if } u_0 = 0. \end{cases} \quad (1.2)$$

We now define what we mean by a finite-time blowup solution of equation (1.1).

Definition 1.1.1. A finite-time blowup solution of equation (1.1) is a solution $u(t)$ for which there exists a $0 < T < \infty$ such that

$$\lim_{t \uparrow T} |u(t; u_0)| = \infty,$$

where by $|\cdot|$ we denote the absolute value.

Thus, the norm of a finite-time blowup solution of (1.1) blows up as we approach the blowup time T . For this example, it should be clear that $u(t, u_0)$ as given by (1.2) is a finite-time blowup solution if and only if $u_0 > 0$ and the blowup time $T = \frac{1}{u_0}$. Furthermore, the solution ceases to exist at $t = T$. For simplicity, we choose $u_0 = 1$, in which case $T = 1$.

It is possible to scale out the blowup in a self similar way, i.e. we can write

$$u(t; 1) = \frac{Q(\tau)}{L(t)}, \quad (1.3)$$

where: τ is monotonically increasing from 0 to infinity as t ranges from 0 to T , Q is a bounded function of τ and $L(t)$ vanishes as t tends to the blowup time T . The idea is that $L(t)$, i.e. the denominator in the right hand side of (1.3), is the blowup rate and causes blowup in $u(t; 1)$ because it vanishes as t tends to the blowup time T . Therefore, we also pose the requirement that Q is bounded away from 0 for τ large enough. Substitution into (1.1a) leads to

$$\frac{Q'(\tau) \frac{\partial \tau}{\partial t}}{L(t)} - \frac{QL'(t)}{L(t)^2} = \frac{Q(t)^2}{L(t)^2}. \quad (1.4)$$

The next step is to pose extra requirements such that this equation can be written as an autonomous equation in Q . This step is ad hoc and cannot be guaranteed to work for every type of equation. In this case, we pose the requirement:

$$\frac{\frac{\partial \tau}{\partial t}}{L(t)} = -\frac{L'(t)}{L(t)^2} = \frac{1}{L(t)^2}.$$

From this, we obtain $L(t) = T - t$ and $\tau = -\log(T - t)$. The equation for Q now becomes

$$Q' = Q^2 - Q. \quad (1.5)$$

It follows from equation (1.3) that if we can construct a stationary solution of (1.5) which is not the zero solution and is bounded away from zero for τ large enough, then we have constructed a finite-time blowup solution of equation (1.1). Here, $Q = 1$ is such a stationary solution, so that we indeed recover $u(t) = \frac{1}{1-t}$ through (1.3). In principle, other solutions of (1.5), i.e. non-stationary solutions or solutions that are not bounded away from 0 for τ large enough, can also be studied, but they are, however, not guaranteed to correspond to blowup solutions. This rescaling technique is common in the study of blowup solutions and we will use similar techniques in more complicated problems, see Section 1.2.4.

1.2 The Ginzburg-Landau equation

In this thesis, we will study a finite-time blowup solution of the Ginzburg-Landau equation. The Ginzburg-Landau equation is an amplitude equation and has applications in many fields, among which: nonlinear optics, turbulence, Rayleigh-Bénard convection, superconductivity, superfluidity, Taylor-Couette flow and reaction-diffusion systems, see [5, 10, 11, 32, 43] and the review article [2]. Mathematically, it can be derived from nonlinear partial differential equations on unbounded domains, and it describes the leading order amplitude of a pattern close to a threshold where the pattern appears, i.e. where it bifurcates (see definition below) from a trivial state. In [28], an overview and the validity of this derivation is presented. The derivation in [28] starts with a very general nonlinear partial differential equation of the form:

$$\partial_t u = L(\mu, \partial_x, \partial_y)u + N(\mu, \partial_x, \partial_y, u), \quad (1.6)$$

where u is a function of time t and space $(x, y) \in \mathbb{R}^d \times \Sigma$, Σ is a bounded cross section of a real coordinate space, μ is a real valued parameter, and $N(\mu, \partial_x, \partial_y, \varepsilon u) = O(\varepsilon^2)$. The idea is that $u = 0$ solves this equation and that, through varying the parameter μ , this solution changes stability under the right spectral assumptions (see [28] for more details on the spectral assumptions). A change in the number of stationary solutions of (1.6) or a change in the stability of the stationary solutions through varying the parameter μ is called a bifurcation, see [21] for more information on bifurcations and structures that can appear when a bifurcation takes place. When a so-called Hopf or Turing-Hopf bifurcation takes place, the amplitude A of the most unstable spectral mode is independent of the bounded variable y and its dynamics is (under certain conditions, see [28]) to leading order given by the complex Ginzburg-Landau equation (GL). The GL can be scaled into the form

$$\partial_t A = \tilde{\rho} A + (1 + i\tilde{c})\nabla^2 A - (1 + i\tilde{b})|A|^2 A, \quad (1.7)$$

where $A(x, t)$ is a complex amplitude, $x \in \mathbb{R}^d$ and $t \geq 0$. Furthermore, the parameters $\tilde{b}$, $\tilde{\rho}$ and $\tilde{c}$ are real-valued. Since taking the complex conjugate of this equation results in an equation for $\bar{A}$ of the same form but with the sign of $\tilde{b}$ and $\tilde{c}$ reversed, we choose without loss of generality $\tilde{c} > 0$. The discussed derivation of the GL loses its validity for large amplitudes. Therefore, it is of particular interest to study existence and stability of blowup solutions to the Ginzburg-Landau equation, see for example [37].

Another way to introduce the Ginzburg-Landau equation is through considering functionals of which the minimizers are the stationary solutions of Ginzburg-Landau type equations. Note that the Ginzburg-Landau equation typically has real coefficients in these cases. Further details can be found in [4].

1.2.1 The complex Ginzburg-Landau equation near the Nonlinear Schrödinger limit

Since there is a lot of literature on (finite-time) blowup in Nonlinear Schrödinger equation (NLS), see for example [44] and the references therein, we study the GL near the Nonlinear Schrödinger limit. This limit can be obtained from equation (1.7) if we simultaneously let $\tilde{b}$ and $\tilde{c}$ tend to infinity. When $\tilde{b}$ and $\tilde{c}$ are both of $O\left(\frac{1}{\varepsilon}\right)$, where $0 < \varepsilon \ll 1$, we obtain, after rescaling, an equation of the form:

$$i \frac{\partial \Phi}{\partial t} - i\rho \varepsilon^2 \Phi + (1 - i\varepsilon)\nabla^2 \Phi + (1 + ib\varepsilon)|\Phi|^{2\sigma} \Phi = 0, \quad (1.8)$$

where $\Phi(x, t)$ is a complex amplitude, $x \in \mathbb{R}^d$, $t \geq 0$ and b and ρ are both $O(1)$ parameters. For $\varepsilon = 0$, equation (1.8) is known as the NLS. Since studying a more general nonlinearity is common in the NLS, see for example [44], a parameter $\sigma > 0$ is introduced in (1.8) and we also study a GL with a more general nonlinearity.

As in [9, 36, 40], we consider a GL without a linear term, see also Remark 1.2.3 for a motivation of this choice. Thus, for $\rho = 0$, we obtain:

$$i \frac{\partial \Phi}{\partial t} + (1 - i\varepsilon)\nabla^2 \Phi + (1 + ib\varepsilon)|\Phi|^{2\sigma} \Phi = 0. \quad (1.9)$$

Furthermore, we pose the following initial and boundary conditions

$$\Phi(x, 0) = \Phi_0(x), \quad (1.10a)$$

$$|\Phi| \rightarrow 0 \text{ as } |x| \rightarrow \infty. \quad (1.10b)$$

Although we introduced d as the dimension, we also consider non-integer d . This is very common in the existing literature where radially symmetric blowup solutions are studied, see for example [44] and the references therein. The Laplace operator in a radially symmetric setting reduces to $\partial_r^2 + \frac{d-1}{r}\partial_r$, where $r \geq 0$ is the radial coordinate, and we can then simply interpret d as a non-integer parameter.

The topic of this thesis is the stability of finite-time blowup solutions of the GL with respect to both radially symmetric and non-radially symmetric perturbations. A detailed discussion of the interpretation of non-radially symmetric perturbations for non-integer values of the dimension d will be given in Section 4.1.2. Completely analogous to the example in Section 1.1, we now give the definition of a finite-time blowup solution of (1.9).

Definition 1.2.1. *A finite-time blowup solution of equation (1.9) with boundary and initial conditions given by (1.10) is a function $\Phi(x, t)$ for which there exists a $0 < T < \infty$ such that:*

1.2.1.a $\Phi(x, t)$ exists as a bounded and smooth (enough) solution of (1.9) for $0 \leq t < T$;

1.2.1.b $\Phi(x, t)$ satisfies initial condition (1.10a) and boundary condition (1.10b);

1.2.1.c $\lim_{t \uparrow T} \max_{x \in \mathbb{R}^d} |\Phi(x, t)| = \infty$. Here, $|\cdot|$ denotes the absolute value in the complex plane.

In general, infinite-time blowup solutions of partial differential equations are also studied, for the NLS see for example [27, 31, 38]. However, in this thesis, we do not study infinite-time blowup solutions and, unless stated otherwise, make the following convention: when we refer to blowup solutions, we always mean a finite-time blowup solution.

We make the following assumptions for the parameters d and σ in equation (1.9). Unless stated otherwise, these assumptions are supposed to hold throughout this thesis.

Assumption 1.2.2.

1.2.2.a $d > 1$,

1.2.2.b $2 < \sigma d < 2 + 2\sigma$,

1.2.2.c $0 < \sigma < 2$.

Assumption 1.2.2.a is used in many places in this thesis and especially in Lemma 2.2.6, the fundamental lemma that makes the more abstract existence results as presented in Theorem 2.1.1 more explicit. The restriction stated in Assumption 1.2.2.b is made because (self-similar) blowup can only occur if $\sigma d > 2$ and local existence of solutions is guaranteed if $\sigma d < 2 + 2\sigma$, see [13] equation (3.39), [26] Remark 5.11 and [26] Figure 7.1. Note that although the restriction $\sigma d < 2 + 2\sigma$ is common in the existing literature, see [9, 36, 40], as far as we are aware, it is not clear that local existence is guaranteed to fail for $\sigma d \geq 2 + 2\sigma$. Thus, we leave the possibility open that the results presented in this thesis can be extended beyond the parameter range specified in Assumption 1.2.2.b. Finally, Assumption 1.2.2.c is often made in the existing literature on the NLS, see for example [14]. Assumption 1.2.2.c is also used in Lemma 2.2.6 for technical reasons.

In short, the analysis performed in this thesis can be described as follows: we first establish existence of a blowup solution, then we perturb the blowup solution with a small perturbation, linearize and analyze the corresponding eigenvalue problem. We next describe the context in which we study blowup, describe how this fits into the existing literature and give an overview of the methodology.

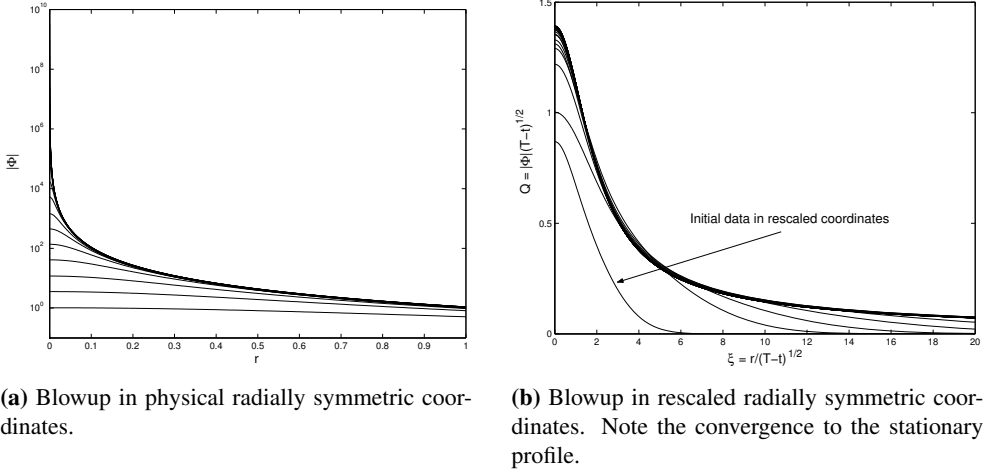


Figure 1.2.1: Numerical simulation of the full GL for $d = 3$, $\sigma = 1$, $\varepsilon = 0.1$, $b = 0$ and $\Phi_0(r) = \exp(-r^2)$. The thin lines represent snapshots of the profile at subsequent times between $t = 0$ and the blowup time T . The bold lines represent the final blowup profile. This is a reproduction of Figure 2.1 in [9].

1.2.2 Blowup in the NLS

Recall that for $\varepsilon = 0$, equation (1.9) is known as the Nonlinear Schrödinger equation (NLS). In the NLS, $\sigma d = 2$ is the so-called critical case: for $\sigma d < 2$ there are no finite-time blowup solutions, for $\sigma d > 2$ there are self-similar finite-time blowup solutions and for $\sigma d = 2$ there are finite-time blowup solutions, but the blowup is not necessarily self-similar, see [38, 44] for an overview of the literature on blowup in the NLS. As already noted, the critical dimension also appears in the parameter range for d and σ specified in Assumption 1.2.2.b.

1.2.3 Blowup in the GL

Upon studying the GL in the NLS limit, blowup solutions of a similar type as those arising in the NLS can be found. Existence of blowup solutions is studied in [9, 36, 39, 40]. Stability of blowup solutions is studied in [9, 36] with numerical methods. After transforming to radially symmetric coordinates, Figure 1.2.1a shows an example of a blowup solution in the GL. Similar to the introductory example in Section 1.1, we will introduce a rescaling such that the blowup solution corresponds to the stationary state of a rescaled differential equation, see Section 1.2.4. Figure 1.2.1b shows blowup solution presented in Figure 1.2.1a in rescaled coordinates.

1.2.4 The dynamical rescaling

In this section, we discuss a rescaling that scales a radially symmetric finite-time blowup solution of (1.9) into a solution that converges to a stationary profile in infinite time, see Figure 1.2.1. A radially symmetric solution of equation (1.9) satisfies

$$i\Phi_t + (1 - i\varepsilon)\left(\Phi_{rr} + \frac{d-1}{r}\Phi_r\right) + (1 + ib\varepsilon)|\Phi|^{2\sigma}\Phi = 0, \quad (1.11)$$

where $r \geq 0$ is the radial coordinate. As in [9, 36, 39], we introduce the transformation

$$\xi := \frac{r}{L(t)} \quad (1.12a)$$

$$\tau := \int_0^t \frac{1}{L(s)^2} ds, \quad (1.12b)$$

$$Q(\xi, \tau) := e^{-i\omega\tau} L(t)^{\frac{1}{\sigma}} \Phi(x, t). \quad (1.12c)$$

We will refer to the (r, t) -coordinates as the physical coordinates and to the (ξ, τ) -coordinates as the rescaled coordinates. From (1.11), we obtain

$$iQ_\tau + (1 - i\varepsilon) \left(Q_{\xi\xi} + \frac{d-1}{\xi} Q_\xi \right) - \omega Q + i \frac{a(\tau)}{\sigma} Q + ia(\tau)\xi Q_\xi + (1 + i\varepsilon)|Q|^{2\sigma} Q = 0, \quad (1.13a)$$

$$a(\tau) = -L \frac{dL}{dt}. \quad (1.13b)$$

And, the boundary conditions are given by

$$Q_\xi(0, \tau) = 0, \quad (1.14a)$$

$$|Q(\xi, \tau)| \rightarrow 0 \text{ as } \xi \rightarrow \infty. \quad (1.14b)$$

Note that condition (1.14a) ensures that when the solution of (1.13a), i.e. a function of $\xi \geq 0$ and τ , is interpreted as a radially symmetric function in $\mathbb{R}^d$, it is smooth in the origin.

We now show that if we assume that $L(t)$ vanishes as t tends to the blowup time T , $a(\tau)$ is a positive constant and that Q is a non-zero solution of

$$(1 - i\varepsilon) \left(Q_{\xi\xi} + \frac{d-1}{\xi} Q_\xi \right) - \omega Q + i \frac{a}{\sigma} Q + ia\xi Q_\xi + (1 + i\varepsilon)|Q|^{2\sigma} Q = 0 \quad (1.15)$$

with boundary conditions

$$Q_\xi(0) = 0, \quad (1.16a)$$

$$|Q(\xi)| \rightarrow 0 \text{ as } \xi \rightarrow \infty, \quad (1.16b)$$

then $Q(\xi)$ corresponds to a finite time blowup solution of (1.11). For constant $a > 0$, we obtain from (1.12) and (1.13b) that

$$L(t) = \sqrt{2a(T-t)} > 0, \quad (1.17a)$$

$$\tau = -\frac{1}{2a} \log \frac{T-t}{T}, \quad (1.17b)$$

where T is the finite blowup time of the problem. Now, first, observe that as t tends to the blowup time T , i.e. as τ tends to infinity, $L(t)$ tends to zero. Second, since the stationary solution is assumed not to be the zero solution, it is non-zero at least one point ξ_0 . And, since $|Q(\xi_0, \tau)|$ tends to a non-zero value as τ tends to infinity, $|\Phi(x, t)| = \left| \frac{Q(\xi, \tau)}{L(t)} \right|$ blows up as t tends to the blowup time T . Note that, as required, $L(t)$, see (1.17a), indeed vanishes as t tends to the blowup time T . Therefore, it follows from (1.12) that a stationary profile in rescaled coordinates will, apart from blowing up, also contract towards the origin in physical coordinates.

Rescaling (1.12) plays, in the following way, a similar role as the rescaling as the introductory example discussed in Section 1.1, see also equation (1.4). First, the rescaling transforms the original differential equation, i.e. (1.11), into a rescaled differential equation, i.e. (1.13a). Then, extra conditions are imposed such that the resulting differential, i.e. (1.13a), is autonomous when seen as an equation in τ . After that, stationary profiles of (1.13a) correspond to blowup solutions of (1.11).

Now that we have explained the dynamic rescaling, we motivate why we study a GL without a linear term. The linear term $-i\rho\varepsilon^2\Phi$ in (1.8) would have produced an extra term $-i\rho\varepsilon^2L(t)^2Q$ in (1.13a) which varies explicitly in time. Since $L(t) = \sqrt{2aT}e^{-a\tau}$, this term decays exponentially fast as τ tends to infinity. Therefore, this term is very small close to the blowup time. This motivates the choice to first fully understand existence and stability of blowup solutions in a GL without a linear term. We formulate this in a remark.

Remark 1.2.3. *The linear term $-i\rho\varepsilon^2\Phi$ in (1.8), produces an extra term $-i\rho\varepsilon^2L(t)^2Q$ in (1.13a). This term decays exponentially fast as τ tends to infinity and is thus expected to be small close to the blowup time. This motivates the choice to (first) study a GL without this linear term.*

1.3 Existence and numerical stability

1.3.1 Numerical results on existence and stability of blowup solutions

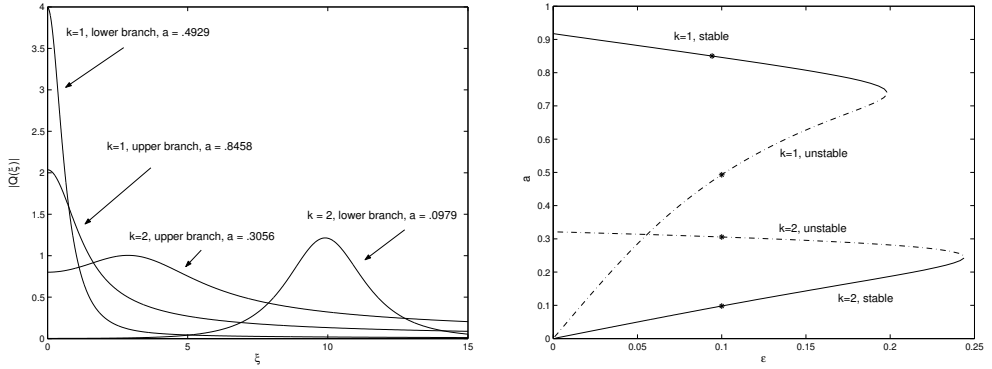
In this thesis, we focus on one of the types of blowup solutions studied in [9] which will be called the m_2 -solution.

Definition 1.3.1. *The solution of which we study existence and stability in this thesis will be called the m_2 -solution and comes in two types: m_2^{upper} and m_2^{lower} , see Definition 2.2.2 for a precise formulation.*

In [9], numerical and asymptotic methods are employed to study the existence of radially symmetric blowup solutions (including the m_2 -solution) of equation (1.11). And, numerical stability with respect to radially symmetric perturbations is also studied. In this section, we focus on the numerical results. Figure 1.3.1a shows several stationary solutions of equation (1.15) for $d = 3$, $b = 0$, $\omega = 1$ and $\sigma = 1$. These solutions thus correspond to blowup solutions of equation (1.11). Apart from at the origin, blowup can also take place at $\xi = \frac{\kappa}{a}$ for some $\kappa > 0$. These solutions correspond in Figure 1.3.1a to the ones with a maximum away from the origin and are labeled with “ $k = 2$ ”. Since for fixed ξ the physical coordinate r tends to zero as t tends to the blowup time T , solutions of which the blowup takes place at $\xi = \frac{\kappa}{a}$ look in physical coordinates like a contracting ring of which the absolute value blows up as t tends to the blowup time T .

Different blowup solutions of equation (1.11) require a different rescaling to scale out the blowup. Figure 1.3.1b gives the relation between ε and a for given d , b and σ , i.e. the figure shows the relation between ε and the rescaling L , depending on a , which selects the blowup solution that can be studied with that particular rescaling. For $d = 3$, $b = 0$, $\omega = 1$ and $\sigma = 1$, the m_2 -solutions correspond to solutions on to the lower part of the $k = 2$ -branch in Figure 1.3.1b. In Figure 1.3.1a, the solution which is small at the origin and with its maximum away from the origin is an m_2 -solution.

In [9], numerical stability of the m_2 -solution with respect to radially symmetric perturbations is studied by full PDE-simulation of equation (1.11). Solid and dashed lines in Figure



(a) Blowup solutions of the GL in the rescaled radially symmetric coordinates.

(b) Solution branches in the (ε, a) -plane.

Figure 1.3.1: Blowup solutions and solution branches of the GL in rescaled radially symmetric coordinates. Note that the blowup rescaling (1.12) depends on the parameter a through (1.17). The positive integer k indicates the number of maxima if we extend the solution, interpreted as a function of a one dimensional variable $\xi \geq 0$, symmetrically to the interval $(-\infty, \infty)$. The solutions corresponding to the *'s in Figure 1.3.1b are shown in Figure 1.3.1a. In both figures $d = 3$, $b = 0$, $\omega = 1$ and $\sigma = 1$. These figures are a reproduction of Figures 1.1 and 1.2 in [9].

1.3.1b represent respectively (numerically) stable and unstable solutions. Particularly surprising are the stability results of the m_2 -solution on the lower part of the $k = 2$ -branch in Figure 1.3.1b. Although not always the case (see for example [29]), it is common for ring solutions of PDE-models to be unstable, see [6, 8, 12]. Moreover, one might expect that $k = 1$ -solutions are physically more likely to be observed than ring solutions, i.e. the m_2 -solution. Therefore, in the range $0.2 \lesssim \varepsilon \lesssim 0.25$ the situation becomes even more interesting, because there the $k = 1$ -solutions do not, but the m_2 -solution does exist.

Studying blowup solutions with merely numerical methods is tricky, especially when the blowup takes place in finite time and when the spacial domain is infinite. For this reason, it is essential that methods are devised by which blowup behavior can be studied analytically.

1.3.2 Analytic existence results

The m_2 -solution, as defined by Definition 1.3.1, corresponds to solutions on the lower part of the $k = 2$ -branch. We will focus on the part of the $k = 2$ -branch where both ε and a are small. Existence of the m_2 -solution has been studied with asymptotic methods in [9]. In [40], existence has been studied with analytic methods and it is shown that the m_2 -solution is, actually, only one of four ring-like solutions that equation (1.15) admits if a and ε are both small. Recently, an asymptotic construction of one of the other three ring-like solutions has been presented in [39], see also Theorem 2.1.1 and Section 2.5.1 for more details. In the remainder of this thesis, however, we limit the discussion to the m_2 -solution.

1.3.2.1 Regions and assumptions used in the asymptotic construction

The rigorous existence result of the m_2 -solution relies heavily on insights obtained by asymptotic methods. We now present several assumptions, observations and definitions needed for further discussion.

Upon rescaling τ , ξ , a and Q in equation (1.15) or (1.13a), it is possible to scale ω to

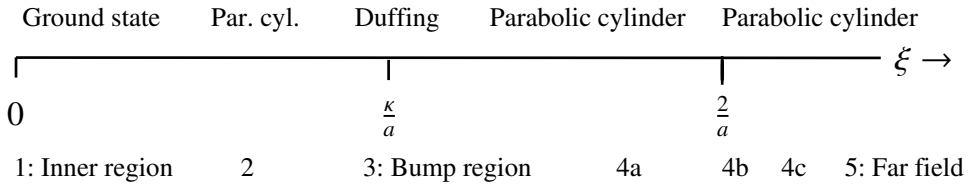


Figure 1.3.2: Overview of all the different regions in the asymptotic construction. Furthermore, the type of equation to which equation (1.15) reduces at leading order is indicated for all the regions.

1. Alternatively, ω can be left free and we can rescale a , ω and Q such that $|Q(0,0)| = 1$. Without loss of generality, we make the following assumption:

Assumption 1.3.2. *Unless stated otherwise, we set $\omega = 1$.*

We continue with the observation that equation (1.15) is phase invariant. This leads to the following conclusion.

Remark 1.3.3. *Since (1.15) is phase invariant, every solution is part of a family of solutions.*

Based on the numerically suggested linear behavior between ε and a when they are both small (see Figure 1.3.1b), we introduce the following scaling.

Assumption 1.3.4. $\varepsilon = Ka$ for some $0 < K = O(1)$.

The asymptotic construction is performed in the small parameter a . Thus, from now on we assume:

Assumption 1.3.5. $0 < a, \varepsilon \ll 1$.

It might seem counterintuitive to, as in Assumption 1.3.4, write ε as a function of a , this is done for historical reasons: for example, in the numerical analysis presented in [9], the parameter a is tuned until a blowup solution is detected.

Finally, in the asymptotic construction the half-line $\xi \geq 0$ is divided into several regions. Since both the asymptotic construction of the m_2 -solution and the study of stability of the m_2 -solution relies on this division into regions, we present it here. Figure 1.3.2 shows the distinguished regions used in the asymptotic construction. Furthermore, it also indicates the type of equation to which (1.15) reduces at leading order in the indicated region. We now define the regions used in the asymptotic construction.

Definition 1.3.6.

1.3.6.a *In the inner region, i.e. the region corresponding to Region 1 in Figure 1.3.2, $\xi \ll \frac{1}{a}$.*

1.3.6.b *In the bump region, i.e. the region corresponding to Region 3 in Figure 1.3.2, $|\xi - \frac{\kappa}{a}| \ll \frac{1}{a}$. This is the region in which the ring solutions on the lower part of the $k = 2$ -branch have their maximum.*

1.3.6.c *In the far field, i.e. the region corresponding to Region 5 in Figure 1.3.2, $|\xi - \frac{2}{a}| \gg \frac{1}{a}$. These three regions do not overlap, therefore, in between these regions, there are transition regions which are called Region 2 and Region 4. In the transition regions, the equation reduces to a parabolic cylinder equation with slowly varying coefficients, see Section 2.3.*

1.3.6.d In Region 2, i.e. the transition region between the inner region and the bump region, we have $\xi \gg 1$, $\xi < \frac{\kappa}{a}$ and $|\xi - \frac{\kappa}{a}| \gg 1$.

At the point $\xi = \frac{2}{a}$ in transition Region 4, the parabolic cylinder equation has a turning point, see Section 2.3.6. Therefore, we split up Region 4 into three parts: 4a, 4b and 4c.

1.3.6.e Region 4a consists of all ξ for which $\xi - \frac{\kappa}{a} > 0$, $|\xi - \frac{\kappa}{a}| \gg 1$, $\xi - \frac{2}{a} < 0$ and $|\xi - \frac{2}{a}| \gg 1$.

1.3.6.f In Region 4b, $|\xi - \frac{2}{a}| \ll a^{-\frac{2}{3}}$;

1.3.6.g For Region 4c, we require $\xi - \frac{2}{a} > 0$, $|\xi - \frac{2}{a}| \gg 1$ and $\xi \ll \frac{1}{a^2}$.

1.3.2.2 Extension of the existence results

In Chapter 2, we extend the existence results for the m_2 -solution as presented in [9] and [40]. An important discovery is the existence of a saddle-node bifurcation that the system undergoes as we pass (for $\sigma = 1$) the curve $d^2 + 24bd - 32b + 16b^2 = 0$ in the strip $2 < d < 3$ of the (d, b) -plane, see also Figure 2.2.1 and Corollary 2.2.7. The solutions on the two branches of the saddle-node bifurcation will be called m_2^{upper} and m_2^{lower} .

Definition 1.3.7. *The solutions on the upper and lower branch of the saddle-node bifurcation, i.e. the branches with the largest and smallest value of κ , as denoted in Figure 2.2.1 and Corollary 2.2.7 will be called respectively m_2^{upper} and m_2^{lower} , see Definition 2.2.8 for a precise formulation.*

Note that Figure 1.3.1b does not display the saddle-node bifurcation, and, therefore, m_2^{upper} and m_2^{lower} should not be confused with solutions on the upper and lower part of the $k = 2$ -branch in Figure 1.3.1b! Since, as can be verified, the saddle-node bifurcation takes place for values $b < 0$ and the case $b < 0$ is not included in [9] and [40], this can be seen an extension of the existence results to a wider parameter range.

We also present in Chapter 2 a non-trivial higher order extension of the asymptotic construction presented [9] and a detailed matching procedure. The higher order extension is needed for the spectral stability analysis we perform in Chapter 3.

1.4 Stability

In Chapters 3 and 4, we study the stability of the m_2 -solution with analytic methods. Our goal is to obtain explicit stability results through using the asymptotic construction of these solutions as presented in [9] and further extended in Chapter 2.

After adding a perturbation V to the blowup solution in rescaled variables and linearization, we obtain a linear evolution equation of the form

$$V_\tau = L_V(a)V,$$

where $L_V(a)$ is a matrix containing second order partial differential operators and V is a vector with complex valued functions.

Definition 1.4.1. *A (rescaled blowup) solution $Q(\xi)$ of equation (1.15) is called linearly stable if for every allowed perturbation V_0 , the solution $V(\tau)$ of the linearized equation $V_\tau = L_V(a)V$ with $V(0) = V_0$ vanishes (in an appropriate norm) as τ tends to infinity.*

By “allowed”, we mean allowed according to an appropriate criterion in this definition, e.g. through choosing an appropriate function space.

Definition 1.4.2. *A rescaled blowup solution $Q(\xi)$ of equation (1.15) is called spectrally stable if the spectrum of the linear operator $L_V(a)$ is a subset of $\{c \in \mathbb{C} : \text{Re } c \leq 0\}$.*

The spectrum of $L_V(a)$ does not solely consist of eigenvalues (of finite multiplicity), i.e. isolated values of $\lambda \in \mathbb{C}$ for which the equation

$$L_V(a)V = \lambda V \quad (1.18)$$

has a solution V in the appropriately chosen function space and for which the dimension of the kernel of $L_V(a) - \lambda$ is finite. This part of the spectrum is also referred to as discrete spectrum. The part of the spectrum that does not correspond to eigenvalues (of finite multiplicity) is defined as essential spectrum, see [19]. Although the relation between linear and spectral stability may be subtle, see for example Chapter 5 of [19], it is straightforward to conclude the following for eigenvalues. Suppose we construct an eigenfunction V , i.e. an explicit solution V of the eigenvalue problem with corresponding eigenvalue λ , then the function $e^{\lambda\tau}V$ solves $V_\tau = L_V(a)V$. From this relation, we conclude the following.

Remark 1.4.3. *If V is an eigenfunction corresponding to an eigenvalue λ with positive real part, then the solution Q is both linearly and spectrally unstable.*

1.4.1 Spectral stability with respect to radially symmetric perturbations

In Chapter 3, we study spectral stability with respect to radially symmetric perturbations of the m_2 -solution. Thus, the perturbation V then only depends on the radial coordinate ξ . We will show that the m_2^{lower} -solution is spectrally and linearly unstable with respect to radially symmetric perturbations. Furthermore, for the m_2^{upper} -solution, we do not find any spectrum that leads to instability. See Figure 3.4.1a and Theorem 3.4.1 for further details.

1.4.1.1 The unperturbed NLS spectrum

For our stability analysis, we use the same regions as in the asymptotic construction of the solution. The use of the asymptotic construction and the associated division in spatial regions, see Figure 1.3.2, in the stability analysis is inspired by an example presented in [23]. Here, the author studies with Evans function techniques how zero eigenvalues are perturbed in a perturbed NLS:

$$i\partial_\tau S + \partial_s^2 S - \alpha S + |S|^{2\sigma} S = aR(s, S, \bar{S}), \quad (1.19)$$

where $S(s, \tau)$ is a complex valued function, $\tau \geq 0$ is the temporal coordinate, $s \in \mathbb{R}$ is the spatial coordinate, α is a positive real valued constant and $R(s, S, \bar{S})$ is a nonlinearity. Although explicit theory in [23] is only developed for equation (1.19) with various special forms of R , the general framework presented is much broader. We can transform equation (1.13a) in the bump region into equation (1.19) in the following way. First, zoom in on the bump region with the translation $s = \xi - \frac{\kappa}{a}$. Second, by setting $Q(s, \tau) = e^{-\frac{1}{2}i\kappa_0 s} S(s, \tau)$, where κ_0 is a leading order approximation of κ , the leading order equation will be in self-adjoint form, i.e. the first order derivative vanishes. Furthermore, with this transformation $S(s, \tau)$ satisfies a PDE of the same form as equation (1.19), see also equation (2.10). Outside the bump region, this transformation is still valid, but then the perturbation $R(s, S, \bar{S})$ will, as we will show, no longer be small. This will be covered in more detail in Chapter 3. Nevertheless, eigenvalue problem (1.18) can be transformed with a similar transformation. The resulting eigenvalue problem will be written as:

$$L(a)W = \lambda W, \quad (1.20)$$

The unperturbed NLS spectrum, i.e. the spectrum of $L(0)$, is known and consists of an eigenvalue $\lambda = 0$ with algebraic multiplicity 4 and geometric multiplicity 2, see [23, 25, 45].

Furthermore, unperturbed essential spectrum consists of the imaginary axis with the section around the origin removed. As a becomes non-zero, an a priori unknown number of these 4 eigenvalues will move away from $\lambda = 0$. If one of these eigenvalues has positive real part for $a \neq 0$, this implies spectral (and linear, see Remark 1.4.3) instability. Similarly, the essential spectrum also perturbs and might lead to spectral instability.

1.4.1.2 The perturbed NLS spectrum and Evans function techniques

In Chapter 3, the unperturbed NLS spectrum will be taken as a starting point and we will limit the study to the part of the spectrum that can be seen as a perturbation of the unperturbed NLS spectrum. Since the stability analysis will use the asymptotic construction of the m_2 -solution, the regions distinguished in the asymptotic construction of the m_2 -solution, see Definition 1.3.6, also play a role in the stability analysis.

To study the perturbed NLS spectrum, we will study two types of solutions of eigenvalue problem (1.20): solutions that are exponentially localized in the bump region and solutions that remain bounded in the bump region. In the second case, we show using asymptotic methods that the perturbed NLS essential spectrum is contained in the subset $\{z \in \mathbb{C} : \operatorname{Re}(z) \leq 0\}$. This implies that the perturbed NLS essential spectrum does not lead to spectral instability, see Theorem 3.3.3.

Remark 1.4.4. *The term localized as a property of real or complex valued functions is not precisely defined, but is often used to describe qualitative behavior of functions. With localized, we mean vanishing at infinity. With localized in a region, we mean that the function is small in the complement of the region, where small will dependent on the context. See [18] for a discussion. We will use the term localized in qualitative discussions in this thesis and will avoid the use of the term in theorems.*

To determine the perturbed NLS discrete spectrum, we study solutions that are exponentially localized in the bump region. As in [23], Evans function techniques are used to determine the perturbed NLS discrete spectrum. The Evans function, which is a standard technique for eigenvalue problems, is an analytic function associated to the linear operator $L(a)$ of the eigenvalue problem and its zeros are the eigenvalues of $L(a)$, see [1, 15, 23, 41]. The construction of the Evans function relies on exponential localization of the eigenfunctions of $L(a)$. However, it follows from the asymptotic construction that eigenfunctions, associated to the spectral stability of the m_2 -solution, will decay algebraically in the far field. This is one of the reasons that, although the theory in [23] forms the basis of our study, it is certainly not directly applicable and has to be extended, see Section 3.2.1.

In Section 3.2, we show that the Evans function $E(\lambda, a)$ has a Taylor expansion around $\lambda = 0$ and $a = 0$ of the form

$$E(\lambda, a) = \lambda \left(\frac{\partial_\lambda^4 E(0, 0)}{4!} \lambda^3 + \frac{\partial_a \partial_\lambda^3 E(0, 0)}{3!1!} a \lambda^2 + \frac{\partial_a^2 \partial_\lambda^2 E(0, 0)}{2!2!} a^2 \lambda + \frac{\partial_a^3 \partial_\lambda E(0, 0)}{1!3!} a^3 + o(3) \right), \quad (1.21)$$

where by $o(3)$ we indicate terms that are of higher order compared to terms of degree 3 in a and λ . From the fact that the Evans function vanishes at the eigenvalues, we can determine the leading order location of the eigenvalues near 0 for $0 < a \ll 1$. Note that it also follows from (1.21) that, under the assumption that $\partial_a^3 \partial_\lambda E(0, 0) \neq 0$, 3 eigenvalues move out of $\lambda = 0$ as a becomes non-zero. Expressions for the derivatives of the Evans function needed in (1.21) can be deduced by extending the theory developed in [23].

1.4.1.3 Conclusion

As explained, the m_2 -solution undergoes a saddle-node bifurcation upon (for $\sigma = 1$) passing the curve $d^2 + 24bd - 32b + 16b^2 = 0$ in the strip $2 < d < 3$ of the (d, b) -plane, see also Figure 2.2.1. The conclusion of Chapter 3 will be that the perturbed NLS spectrum does not lead to instability for the m_2^{upper} -solution, i.e. the solution on the upper branch of the saddle-node bifurcation, and that the m_2^{lower} -solution is spectrally (and linearly, see Remark 1.4.3) unstable, i.e. the solution on the lower branch of the saddle-node bifurcation, see Theorem 3.4.1.

1.4.2 Spectral stability with respect to non-radially symmetric perturbations

In Chapter 4, we study stability of the m_2^{upper} -solution with respect to non-radially symmetric perturbations. Since the m_2^{lower} -solution is already unstable with respect to radially symmetric perturbations, its stability with respect to non-radially symmetric perturbations will not be considered. Apart from determining spectral stability with respect to non-radially symmetric perturbations, a second goal in Chapter 4 is to investigate to what extend we can generalize the techniques used in the radially symmetric setting such that they can be applied in the non-radially symmetric setting.

The first step in the non-radially symmetric case is to expand the eigenvalue equation into spherical harmonic modes Y_ℓ^m and to derive an eigenvalue problem for each mode. Each mode will then be analyzed separately.

As in the radially symmetric case, the unperturbed NLS point spectrum will be taken as a starting point. In the second step, we apply the same techniques used in the radially symmetric setting to derive an explicit expressions for a Taylor expansion of the Evans function around $\lambda = 0$ and $a = 0$ for each mode (a priori parametrized by ℓ and m , although the eigenvalue problem turns out to be independent of m):

$$E(\lambda, a, \ell, m) = E(\lambda, a, \ell) = \frac{\partial_\lambda^4 E(0, 0, \ell)}{4!} \lambda^4 + \frac{\partial_a \partial_\lambda^3 E(0, 0, \ell)}{3!1!} \lambda^3 a + \frac{\partial_a^2 \partial_\lambda^2 E(0, 0, \ell)}{2!2!} \lambda^2 a^2 + \frac{\partial_a^3 \partial_\lambda E(0, 0, \ell)}{1!3!} \lambda a^3 + \frac{\partial_a^4 E(0, 0, \ell)}{4!} a^4 + o(4), \quad (1.22)$$

where by $o(4)$ we indicate terms that are of higher order compared to terms of degree 4 in a and λ . From (1.22), the perturbed NLS point spectrum can be determined to leading order. Although it turns out to be sufficient to study the case $\ell = 1$, the case $\ell = 2$ will also be analyzed as an illustration.

Finally, the conclusion of Chapter 4 will be that, if non-radially symmetric perturbations are included, the perturbed NLS spectrum leads to spectral (and linear, see Remark 1.4.3) instability for the m_2^{upper} -solution, see Theorem 4.3.1.

Chapter 2

Existence of ring-like blowup solutions

2.1 Set-up

In this chapter, we study the existence and asymptotic approximations of a ring-like self-similar blowup solutions of (1.9), initial and boundary conditions given by (1.10), called the m_2 -solution. We will start this section with a discussion of the existence of several types of blowup solutions of (1.9) and a list of defining properties of the m_2 -solution is given in Assumption 2.2.1. A precise definition of the m_2 -solution will be given in Definition 2.2.2.

As explained in Section 1.2.4, ring-like self-similar blowup solutions of (1.9) may appear if Q satisfies

$$iQ_\tau + (1 - i\varepsilon) \left(Q_{\xi\xi} + \frac{d-1}{\xi} Q_\xi \right) - Q + i\frac{a}{\sigma} Q + ia\xi Q_\xi + (1 + ib\varepsilon)|Q|^{2\sigma} Q = 0, \quad (2.1)$$

where $\xi \geq 0$, the given initial condition is denoted by $Q(0, \tau)$ and boundary conditions are given by

$$Q_\xi(0, \tau) = 0, \quad (2.2a)$$

$$|Q(\xi, \tau)| \rightarrow 0 \text{ as } \xi \rightarrow \infty. \quad (2.2b)$$

Ring-like blowup solutions of (1.9) now correspond to stationary solutions of (2.1). As denoted in Section 1.2.4, equation (1.9) is phase invariant and every solution is thus part of a family of solutions. Let Ψ_0 be the phase of the m_2 -solution in the origin, i.e. the phase of $Q(0)$. Of course, we could set $\Psi_0 = 0$, but we prefer to set Ψ_0 to such a value that the leading order of a rescaling of the solution is real-valued in the bump region, see Section 2.2.2 and expression (2.67). Thus, the stationary bump solutions we study in this chapter satisfy

$$\left\{ \begin{aligned} (1 - i\varepsilon) \left(Q_{\xi\xi} + \frac{d-1}{\xi} Q_\xi \right) - Q + i\frac{a}{\sigma} Q + ia\xi Q_\xi + (1 + ib\varepsilon)|Q|^{2\sigma} Q &= 0, \end{aligned} \right. \quad (2.3a)$$

$$\left\{ \begin{aligned} Q_\xi(0) &= 0, \quad \frac{Q(0)}{|Q(0)|} = e^{i\Psi_0} \text{ and } |Q(\xi)| \rightarrow 0 \text{ as } \xi \rightarrow \infty, \end{aligned} \right. \quad (2.3b)$$

where $b \in \mathbb{R}$, $0 < \varepsilon \ll 1$, $0 \leq \Phi_0 < 2\pi$, Q is a complex-valued function of ξ and the valid parameter range for d and σ is as defined in Assumption 1.2.2

We are interested in the existence of the ring-like solutions on the lower part of the $k = 2$ -branch, see Figure 1.3.1. Rigorous existence results for solutions to (2.3) can be found in [40]. From these rigorous existence results, it follows that there exist four $k = 2$ -solutions, i.e. solutions with two maxima on the real line (one of these four will be the m_2 -solution). We reformulate the main result presented in [40] in such a way that it directly applies to our setting.

Theorem 2.1.1. *Let $0 < a \ll 1$, $0 \leq \Psi_0 < 2\pi$, $b > 0$ and $2 < d < 4$ be given and let $\sigma = 1$. If there exist $0 < K_0 = O(1)$ and $0 < \eta < 2$ such that:*

$$2.1.1.a \quad \frac{d-2}{2} - K_0 - \frac{2}{3}(2b-1)K_0 \left(1 - \frac{\eta^2}{4}\right) = 0,$$

$$2.1.1.b \quad \frac{2}{3} \left(\frac{1-d}{\eta} + K_0 \eta \right) + \frac{\eta}{2} = 0,$$

then there exists an $n_0(a)$ such that for $n \leq n_0(a)$ there exist $2n$ unique n -bump solutions of equation (2.3), i.e. solutions with $\frac{n}{2}$ maxima on the half line $\xi \geq 0$, with

$$2.1.1.c \quad \varepsilon = K_0 a + o(a),$$

$$2.1.1.d \quad n \text{ is an even integer.}$$

Furthermore, $n_0(a)$ increases as the parameter a decreases. The maxima of a particular solution are located between $\xi = 0$ and $\xi = \xi_{\max} = \frac{2-\sqrt{a}}{a}$ and are $O(-\log a)$ apart. For $n-1$ of the $2n$ solutions $|Q(\xi_{\max})| = O(a^{\frac{3}{8}})$ and the last maximum of these last $n-1$ solutions occurs near $\xi_{\max}$ with $|Q| = \sqrt{2}a^{\frac{1}{4}}(1 - \frac{1}{8}\sqrt{a}) + o(a^{\frac{3}{4}})$.

Proof. See Appendix A. □

According to Theorem 2.1.1, there exist four different 2-bump solution on the lower part of the $k = 2$ -branch in the limit $\varepsilon \rightarrow 0$, i.e. the part of the $k = 2$ -branch where both a and ε are small, see Figure 1.3.1. In Section 2.2, we present a higher order asymptotic construction of one of these four 2-bump solutions called the m_2 -solution, which extends the less detailed leading order asymptotic construction in [9]. Furthermore, the m_2 -solution has also been observed numerically in [9]. This construction extends Theorem 2.1.1 for one of these solutions to certain negative values of b , see also Corollary 2.2.7. In Section 2.5.1, we discuss the relation between the constructed solution and the four different 2-bump solution on the lower part of the $k = 2$ -branch that exist according to Theorem 2.1.1. Recently, the asymptotic construction for another of these 4 blowup solutions has been presented in [39], see also Section 2.5.1.

We point out that Condition 2.1.1.a together with Condition 2.1.1.b defines a system of equations from which K_0 and η can be solved as a function of d and b . The exact same system of equations also plays a role in the asymptotic construction presented in Section 2.2 and arises as a solvability condition when one requires that the $O(a)$ correction to the leading order solution is bounded. Lemma 2.2.6 gives conditions on d and b under which this system of equations has a unique solution.

2.2 Asymptotic construction of a ring-like blowup solution

In this section, we present an asymptotic construction of a ring-like solution, introduced as the m_2 -solution in Section 2.1, to the ordinary differential equation (2.3a) that goes beyond the existence results and asymptotic construction in the currently existing literature, see [9, 39, 40]. If convenient, we will restrict ourselves, as will be explicitly indicated in the text, to the case $\sigma = 1$ to avoid technical difficulties. We closely follow the approach in [7, 9]. This section extends the known results in three ways. First, we extend the analysis in the bump

region, the region near the peak, to higher order. Higher order expressions are needed in the study of the stability of the m_2 -solution, see Chapters 3 and 4. Second, we give details of the matching procedure for the GL. So far, these details have only been presented for the NLS, see [7]. And finally, we extend the asymptotic results in [9] for certain negative values of b , which leads to the discovery of a saddle-node bifurcation, see Corollary 2.2.7.

Assumption 2.2.1. *The m_2 -solution we construct is assumed to have the following properties:*

2.2.1.a *$Q(0)$ is exponentially small in a .*

2.2.1.b *The absolute value of Q attains its maximum at $\xi = \frac{\kappa}{a}$ for some $O(1)$ constant $0 < \kappa < 2$. This assumption is based upon the proof of Theorem 2.1.1 (see [40] and Appendix A) and the numerical simulations in [9]. The requirement $\kappa < 2$ will be used in Section 2.3.1.*

2.2.1.c *As also stated in Assumption 1.3.4 and restated here to make the list of assumptions complete, we assume $\varepsilon = Ka$, with $0 < K = O(1)$. Theorem 2.1.1 shows that for $0 < \varepsilon \ll 1$ the relation between a and ε is linear to leading order, see also the lower part of the $k = 2$ -branch in Figure 1.3.1. For consistency with existing literature we prefer to work with the parameter a instead of ε as our small parameter, see [7, 9, 36, 40].*

2.2.1.d *We assume that the following expansions in the small parameter a exist and are well-ordered:*

$$Q(\xi) = Q_0(\xi) + aQ_1(\xi) + a^2Q_2(\xi) + o(a^2), \quad (2.4a)$$

$$\kappa = \kappa_0 + a\kappa_1 + a^2\kappa_2 + o(a^2), \quad (2.4b)$$

$$K = K_0 + aK_1 + a^2K_2 + o(a^2). \quad (2.4c)$$

As will be covered in more detail in Section 2.5.1, these four assumptions select for $b > 0$ one particular solution from the four 2-bump solutions that exist according to Theorem 2.1.1. One of the other solutions attains its maximum $o(\frac{1}{a})$ close to $\xi = \frac{2}{a}$, i.e. for this solution $\kappa = 2$ to leading order. This is inconsistent with Assumption 2.2.1.b and hence we do not construct it here. For the other two solutions, Assumption 2.2.1.d does not hold.

The solution studied in this chapter will be called the m_2 -solution.

Definition 2.2.2. *A solution that satisfies equation (2.3) and satisfies Assumption 2.2.1 will be called an m_2 -solution of equation (2.3).*

Note that for $b \leq 0$, existence and uniqueness of the m_2 -solution is not implied by Theorem 2.1.1 as it is presented here and will be investigated with asymptotic methods in Section 2.2.2.3, see also Corollary 2.2.7.

In the forthcoming sections, we present an asymptotic construction of the m_2 -solution in each of the regions described in Section 1.3.2.1, see also Figure 1.3.2. After that we will perform a matching procedure in a scale in which the regions overlap.

2.2.1 Inner region

Since we assumed $|Q(0)| \ll 1$, see Assumption 2.2.1.a, we conclude that in a region close to $\xi = 0$ we have $|Q|^{2\sigma} \ll 1$. For $|Q|^{2\sigma} \ll 1$ and $\xi \ll \frac{1}{a}$, equation (2.3a) reduces to leading order to

$$\xi^2 Q_{\xi\xi} + \xi(d-1)Q_\xi - \xi^2 Q = 0. \quad (2.5)$$

The transformation $Q = \xi^{\frac{2-d}{2}} \tilde{Q}$ reduces this equation to the modified Bessel equation:

$$\xi^2 \tilde{Q}_{\xi\xi} + \xi \tilde{Q}_{\xi} - \left(\xi^2 + \left(\frac{d-2}{2} \right)^2 \right) \tilde{Q} = 0,$$

see [33] equation (10.25.1). Therefore, the solution to equation (2.5) is given by

$$Q(\xi) = C_I \xi^{\frac{2-d}{2}} I_{\frac{d-2}{2}}(\xi) + C_K \xi^{\frac{2-d}{2}} K_{\frac{d-2}{2}}(\xi), \quad (2.6)$$

where C_I and C_K are constants and $I_{\frac{d-2}{2}}$ and $K_{\frac{d-2}{2}}$ are modified Bessel functions of the first and second kind respectively, see [33] Section 10.25. Since $\xi^{\frac{2-d}{2}} K_{\frac{d-2}{2}}$ blows up at $\xi = 0$, we require that

$$C_K = 0.$$

Note that with this requirement also the boundary condition $Q_{\xi}(0) = 0$ is satisfied. Now, from (2.6), it follows that

$$Q(0) = \frac{2^{\frac{2-d}{2}}}{\Gamma\left(\frac{d}{2}\right)} C_I. \quad (2.7)$$

Also, note that for $Q(0)$ to be exponentially small in a , our analysis must show that C_I is exponentially small in a . We will derive this in Section 2.3.2, where we will match this solution in the inner region with a solution in Region 2 by using the so-called intermediate scale $\xi = a^{-\gamma}t$ with $0 < \gamma < 1$ and $t = O(1)$, i.e. a scale in which Region 1 and Region 2 overlap, see Definition 1.3.6. From the asymptotic behavior of the modified Bessel function for large ξ , see [33] equation (10.40.1), it follows that the leading order behavior of Q , as given by (2.6), in the intermediate scale is given by

$$Q(a^{-\gamma}t) = C_I (a^{-\gamma}t)^{\frac{2-d}{2}} \sqrt{\frac{2}{\pi a^{-\gamma}t}} \exp(a^{-\gamma}t). \quad (2.8)$$

2.2.2 Bump region

In this section, we construct an asymptotic expansion of the m_2 -solution which is localized in the bump region, i.e. the region that corresponds to Region 3 in Definition 1.3.6.b. We defined κ such that $|Q|$ attains its maximum at $\xi = \frac{\kappa}{a}$, see Assumption 2.2.1.b. This motivates the translation

$$\xi = \frac{\kappa}{a} + s, \quad (2.9)$$

where the bump region is restricted by $|\xi - \frac{\kappa}{a}| = |s| \ll \frac{1}{a}$. Furthermore, we assumed a linear relation between a and ε : $\varepsilon = Ka$, see Assumption 2.2.1.c. Upon using (2.9), equation (2.3a) for Q becomes

$$Q_{ss} + i\kappa Q_s - Q + |Q|^{2\sigma} Q = iKaQ_{ss} - a(1 - iKa) \frac{d-1}{\kappa + as} Q_s - iasQ_s - i\frac{a}{\sigma} Q - ibKa|Q|^{2\sigma} Q.$$

The Liouville transformation $Q(s) = e^{-\frac{1}{2}i\kappa_0 s} S(s)$ eliminates the first order derivatives at leading order. This leads to

$$\begin{aligned} S'' - \left(1 - \frac{\kappa_0^2}{4}\right) S + |S|^{2\sigma} S &= iKa \left[S'' - i\kappa_0 S' - \frac{\kappa_0^2}{4} S + a \frac{d-1}{\kappa + as} \left(S' - \frac{i\kappa_0}{2} S \right) \right] \\ &\quad - i((\kappa - \kappa_0) + as) \left(S' - \frac{i\kappa_0}{2} S \right) - \frac{ia}{\sigma} S - a \frac{d-1}{\kappa + as} \left(S' - \frac{i\kappa_0}{2} S \right) - ibKa|S|^{2\sigma} S, \end{aligned} \quad (2.10)$$

where prime denotes differentiation with respect to the translated space variable s . In this equation, we pose a regular expansion for $S(s)$:

$$S(s) = S_0(s) + aS_1(s) + a^2S_2(s) + O(a^3).$$

This is similar to the expansion (2.4a) for Q , and we have that $Q_i(s) = e^{-\frac{1}{2}ik_0s}S_i(s)$.

2.2.2.1 Leading order analysis

The equation for S_0 is given by

$$S_0'' - \alpha S_0 + |S_0|^{2\sigma} S_0 = 0, \quad (2.11)$$

where

$$\alpha := 1 - \frac{\kappa_0^2}{4}. \quad (2.12)$$

We now exploit the phase invariance and require that the solution S_0 is real-valued. This requirement fixes Ψ_0 , the phase of $Q(0)$ as introduced in Section 2.1, for which we derive an explicit formula in Section 2.3.3. For S_0 real, equation (2.11) is known as the Duffing equation. The only positive localized solution of the Duffing equation is given by

$$S_0(s) = ((\sigma + 1)\alpha)^{\frac{1}{2\sigma}} \operatorname{sech}^{\frac{1}{\sigma}}(\sigma\sqrt{\alpha}s), \quad (2.13)$$

see [16].

2.2.2.2 First order analysis

Taking the real and imaginary part of equation (2.10) at $O(a)$ leads to equations for the real and imaginary part of S_1 . For convenient notation, we define the following differential operators L_1 and L_2 :

$$L_1 u := (\partial_s^2 - \alpha + (2\sigma + 1)S_0^{2\sigma})u, \quad (2.14a)$$

$$L_2 u := (\partial_s^2 - \alpha + S_0^{2\sigma})u. \quad (2.14b)$$

In terms of L_1 and L_2 , the equations for the real and imaginary part of S_1 are given by

$$L_1 \operatorname{Re} S_1 = f(s) := K_0 \kappa_0 S_0' - \frac{\kappa_0}{2}(s + \kappa_1)S_0 - \frac{d-1}{\kappa_0}S_0', \quad (2.15a)$$

$$L_2 \operatorname{Im} S_1 = g(s) := K_0 \left(S_0'' - \frac{\kappa_0^2}{4}S_0 \right) - (s + \kappa_1)S_0' - \frac{S_0}{\sigma} + \frac{d-1}{2}S_0 - bK_0 S_0^{2\sigma+1}. \quad (2.15b)$$

Solutions to the homogeneous part of equation (2.15a) are given by

$$\psi_1(s) := S_0', \quad (2.16a)$$

$$\psi_2(s) := \begin{cases} S_0' \left(\int_{\tilde{c}_1}^s \frac{1}{S_0^2} d\tilde{s} \right) & \text{if } s > 0, \\ S_0' \left(\int_{\tilde{c}_2}^s \frac{1}{S_0^2} d\tilde{s} \right) & \text{if } s < 0, \\ \tilde{c}_3 & \text{if } s = 0, \end{cases} \quad (2.16b)$$

where $\tilde{c}_1 > 0$ and $\tilde{c}_2 < 0$. It can easily be verified that the conditions $\psi_2(s)$ is smooth in $s = 0$ and $\psi_2(s)$ is an even function of s uniquely determine the constants $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{c}_3$. In particular, for $\sigma = 1$, it follows that $\tilde{c}_3 = 0$, $\tilde{c}_2 = -\tilde{c}_1$ and $\tilde{c}_1$ is the unique positive root of

$$\frac{1}{4} \sinh(2\sqrt{\alpha}\tilde{c}_1) - \coth(\sqrt{\alpha}\tilde{c}_1) + \frac{3\tilde{c}_1\sqrt{\alpha}}{2} = 0.$$

Note that ψ_1 and ψ_2 are constructed in such a way that

$$W(\psi_1, \psi_2) = 1, \quad (2.17)$$

where by $W(\psi_1, \psi_2)$ we denote the Wronskian of ψ_1 and ψ_2 . Furthermore, ψ_1 is localized and ψ_2 is unbounded. By using variation of constants and (2.17), we obtain the general solution of equation (2.15a):

$$\operatorname{Re} S_1(s) = B_1 \psi_1 + B_2 \psi_2 - \psi_1 \int_0^s \psi_2 f \, d\tilde{s} + \psi_2 \int_0^s \psi_1 f \, d\tilde{s}, \quad (2.18)$$

where B_1 and B_2 are real constants.

Similarly, solutions to the homogeneous part of equation (2.15b) are given by

$$\phi_1(s) := S_0, \quad (2.19a)$$

$$\phi_2(s) := S_0 \int_0^s \frac{1}{S_0^2} \, d\tilde{s}. \quad (2.19b)$$

Again, ϕ_1 and ϕ_2 are constructed in such a way that

$$W(\phi_1, \phi_2) = 1. \quad (2.20)$$

And also, ϕ_1 is localized and ϕ_2 is unbounded. Thus, the solution to equation (2.15b) is given by

$$\operatorname{Im} S_1(s) = C_1 \phi_1 + C_2 \phi_2 - \phi_1 \int_0^s \phi_2 g \, d\tilde{s} + \phi_2 \int_0^s \phi_1 g \, d\tilde{s}, \quad (2.21)$$

where C_1 and C_2 are real constants.

2.2.2.3 A saddle-node bifurcation

We require S to be localized. To achieve this, conditions are imposed on f , g , B_2 and C_2 . We formulate these conditions in the following two lemmas.

Lemma 2.2.3. *Let L_1 be defined by (2.14a) and let $f(s)$ be a continuous function with asymptotic behavior $f = c_{\pm} s^n e^{\mp \sqrt{\alpha} s} + o(s^n e^{\mp \sqrt{\alpha} s})$ as s tends to $\pm\infty$. Then, the solution u of $L_1 u = f$ is given by (2.18). Furthermore, the solution vanishes at infinity iff the following two conditions are satisfied:*

$$2.2.3.a \quad B_2 = - \int_0^{\infty} \psi_1 f_{\text{even}} \, d\tilde{s},$$

$$2.2.3.b \quad \int_0^{\infty} \psi_1 f_{\text{odd}} \, d\tilde{s} = 0,$$

where $f_{\text{even}}(s) = \frac{f(s)+f(-s)}{2}$ and $f_{\text{odd}}(s) = \frac{f(s)-f(-s)}{2}$. And finally, if both 2.2.3.a and 2.2.3.b are satisfied, then the solution $u(s)$ of $L_1 u = f$ satisfies $u(s) = c_u s^{n+1} e^{\mp \sqrt{\alpha} s} + o(s^{n+1} e^{\mp \sqrt{\alpha} s})$ as s tends to $\pm\infty$ for some constant c_u .

Proof. From (2.13), it follows that $\psi_1 = S'_0 = \mp c_1 e^{\mp \sqrt{\alpha} s} + o(e^{\mp \sqrt{\alpha} s})$ as s tends to $\pm\infty$ for some constant c_1 . And also $\psi_2 = S'_0 \int_0^s \frac{1}{S_0^2} \, d\tilde{s} = c_2 e^{\pm \sqrt{\alpha} s} \pm c_3 s e^{\mp \sqrt{\alpha} s} + o(s^n e^{\mp \sqrt{\alpha} s})$ as s tends to $\pm\infty$, where c_2 and c_3 are constants. By considering the second and fourth term in (2.18) together, we obtain

$$\psi_2 \left(B_2 + \int_0^s \psi_1 f \, d\tilde{s} \right) = \left(c_2 e^{\pm \sqrt{\alpha} s} \pm c_3 s e^{\mp \sqrt{\alpha} s} \right) \left(B_2 \mp c_4 s^n e^{\mp 2 \sqrt{\alpha} s} + \int_0^{\pm\infty} \psi_1 f \, d\tilde{s} + o(s^n e^{\mp 2 \sqrt{\alpha} s}) \right), \quad (2.22)$$

for some constant c_4 . A similar expression can be obtained for the first and third term in (2.18). From these expressions the results follows upon considering the two limits $s \rightarrow \pm\infty$. $\square$

Lemma 2.2.4. *Let L_2 be defined by (2.14b) and let $g(s)$ be a continuous function with asymptotic behavior $g = c_{\pm} s^n e^{\mp \sqrt{\alpha} s} + o(s^n e^{\mp \sqrt{\alpha} s})$ as s tends to $\pm\infty$. Then, the solution $u(s)$ of $L_2 u = g$ is given by (2.21). Furthermore, the solution vanishes at infinity iff the following two conditions are satisfied:*

$$2.2.4.a \quad C_2 = - \int_0^{\infty} \phi_1 g_{\text{odd}} d\tilde{s},$$

$$2.2.4.b \quad \int_0^{\infty} \phi_1 g_{\text{even}} d\tilde{s} = 0,$$

where $g_{\text{even}}(s) = \frac{g(s)+g(-s)}{2}$ and $g_{\text{odd}}(s) = \frac{g(s)-g(-s)}{2}$. And finally, if both 2.2.4.a and 2.2.4.b are satisfied, then the solution $u(s)$ of $L_2 u = g$ satisfies $u(s) = c_u s^{n+1} e^{\mp \sqrt{\alpha} s} + o(s^{n+1} e^{\mp \sqrt{\alpha} s})$ as s tends to $\pm\infty$ for some constant c_u .

Proof. The proof is similar to the proof of Lemma 2.2.3. $\square$

Remark 2.2.5. *Note that it also follows from expression (2.22) in the proof of Lemma 2.2.3 that the only bounded solutions of $L_1 u = f$ are localized solutions. Similarly, the only bounded solutions of $L_2 u = g$ are also localized solutions.*

Upon using 2.2.3.a and 2.2.4.a, we obtain

$$B_2 = \frac{\kappa_0 \kappa_1}{2} \int_0^{\infty} \psi_1 S_0 d\tilde{s} \quad (2.23)$$

and

$$C_2 = \kappa_1 \int_0^{\infty} \phi_1 S'_0 d\tilde{s}. \quad (2.24)$$

Furthermore, 2.2.3.b and 2.2.4.b lead to a system of equations for κ_0 and K_0 :

$$\left(K_0 \kappa_0 + \frac{1-d}{\kappa_0} \right) \int_0^{\infty} S_0'^2 d\tilde{s} = - \frac{\kappa_0}{4} \int_0^{\infty} S_0^2 d\tilde{s}, \quad (2.25a)$$

$$K_0 \int_0^{\infty} S_0'^2 d\tilde{s} = \frac{1}{2} \left(d - 1 - \frac{2}{\sigma} - \frac{\kappa_0^2}{2} K_0 + 1 \right) \int_0^{\infty} S_0^2 d\tilde{s} - b K_0 \int_0^{\infty} S_0^{2\sigma+2} d\tilde{s}. \quad (2.25b)$$

Note that C_2 and B_2 depend on κ_1 . A similar analysis at next order leads to a system of equations for K_1 and κ_1 .

For $\sigma = 1$, the integrals in (2.25) can be evaluated explicitly which results in

$$I_1 := 4\alpha \left(-K_0 \kappa_0^2 + d - 1 \right) - 3\kappa_0^2 = 0, \quad (2.26a)$$

$$I_2 := -6d + 4\alpha(4b+1)K_0 + 3 \left(K_0 \kappa_0^2 + 4 \right) = 0. \quad (2.26b)$$

These equations differ from [9] formula (6.22) due to an error in formula (6.12) in [9]. Since in [9] only the case $\kappa_0 = 1$ is considered, the conclusions drawn in [9] remain valid.

Note that, equation (2.26a) is exactly the same as Condition 2.1.1.b of Theorem 2.1.1 (η corresponds to κ_0), and that equation (2.26b) is exactly the same as Condition 2.1.1.a of

Theorem 2.1.1. Theorem 2.1.1 does not state any conditions on d and b under which this system of equations has a unique solution. The following lemma gives conditions on d and b under which we can solve κ_0 and K_0 uniquely from (2.25a) and (2.25b).

Lemma 2.2.6. *Under validity of Assumption 1.2.2, the system of equations (2.25) has a solution (κ_0, K_0) satisfying $\kappa_0 > 0$, $\alpha = 1 - \frac{\kappa_0^2}{4} > 0$ and $K_0 > 0$ iff one of the following conditions is satisfied:*

$$2.2.6.a \quad b > -\frac{W'}{W_2},$$

$$2.2.6.b \quad b \leq -\frac{W'}{W_2}, \left(\frac{4}{\sigma} - (d+2) - b \frac{W_2}{W'} \right)^2 + 4(d-1) \left(1 + b \frac{W_2}{W'} \right) \geq 0 \text{ and } \frac{2}{\sigma} < d < \frac{4}{\sigma} - 1, \\ \text{where}$$

$$W_1 := (1 + \sigma)^{\frac{1}{\sigma}} \int_0^\infty \operatorname{sech}^{\frac{2}{\sigma}}(\sigma t) dt, \quad (2.27a)$$

$$W_2 := (1 + \sigma)^{\frac{1+\sigma}{\sigma}} \int_0^\infty \operatorname{sech}^{\frac{2+2\sigma}{\sigma}}(\sigma t) dt, \quad (2.27b)$$

$$W' := (1 + \sigma)^{\frac{1}{\sigma}} \int_0^\infty \left(\frac{d}{dt} \operatorname{sech}^{\frac{1}{\sigma}}(\sigma t) \right)^2 dt \quad (2.27c)$$

Furthermore, the solution is unique iff one of the following conditions is satisfied:

$$2.2.6.c \quad b > -\frac{W'}{W_2},$$

$$2.2.6.d \quad b = -\frac{W'}{W_2} \text{ and } \frac{2}{\sigma} < d < \frac{4}{\sigma} - 1,$$

$$2.2.6.e \quad \left(\frac{4}{\sigma} - (d+2) - b \frac{W_2}{W'} \right)^2 + 4(d-1) \left(1 + b \frac{W_2}{W'} \right) = 0 \text{ and } \frac{4}{\sigma} - (d+2) - b \frac{W_2}{W'} > 0.$$

If a valid solution exists but is not unique, then there exist exactly 2 solutions.

Proof. Using (2.13), we find

$$\begin{aligned} \int_0^\infty S_0^2 d\tilde{s} &= \alpha^{\frac{1}{\sigma} - \frac{1}{2}} W_1, \\ \int_0^\infty S_0^{2+2\sigma} d\tilde{s} &= \alpha^{\frac{1}{\sigma} + \frac{1}{2}} W_2, \\ \int_0^\infty S_0'^2 d\tilde{s} &= \alpha^{\frac{1}{\sigma} + \frac{1}{2}} W', \end{aligned}$$

where W_1 , W_2 and W' are defined by (2.27). Substitution into (2.25) leads to

$$\left(K_0 \kappa_0 + \frac{1-d}{\kappa_0} \right) \alpha W' = -\frac{\kappa_0}{4} W_1, \quad (2.28a)$$

$$K_0 \alpha W' = \left(\frac{d}{2} - \frac{1}{\sigma} - \frac{\kappa_0^2}{4} K_0 \right) W_1 - b K_0 \alpha W_2. \quad (2.28b)$$

By eliminating K_0 from the first of the two equations and substituting $z = \frac{3\kappa_0^2}{4\alpha} = \frac{3\kappa_0^2}{4-\kappa_0^2}$, it follows that solving (2.28) is equivalent with solving the system

$$K_0 = \frac{1}{3\kappa_0^2} \left(3(d-1) - \frac{W_1}{W'} \right), \quad (2.29a)$$

$$0 = z^2 - 3 \left(\frac{4W'}{W_1 \sigma} - (d+2) \frac{W'}{W_1} - b \frac{W_2}{W_1} \right) z - 9(d-1) \left(\frac{W'^2}{W_1^2} + b \frac{W_2 W'}{W_1^2} \right). \quad (2.29b)$$

Now, a solution (κ_0, K_0) of the system defined by (2.28) that satisfies the conditions $K_0 > 0$, $\kappa_0 > 0$ and $1 - \frac{\kappa_0^2}{4} > 0$ corresponds to a positive solution (z, K_0) of the system defined by (2.29). Let z_+ and z_- be the two solutions of (2.29b), then we have

$$z_+ z_- = -9(d-1) \frac{W_2 W'}{W_1^2} \left(\frac{W'}{W_2} + b \right). \quad (2.30)$$

Since from Assumption 1.2.2.a it follows that $d > 1$, the sign of $z_+ z_-$ is determined by the sign of $b + \frac{W'}{W_2}$. In order to prove statements 2.2.6.a to 2.2.6.e, we will analyze the three different cases $b > \frac{W'}{W_2}$, $b < \frac{W'}{W_2}$ and $b = \frac{W'}{W_2}$ separately. For each of these cases, we will study under which conditions (2.29b) has a real positive solution and then verify whether $K_0 > 0$ for this positive real solution.

The case $b > \frac{W'}{W_2}$: In this section, we will prove statements 2.2.6.a and 2.2.6.c. If $b > \frac{W'}{W_2}$, it follows that the right hand side of (2.30) is negative. And thus, since the coefficients of z in (2.29b) are real, there exists a unique positive solution z_+ of (2.29b). It remains to show that also $K_0 > 0$ for this particular value of z_+ . From (2.29a), it follows that

$$z_+ < \frac{3W'}{W_1} (d-1) \quad (2.31)$$

should hold. We prove (2.31) by proving a slightly more general statement, since that will be of use later in the proof. Let z_+ and z_- be the two real of (2.29b). After solving $z_{\pm}$ from (2.29b), with the quadratic formula for solving second order polynomial equations, and substitution into (2.31), many terms drop out and after some algebra we obtain

$$\pm \sqrt{\left(\frac{4}{\sigma} - d - 2 - b \frac{W_2}{W'} \right)^2 + 4(d-1) \left(1 + b \frac{W_2}{W'} \right)} < 2d - 2 - \left(\frac{4}{\sigma} - d - 2 - b \frac{W_2}{W'} \right), \quad (2.32)$$

where the inequality should hold for both the plus and the minus sign. We now show that the absolute value of the left hand side is smaller than the absolute value of the right hand side. If the right hand side of (2.32) is positive, then inequality (2.32) is equivalent with

$$0 < \frac{(d-1)(\sigma d - 2)}{\sigma}.$$

From Assumption 1.2.2, it follows that this inequality holds. We conclude that the condition $K_0 > 0$ is satisfied for a positive solution z of (2.29b) provided that the right hand side of (2.32) is positive, i.e.

$$2d - 2 - \left(\frac{4}{\sigma} - d - 2 - b \frac{W_2}{W'} \right) > 0. \quad (2.33)$$

Now, the equation

$$2d - 2 - \left(\frac{4}{\sigma} - d - 2 - b \frac{W_2}{W'} \right) = 0 \quad (2.34)$$

defines a straight line in the (d, b) -plane and we claim that this line lies below the line $b = -\frac{W'}{W_2}$ if Assumption 1.2.2 holds. This fact can be verified as follows. We write (2.34) as

$$3d = -\frac{W_2}{W'} b + \frac{4}{\sigma}. \quad (2.35)$$

Note that this line has a negative slope in the (d, b) -plane. Calculating the intersection point with the line $b = \frac{W_2}{W'}$ gives $d = \frac{4+\sigma}{3\sigma}$. It can easily be verified that $\frac{4+\sigma}{3\sigma} < \frac{2}{\sigma}$ for $0 < \sigma < 2$ and

$\frac{4+\sigma}{3\sigma} \leq 1$ for $\sigma \geq 2$. Upon using Assumption 1.2.2.b and 1.2.2.a, we conclude that the line defined by (2.35) lies indeed below the line $b = -\frac{W'}{W_2}$. It now follows that inequality (2.33) holds if $b > \frac{W_2}{W'}$ and Assumption 1.2.2 holds. In particular, we have now proved the existence of a unique solution for $b > -\frac{W'}{W_2}$, i.e. 2.2.6.a and 2.2.6.c.

The case $b = -\frac{W'}{W_2}$: When $b = -\frac{W'}{W_2}$, it follows from (2.30) that $z_+z_- = 0$. From (2.29b), we obtain that (without loss of generality) $z_- = 0$ and $z_+ = \frac{12W'}{W_1\sigma} - 3(d+2)\frac{W'}{W_1} - 3b\frac{W_2}{W_1}$. Thus, z_+ is positive iff $d < \frac{4}{\sigma} - 1$. Finally, to prove that $K_0 > 0$, we can use exactly the same reasoning as we used in the case $b > -\frac{W'}{W_2}$, see also the argument below equation (2.31). This proves 2.2.6.d and part of 2.2.6.b.

The case $b < -\frac{W'}{W_2}$: In this section, we prove the remaining parts of the lemma: 2.2.6.b and 2.2.6.e. When $b < -\frac{W'}{W_2}$, it follows from (2.30) that $z_+z_- > 0$. Thus, the possibilities are: a pair of complex conjugate solutions, two positive solutions or two negative solutions. If the discriminant of (2.29b) is nonnegative, the solutions are real-valued. Upon using (2.29b), we obtain after some algebraic manipulation that a nonnegative discriminant is equivalent with

$$D(d, b, \sigma) \geq 0, \quad (2.36)$$

where $D(d, b, \sigma)$ is defined by

$$D(d, b, \sigma) := \left(\frac{4}{\sigma} - (d+2) - b\frac{W_2}{W'} \right)^2 + 4(d-1) \left(1 + b\frac{W_2}{W'} \right). \quad (2.37)$$

For each value of σ , (2.36) defines a region in the (d, b) -plane.

We now investigate whether the curve

$$D(d, b, \sigma) = 0 \quad (2.38)$$

can have an intersection with the line $b = -\frac{W'}{W_2}$ in the strip $\frac{2}{\sigma} < d < \frac{2+2\sigma}{\sigma}$ of the (d, b) -plane, see Assumption 1.2.2.b. For this purpose, we define

$$b_{\max} = \max \left\{ b \in \mathbb{R} \left| \begin{array}{l} \exists d(b) \in \left[\frac{2}{\sigma}, \frac{2+2\sigma}{\sigma} \right] \text{ such that } (d, b) \text{ lies on the curve} \\ D(d, b, \sigma) = 0, \text{ where } D(d, b, \sigma) \text{ is defined by (2.37)} \end{array} \right. \right\}.$$

Since the curve $D(d, b, \sigma) = 0$ is smooth, there are three possibilities $d(b_{\max}) = \frac{2}{\sigma}$, $d(b_{\max}) = \frac{2+2\sigma}{\sigma}$ or on the curve $\frac{\partial b}{\partial d}(d_{\max} = d(b_{\max}), b_{\max}) = 0$. We calculate

$$0 = \frac{dD}{dd} \Big|_{\frac{\partial b}{\partial d}=0} = -2 \left(\frac{4}{\sigma} - (d+2) - b\frac{W_2}{W'} \right) - 4 \left(1 + \frac{W_2}{W'} b \right). \quad (2.39)$$

Together, (2.38) and (2.39) form a system of two equations from which we can solve d and b . We obtain two solutions to this system of equations: $(b, d) = \left(-\frac{W'}{W_2}, \frac{4}{\sigma} - 1 \right)$ or $(b, d) = \left(2\frac{W'}{W_2} \left(\frac{1}{\sigma} - 1 \right), 2 \left(\frac{1}{\sigma} - 1 \right) \right)$. Since the inequality $2 \left(\frac{1}{\sigma} - 1 \right) < \frac{2}{\sigma}$ holds, the latter solution is discarded upon using Assumption 1.2.2.b. We now calculate the intersection points with the boundary of the strip $\frac{2}{\sigma} < d < \frac{2+2\sigma}{\sigma}$ and obtain the three intersection points: $(b, d) = \left(-\frac{2W'}{\sigma W_2}, \frac{2}{\sigma} \right)$ and $(b, d) = \left(\frac{W'}{\sigma W_2} \left(-2(1+3\sigma) \pm 4\sqrt{\sigma(\sigma+2)} \right), \frac{2+2\sigma}{\sigma} \right)$. With basic calculus it can be verified that these three boundary points lie below the line $b = -\frac{W'}{W_2}$ for $0 < \sigma < 2$. And moreover, it can be verified that the point $(b, d) = \left(-\frac{2W'}{\sigma W_2}, \frac{2}{\sigma} \right)$ is the only point in the closure of the strip where $\frac{\partial d}{\partial b}$ vanishes. This point lies on the line $b = -\frac{W'}{W_2}$.

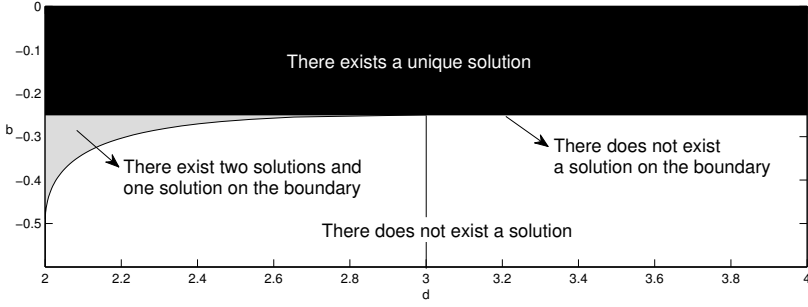


Figure 2.2.1: For $\sigma = 1$, the regions in the strip $2 < d < 4$ of the (d, b) -plane for which the system of equations (2.25) has a solution (κ_0, K_0) satisfying $\kappa_0 > 0$, $\alpha := 1 - \frac{\kappa_0^2}{4} > 0$ and $K_0 > 0$, see Lemma 2.2.6 and Corollary 2.2.7. In particular, the line $b = -\frac{W'}{W_2} = -\frac{1}{4}$ and the shape of the curve $D(d, b, \sigma = 1) = 0$, as defined by (2.43), in the region $2 < d < 3$ and $-\frac{1}{2} < b < -\frac{1}{4}$ are shown.

We now calculate the part of the half strip defined by $\frac{2}{\sigma} < d < \frac{2+2\sigma}{2}$ and $b < -\frac{W'}{W_2}$ of the (d, b) -plane where

$$K_0 > 0 \quad (2.40)$$

holds. Following the reasoning as presented in the case $b > -\frac{W'}{W_2}$, it follows that we should verify whether (2.33) holds. It can easily be verified that $(b, d) = \left(-\frac{2W'}{\sigma W_2}, \frac{2}{\sigma}\right)$ is the only intersection point of the straight line defined by (2.34) with the curve $D = 0$ defined by (2.38) in the closure of the strip $\frac{2}{\sigma} < d < \frac{2+2\sigma}{\sigma}$ of the (d, b) -plane. Note that this is precisely the point on the curve $D = 0$ where $\frac{\partial d}{\partial b}$ vanishes.

If (2.36) holds, then $z_{\pm}$ are real-valued and are either both positive or both negative. If in addition to (2.36),

$$\frac{z_+ + z_-}{3} = \frac{4}{\sigma} - (d + 2) - b \frac{W_2}{W'} > 0 \quad (2.41)$$

holds, then both $z_{\pm}$ are real-valued and positive. Using (2.37), it can be seen that $\frac{4}{\sigma} - (d + 2) - b \frac{W_2}{W'} = 0$ and $D = 0$ implies $b = -\frac{W'}{W_2}$. Thus the three curves $D = 0$, $\frac{4}{\sigma} - (d + 2) - b \frac{W_2}{W'} = 0$ and $b = -\frac{W'}{W_2}$ intersect at the point $(d, b) = \left(\frac{4}{\sigma} - 1, -\frac{W'}{W_2}\right)$, i.e. the point on the curve $D = 0$ where $\frac{\partial b}{\partial d} = 0$. We now combine the results derived above to formulate when simultaneously (2.36), (2.40) and (2.41) hold. This results in the statements 2.2.6.b and 2.2.6.e of the lemma. $\square$

For $\sigma = 1$, we summarize the statement of Lemma 2.2.6 in Figure 2.2.1. We have shown that the system (2.26) has a solution if $(d, b) \in R$, where the region R is defined by

$$R := \left\{ (d, b) \in \mathbb{R}^2 : \begin{array}{l} 2 < d \leq 4 \text{ and } d^2 + 8b(3d - 4 + 2b) \geq 0 \\ \text{or } 3 < d < 4 \text{ and } b > -\frac{1}{4} \end{array} \right\}. \quad (2.42)$$

From Lemma 2.2.6, we obtain the follow corollary.

Corollary 2.2.7. *Let $\sigma = 1$. Then, the system (2.3) combined with Assumption 2.2.1 undergoes a saddle-node bifurcation upon passing the curve*

$$D(d, b, \sigma = 1) = d^2 + 24bd - 32b + 16b^2 = 0 \quad (2.43)$$

in the strip $2 < d < 3$ of the (d, b) -plane, see also Figure 2.2.1. Here D is defined by (2.37).

Proof. In Section 2.3, we show that the asymptotic construction can be extended outside the bump region such that Assumption 2.2.1 holds. For further details on the validity of asymptotic construction, see Section 2.5.2.

The result follows directly from Lemma 2.2.6. It remains to show that evaluation of $D(d, b, \sigma) = 0$ at $\sigma = 1$ leads to (2.43). For $\sigma = 1$, we obtain from (2.27) that

$$W_1|_{\sigma=1} = 2, \quad W_2|_{\sigma=1} = \frac{8}{3} \quad \text{and} \quad W'|_{\sigma=1} = \frac{2}{3}.$$

Using this in (2.37) gives (2.43). $\square$

At the qualitative level, the Figure 2.2.1 would look similar for each $0 < \sigma < 2$ and we will refer to solutions on the upper branch of the saddle-node bifurcation as m_2^{upper} and to solutions on the lower branch as m_2^{lower} .

Definition 2.2.8. *If system (2.3) combined with Assumption 2.2.1 undergoes a saddle-node bifurcation upon passing the curve $D(d, b, \sigma) = 0$ as defined by (2.37). Then the solution on the upper branch of the saddle-node bifurcation, i.e. the branch with the highest value of κ_0 , will be called m_2^{upper} and the solutions on the lower branch of the saddle-node bifurcation will be called m_2^{lower} .*

Remark 2.2.9. *Note that for $\sigma = 2$, which is not in parameter range we study (see Assumption 1.2.2.c), the gray region in Figure 2.2.1 contracts into the point $(d, b) = (\frac{2}{\sigma}, -\frac{W'}{W_2})$. We leave a further analysis of this to future research.*

For σ close to 1 and $b > \max(-\frac{W'}{W_2}, -\frac{1}{4})$, we can find approximate expressions for the solutions κ_0 and K_0 of (2.25). We introduce the expansions

$$\begin{aligned} b &= b_0 + \delta_1, \\ d &= d_0 + \delta_2, \\ \sigma &= 1 + \delta_3, \\ \kappa_0 &= \kappa_{00} + \kappa_{01}\delta_1 + \kappa_{02}\delta_2 + \kappa_{03}\delta_3 + o(\delta_1) + o(\delta_2) + o(\delta_3), \end{aligned}$$

where $|\delta_i| \ll 1$ and $(d_0, b_0) \in R$ as defined by (2.42). We derive that

$$\begin{aligned} \int_0^\infty \text{sech}^{\frac{2}{\sigma}} x \, dx &= 1 + (2 - 2 \ln 2)\delta_3 + o(\delta_3), \\ \int_0^\infty \left(\frac{d}{dx} \text{sech} x \right)^{\frac{2}{\sigma}} dx &= \frac{1}{3} + \left(\frac{8}{9} - \frac{2}{3} \ln 2 \right) \delta_3 + o(\delta_3), \\ \int_0^\infty \text{sech}^{2+\frac{2}{\sigma}} x \, dx &= \frac{2}{3} + \left(\frac{10}{9} - \frac{4}{3} \ln 2 \right) \delta_3 + o(\delta_3). \end{aligned}$$

For $d_0 = 2$, we find

$$\kappa_{00} = 1, \quad \kappa_{01} = 0, \quad \kappa_{02} = \frac{9}{8} \frac{b_0}{3b_0 + 1}, \quad \kappa_{03} = \frac{3b_0 - 2}{12b_0 + 4}, \quad K_0 = \frac{\delta_2 + 2\delta_3}{3b_0 + 1} + o(\delta_1) + o(\delta_2) + o(\delta_3). \quad (2.44)$$

This result agrees with expression (6.25) in [9]. For $d_0 = 3$ and $b_0 = 0$, we find

$$\kappa_{00} = 1, \quad \kappa_{01} = \frac{3}{4}, \quad \kappa_{02} = 0, \quad \kappa_{03} = -\frac{1}{4}, \quad K_0 = \frac{6 - 21\delta_1 + 6\delta_2 + 11\delta_3}{6} + o(\delta_1) + o(\delta_2) + o(\delta_3). \quad (2.45)$$

Also for b large, we can find approximate expressions for the solutions κ_0 and K_0 of (2.25). In order to do so, we assume $1 \ll b \ll \frac{1}{a}$. With this assumption, b can be of $O(a^{-\gamma})$ with $0 < \gamma < 1$. However, since equation (2.11) still includes all terms up to $o(1)$ and equation (2.15) includes all non leading order terms up to $o(a)$, the $O(a)$ analysis presented in Section 2.2.2.2 is still valid. For b large, we obtain from (2.29b) that

$$z \approx \frac{3(d-1)W'(\sigma)}{W_1(\sigma)}.$$

Using that $z = \frac{3\kappa_0^2}{4-\kappa_0^2}$ and (2.29a) leads to

$$\kappa_0 \approx \sqrt{\frac{4(d-1)W'(\sigma)}{(d-1)W'(\sigma) + W_1(\sigma)}}, \quad (2.46a)$$

$$K_0 \approx 0. \quad (2.46b)$$

Note that since (2.46b) is only a leading order approximation for K_0 , this does not contradict the inequality $K > 0$ as stated in Assumption 2.2.1.c. For $\sigma = 1$, (2.46) can be simplified to

$$\kappa_0 \approx 2\sqrt{\frac{d-1}{d+2}}, \quad (2.47a)$$

$$K_0 \approx 0. \quad (2.47b)$$

So far, we have determined that if B_2 is given by (2.23) and C_2 is given by (2.24), then conditions under which (2.15) has a bounded solution are given by Lemma 2.2.6. We now determine the constants B_1 and C_1 in (2.18) and (2.21).

First, we determine B_1 . We defined κ such that $|Q|^2 = |S|^2$ attains its maximum at $\xi = \frac{\kappa}{a}$, i.e. at $s = 0$, see Assumption 2.2.1.b. We deduce

$$\begin{aligned} 0 = \partial_s |S|^2 \Big|_{s=0} &= 2\operatorname{Re} S \operatorname{Re} S' + 2\operatorname{Im} S \operatorname{Im} S' = 2S_0 S'_0 + 2a(S'_0 \operatorname{Re} S_1 + S_0 \operatorname{Re} S'_1) \\ &\quad + 2a^2(\operatorname{Re} S_1 \operatorname{Re} S'_1 + \operatorname{Im} S_1 \operatorname{Im} S'_1 + S_0 \operatorname{Re} S'_2 + \operatorname{Re} S_2 S'_0) + O(a^3) \Big|_{s=0}. \end{aligned} \quad (2.48)$$

We obtain, upon setting the $O(a)$ -term in (2.48) to zero and using $S'_0(0) = 0$ and $S_0(0) \neq 0$, that $\operatorname{Re} S'_1(0) = 0$. Using (2.18) leads to $B_1 = 0$. Similar arguments can be used to determine constants at higher order.

We will now argue that C_1 can be chosen freely because of the phase invariance of equation (2.10). So far, we have only used this freedom at leading order. Suppose that $S(s)$ solves (2.10), then, by the phase invariance of equation (2.10), $e^{i\Psi_b} S(s)$ also solves (2.10), where by Ψ_b we indicate a phase shift in the bump region. Expanding Ψ_b as

$$\Psi_b = \Psi_{b,0} + a\Psi_{b,1} + O(a^2)$$

leads to

$$\begin{aligned} e^{i\Psi_b} S(s) &= e^{i(\Psi_{b,0} + a\Psi_{b,1})} (S_0(s) + aS_1(s)) + O(a^2) \\ &= e^{i\Psi_{b,0}} (1 + ia\Psi_{b,1}) (S_0(s) + aS_1(s)) + O(a^2) \\ &= e^{i\Psi_{b,0}} (S_0(s) + aS_1(s) + ia\Psi_{b,1}S_0(s)) + O(a^2). \end{aligned}$$

We conclude that a perturbation $a\Psi_{b,1}$ in the phase corresponds to an extra term $i\Psi_{b,1}S_0(s)$ at $O(a)$. Since $\phi_1 = S_0$, it follows that C_1 can still be chosen freely.

2.2.2.4 Higher Order Analysis

We proceed with the analysis of equation (2.10) at $O(a^2)$. The expressions for $\text{Re } S_2$ and $\text{Im } S_2$ are needed to evaluate expressions for the stability analysis, see Chapter 3 and Section 3.2.6. Furthermore, the analysis at this order determines κ_1 and K_1 . For technical reasons, we make the following assumption.

Assumption 2.2.10. *In Section 2.2.2.4, we restrict ourselves to the case $\sigma = 1$.*

At $O(a^2)$, taking the real and imaginary part of (2.10) leads to

$$L_1 \text{Re } S_2 = h_1(s), \quad (2.49a)$$

$$L_2 \text{Im } S_2 = h_2(s), \quad (2.49b)$$

where L_1 and L_2 are defined by (2.14),

$$\begin{aligned} h_1(s) = & K_1 \kappa_0 S'_0 - \frac{d-1}{\kappa_0} \left(\text{Re } S'_1 + \frac{\kappa_0}{2} \text{Im } S_1 \right) + \frac{d-1}{\kappa_0^2} (s + \kappa_1) S'_0 + K_0 (-\text{Im } S''_1 \\ & + \frac{\kappa_0^2}{4} \text{Im } S_1 + \kappa_0 \text{Re } S'_1 + \frac{d-1}{2} S_0) + (s + \kappa_1) \left(\text{Im } S'_1 - \frac{\kappa_0}{2} \text{Re } S_1 \right) + \text{Im } S_1 \\ & - \kappa_2 \frac{\kappa_0}{2} S_0 - S_0 \left(3 (\text{Re } S_1)^2 + (\text{Im } S_1)^2 \right) + b K_0 S_0^2 \text{Im } S_1. \end{aligned}$$

and

$$\begin{aligned} h_2(s) = & K_1 \left(S''_0 - \frac{\kappa_0^2}{4} S_0 \right) - \frac{d-1}{\kappa_0} \left(\text{Im } S'_1 - \frac{\kappa_0}{2} \text{Re } S_1 \right) - \frac{d-1}{2\kappa_0} (s + \kappa_1) S_0 \\ & + K_0 \left(\text{Re } S''_1 - \frac{\kappa_0^2}{4} \text{Re } S_1 + \kappa_0 \text{Im } S'_1 + \frac{d-1}{\kappa_0} S'_0 \right) - (s + \kappa_1) \left(\text{Re } S'_1 + \frac{\kappa_0}{2} \text{Im } S_1 \right) \\ & - \text{Re } S_1 - \kappa_2 S'_0 - 2 S_0 \text{Re } S_1 \text{Im } S_1 - 3b K_0 S_0^2 \text{Re } S_1 - b K_1 S_0^3. \end{aligned}$$

Analogous to the solution at $O(a)$, we obtain

$$\text{Re } S_2(s) = F_1 \psi_1 + F_2 \psi_2 - \psi_1 \int_0^s \psi_2 h_1 \, d\tilde{s} + \psi_2 \int_0^s \psi_1 h_1 \, d\tilde{s}, \quad (2.50a)$$

$$\text{Im } S_2(s) = G_1 \phi_1 + G_2 \phi_2 - \phi_1 \int_0^s \phi_2 h_2 \, d\tilde{s} + \phi_2 \int_0^s \phi_1 h_2 \, d\tilde{s}, \quad (2.50b)$$

where F_1 , F_2 , G_1 and G_2 are real constants. In a similar way as we derived $B_1 = 0$, it follows also from (2.48) that $\text{Re } S'_2(0) = 0$, and hence $F_1 = 0$. Similar to C_1 , G_1 is also a free parameter and can be used to fix the phase of the solution. We choose $G_1 = 0$. Since we require S to be localized, we also require S_2 to be localized. From Lemma 2.2.3, we conclude that the even part $h_{1,\text{even}} = \frac{1}{2} (h_1(s) + h_1(-s))$ of h_1 determines F_2 and the odd part $h_{1,\text{odd}} = \frac{1}{2} (h_1(s) - h_1(-s))$ leads to a condition $I_3 = 0$. Similarly, Lemma 2.2.4 implies that the odd part $h_{2,\text{odd}} = \frac{1}{2} (h_2(s) - h_2(-s))$ of h_2 determines G_2 and the even part $h_{2,\text{even}} = \frac{1}{2} (h_2(s) + h_2(-s))$ leads to a condition $I_4 = 0$. We can determine the equations $I_3 = 0$ and

$I_4 = 0$ to be

$$I_3 := 3\kappa_0^4\kappa_1 + 8\alpha^2(K_1\kappa_0^3 + 2(d-1)\kappa_1) \propto \int_0^\infty \psi_1 h_{1,\text{odd}} d\tilde{s} = 0, \quad (2.51a)$$

$$I_4 := (2b-1)(4\alpha(d-1) - 3\kappa_0^2)\kappa_1 - \alpha K_1\kappa_0(3\kappa_0^2 + 4(4b\alpha + \alpha)) \propto \int_0^\infty \phi_1 h_{2,\text{even}} d\tilde{s} = 0, \quad (2.51b)$$

where $\propto$ is the “proportional to” symbol. Furthermore,

$$F_2 = - \int_0^\infty \psi_1 h_{1,\text{even}} d\tilde{s}, \quad (2.52a)$$

$$G_2 = - \int_0^\infty \phi_1 h_{2,\text{odd}} d\tilde{s}. \quad (2.52b)$$

Lemma 2.2.11. *Let $\sigma = 1$, consider the region R as defined by (2.42) and let $(d, b) \in R$. Then, provided that $D(d, b, \sigma = 1) \neq 0$, where D is defined by (2.43), we have $\kappa_1 = K_1 = 0$.*

Proof. Consider the system of equations (2.51). This system is linear in κ_1 and K_1 . Upon using the system of equations (2.26) to eliminate K_0 and κ_0 , it follows that the determinant of the linear system (2.51) vanishes if $D(d, b, \sigma = 1) = 0$. Thus, if $D(d, b, \sigma = 1) \neq 0$, then κ_1 and K_1 can be solved from (2.51) and we obtain $\kappa_1 = K_1 = 0$. $\square$

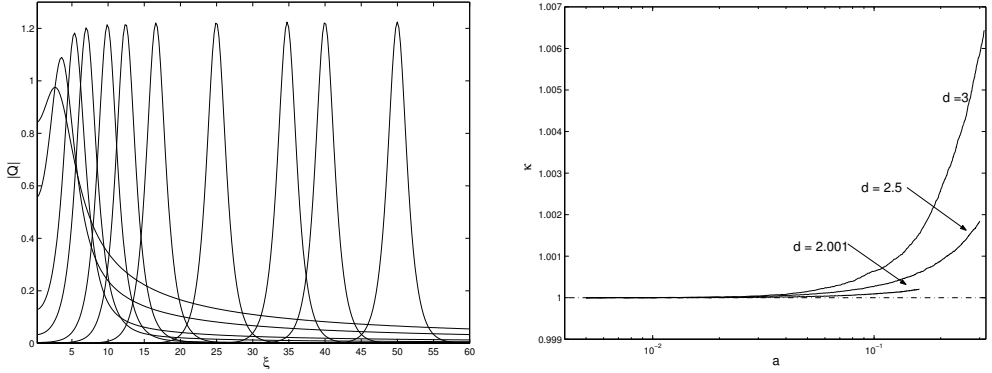
Remark 2.2.12. *From Corollary 2.2.7, it follows that for $\sigma = 1$ equation (2.3) combined with Assumption 2.2.1 undergoes a saddle-node bifurcation upon passing the curve $D(d, b, \sigma = 1) = 0$ in the strip $2 < d < 3$, where D is defined by (2.43). If we combine this with Lemma 2.2.11, then it follows that $K_1 = \kappa_1 = 0$ provided we are not exactly at the saddle-node bifurcation. Thus, the in this section derived higher order asymptotic expansion for $S(s)$ will not be valid at the saddle-node bifurcation. We do not consider the necessary higher order analysis here any further.*

The result of Lemma 2.2.11 agrees with Figure 2.2.2b. This (numerical) figure, taken from [9], shows that the graph of a versus κ is flat as a tends to 0.

Explicitly determining F_2 through brute force evaluation of (2.52a) is an involved procedure. The main difficulty lies in the size of the expression and, thus, a computer algebra system, like Mathematica [47], has to be used. Unfortunately, standard computer algebra systems give little control over the form in which expressions are manipulated. This leads, for example, to the following difficulty. Due to the large number of constants, simply expanding the integrand with a computer algebra system results in hundreds of terms. With standard computer algebra system procedures, this amount cannot easily be reduced. And furthermore, term-by-term integration is not straight forward since individual terms in this expansion are not always integrable: integrating the function $\psi_1 h_1$ term by term from 0 to s results in

$$\int_0^s \psi_1 h_1 d\tilde{s} = A_1 \log \cosh \sqrt{\alpha}s + A_2 \sqrt{\alpha}s \tanh \sqrt{\alpha}s + \text{hundreds of other terms},$$

where A_1 and A_2 are constants and the relation $A_1 + A_2 = 0$ holds upon using the system of equations (2.26). Now, note that $A_1 \log \cosh \sqrt{\alpha}s$ and $A_2 \sqrt{\alpha}s \tanh \sqrt{\alpha}s$ are individually unbounded and not integrable, but if we use the relation $A_1 + A_2 = 0$ the two together produce



(a) The amplitude of the m_2^{upper} -solution of equation (2.3) as a and ε are moving along the $k = 2$ -branch for $d = 3$ and $b = 0$, see Figure 1.3.1b. The peak moves to the right as a tends to 0.

(b) The relation between κ and a for various values of d and $b = 0$. Since the line is flat at the origin it follows $\kappa_1 = 0$.

Figure 2.2.2: These figures are a reproduction of Figure 5 in [9] and are based on numerical simulations.

a bounded and integrable function. Thus, to evaluate (2.52a) we have to integrate hundreds of terms, and also individual terms must be combined in an appropriate way because they are not always integrable individually. After overcoming these numerical complications, we obtain

$$\begin{aligned}
 F_2 = & \frac{1}{2304\alpha\kappa_0^2} \left[64\alpha^3 K_0 \left(4((5-2b)b+7)K_0\kappa_0^2 - 27d + 27 \right) \right. \\
 & - 16\alpha^2 \left(384b(d-1) - 86d^2 + d(64K_0\kappa_0^2 + 175) + 8\kappa_0^2 (9\kappa_0\kappa_2 - 4K_0(K_0\kappa_0^2 + 2)) - 89 \right) \\
 & + 12\alpha\kappa_0^2 \left(12b(d+31) + d(3d+25) - 16K_0\kappa_0^2 - 253 \right) + 9\kappa_0^4(-12b+3d+70) \Big] \\
 & - \frac{(16\alpha^2(d-1) + 3\kappa_0^4)}{12\alpha\kappa_0^2} \log 2.
 \end{aligned}$$

Similar difficulties arise when determining G_2 . We arrive at

$$\begin{aligned}
 G_2 = & -\frac{1}{36\kappa_0} \left[-4\alpha(8b+17)(d-1)K_0 + K_0\kappa_0^2(32\alpha(b+1)K_0 + 12b+39) + 36\alpha\kappa_2\kappa_0 \right] \\
 & - \frac{(2b-1)(4\alpha(d-1) - 3\kappa_0^2) \log 2}{6\alpha\kappa_0}.
 \end{aligned}$$

In the further analysis, we do not need an explicit expression for κ_2 , therefore we will not derive it.

2.2.3 Far field

In this section, we study equation (2.3a) in the far field. From Definition 1.3.6.c, it follows that in the far field $\xi \gg \frac{1}{a}$ holds. From (2.3b), it follows that we want the solution to decay in the far field. Thus, we assume that $|Q|^{2\sigma} \ll 1$. Now, equation (2.3a) reduces to leading order

to

$$(1 - i\varepsilon) \left(Q_{\xi\xi} + \frac{d-1}{\xi} Q_{\xi} \right) - Q + i \frac{a}{\sigma} Q + ia\xi Q_{\xi} = 0. \quad (2.53)$$

If we introduce the transformation

$$z = -\frac{ia}{1-i\varepsilon} \frac{\xi^2}{2}, \quad \beta_1 = \frac{1}{2} \left(\frac{1}{\sigma} + \frac{i}{a} \right), \quad \beta_2 = \frac{d}{2},$$

equation (2.53) becomes

$$zw_{zz} + (\beta_2 - z)w_z - \beta_1 w = 0.$$

This equation is known as the confluent hypergeometric equation or Kummer's equation, see [33] equation (13.2.1). It has two solutions: a slowly oscillating algebraically decaying solution $U(\beta_1, \beta_2, z)$ and a rapidly oscillating exponentially growing solution $V(\beta_1, \beta_2, z) = e^z U(\beta_2 - \beta_1, \beta_2, z)$. Apart from the fact that we are not interested in an exponentially growing solution, we cannot neglect the $|Q|^{2\sigma}$ term in equation (2.3a) if the solution grows exponentially. The asymptotics of the slowly oscillating and decaying solution is at leading order known to be

$$U(\beta_1, \beta_2, z) = z^{-\beta_1} \left(1 + O(|z|^{-1}) \right),$$

see expression (13.7.3) in [33]. From this, we conclude that to leading order

$$Q = \mu \xi^{-\frac{1}{\sigma} - \frac{i}{a}}, \quad (2.54)$$

in the far field, where μ is a complex constant.

2.3 Matching

In Section 2.2, we constructed a solution to (2.3) with asymptotic methods in three regions: the bump region, the inner region and the far field, see Figure 1.3.2. Note that the leading order of this construction is valid for the entire range $0 < \sigma < 2$ to which we restrict ourselves in this thesis, see Assumption 1.2.2.c. In this section, we also construct the solution in Region 4, i.e. the transition region in between the bump region and the far field, and in Region 2, i.e. the transition region in between the bump region and the inner region. Furthermore, we perform a matching procedure, i.e. we show that in a scale in which regions overlap, the solutions of the two different regions are equal to leading order if one chooses undetermined constants appropriately. Since the leading order solutions can be constructed for the entire range $0 < \sigma < 2$ to which we restrict ourselves in this thesis, we will not restrict ourselves to the case $\sigma = 1$ in this section. A similar matching procedure for the NLS is given in less detail in [7].

In between the bump region and the inner region, Region 2, the equation reduces to a parabolic cylinder equation with slowly varying coefficients. Also, in between the far field and the bump region, Regions 4 to 6, the equation reduces to a parabolic cylinder equation with slowly varying coefficients and a turning point at $\xi = \frac{2}{a}$, Region 4b.

In Section 2.3.1, we perform a WKBJ analysis, a method to find asymptotic expansions for linear differential equations with slowly varying coefficients [20], to construct a solution in Region 2. Furthermore, we match the WKBJ solution in Region 2 with the solution in the inner region and the solution in the bump region in Sections 2.3.2 and 2.3.3. This leads to an expression for $Q(0)$ in terms of d , σ , a and κ_0 .

In Section 2.3.4, we perform a WKBJ analysis in Region 4c, and we match this WKBJ solution with a solution that decays in the far field, see Section 2.3.5. In Section 2.3.6, we

analyze the solution around the turning point of the parabolic cylinder equation, i.e. around $\xi = \frac{2}{a}$. The solution around the turning point is matched with the WKB solution of Region 4c, see Section 2.3.7. And finally, we construct a WKB solution in Region 4a, and match this solution with the solution around the turning point and the bump region, see Sections 2.3.8 to 2.3.10. This also leads to an expression for the constant μ as introduced in equation (2.54) depending on d, σ, a and κ_0 .

2.3.1 A WKB analysis in Region 2

In this section, we determine to leading order the solution of equation (2.3a) in Region 2, see Figure 1.3.2 and Definition 1.3.6. Since $|Q| \ll 1$ and $\xi \gg 1$ in this region, we can apply a WKB analysis to obtain a solution. However, the analysis in this section breaks down once ξ is too close to zero, as is the case in the inner region.

Similar to the analysis in the inner region and the far field, it follows that (2.3a) reduces in regions where $|Q| \ll a$ to

$$(1 - i\varepsilon) \left(Q_{\xi\xi} + \frac{d-1}{\xi} Q_{\xi} \right) + ia \left(\xi Q_{\xi} + \frac{1}{\sigma} Q \right) - Q = 0. \quad (2.55)$$

The Liouville transformation

$$Q(\xi) = \xi^{\frac{1-d}{2}} e^{-\frac{1}{4}ia\xi^2/(1-i\varepsilon)} Z(\xi) \quad (2.56)$$

puts this equation in self-adjoint form. For ξ large, we obtain to leading order

$$(1 - i\varepsilon) Z_{\xi\xi} + \left(\frac{a^2 \xi^2}{4(1 - i\varepsilon)} - 1 + \frac{ia}{2} \left(\frac{2}{\sigma} - d \right) \right) Z = 0. \quad (2.57)$$

Note that this equation has a turning point at $\xi = \frac{2}{a}$, i.e. the coefficient of Z vanishes to leading order. This does not play a role in Region 2, but it does play a role if we apply the WKB method on the right hand side of the bump region. We will write Z_2 instead of Z for the solution of equation (2.57) in Region 2.

We want to match solutions to this equation with exponentially decaying solutions in the bump region. From equation (2.13), it follows that this exponential decay takes the form $(C_1 + iC_2)e^{-\sqrt{\alpha}s}$, where $s = \xi - \frac{\kappa}{a}$ and α is defined by (2.12). Note that it is possible to rescale the phase of Z such that in the intermediate region, where we match with the bump region, Z_2 must be real. Therefore, we try to find a real solution of equation (2.57). We split Z_2 in an amplitude $|Z_2|$ and a phase $\arg(Z_2)$ and obtain the system of equations

$$\begin{aligned} |Z_2|_{\xi\xi} - |Z_2| \arg(Z_2)_{\xi}^2 + \left(\frac{a^2 \xi^2}{4} - 1 \right) |Z_2| + \varepsilon \left(|Z_2| \arg(Z_2)_{\xi\xi} + 2|Z_2|_{\xi} \arg(Z_2)_{\xi} \right) &= 0, \\ \arg(Z_2)_{\xi\xi} |Z_2| + 2|Z_2|_{\xi} \arg(Z_2)_{\xi} - \varepsilon \left(|Z_2|_{\xi\xi} - |Z_2| \arg(Z_2)_{\xi}^2 \right) + \varepsilon \frac{a^2 \xi^2}{4} |Z_2| + \frac{a}{2} \left(\frac{2}{\sigma} - d \right) |Z_2| &= 0. \end{aligned}$$

From this system, it follows, ignoring the fact that there is a turning point at $\xi = \frac{2}{a}$ for the moment, that the solution $\arg(Z_2)$ is constant to leading order valid as long as $\xi \ll a^{-\frac{3}{2}}$. For this reason, Z_2 satisfies to leading order

$$\partial_x^2 Z_2 - \frac{4}{a^2} q_2(x) Z_2 = 0, \quad (2.58)$$

where $x = \frac{a\xi}{2}$, $q_2(x) := 1 - x^2$ and Z_2 is real. As claimed in Definition 1.3.6, this equation is a parabolic cylinder equation with slowly varying coefficients, see also Figure 1.3.2. A WKBJ analysis for an equation of this type gives to leading order

$$Z_2 = \frac{1}{q_2(x)^{1/4}} \left(C_{2-} e^{-\frac{2}{a} \int_1^x \sqrt{q_2(s)} ds} + C_{2+} e^{\frac{2}{a} \int_1^x \sqrt{q_2(s)} ds} \right), \quad (2.59)$$

see [20] Chapter 4. Note that we are free to choose the lower integration limit, but it is standard to choose it equal to the zero of $q_2(x)$. This zero of $q_2(x)$ is the turning point, see Section 2.3.6 for an analysis around the turning point. For later reference, we work out the integral and obtain

$$\begin{aligned} \int_1^x \sqrt{q_2(s)} ds &= \int_1^x \sqrt{1-s^2} ds = \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \arcsin x - \frac{\pi}{4} \\ &= \frac{1}{2} x \sqrt{1-x^2} - \frac{1}{2} \arccos x, \end{aligned} \quad (2.60)$$

where we used the identity $\arcsin x + \arccos x = \frac{\pi}{2}$.

2.3.2 Matching the WKBJ solutions with the inner region

In this section, we match the solution in the inner region, as obtained in Section 2.2.1, with the WKBJ solution in Region 2 in the intermediate scale $\xi = a^{-\gamma}t$, where $0 < \gamma < 1$ and $t = O(1)$. In this scale, the leading order behavior of the solution in the inner region is given by (2.8). To write the solution Q_{inner} in the inner region in the same form as the WKBJ solution, we also apply Liouville transformation (2.56) to the solution in the inner region. Combining this with (2.8) gives to leading order

$$Z_{\text{inner}}(a^{-\gamma}t) := (a^{-\gamma}t)^{\frac{d-1}{2}} e^{\frac{1}{4}ia(a^{-\gamma}t)^2/(1-i\epsilon)} Q_{\text{inner}}(a^{-\gamma}t) = \frac{C_I}{\sqrt{2\pi}} e^{a^{-\gamma}t + o(a^{-\gamma})}. \quad (2.61)$$

We proceed with the WKBJ solution in Region 2. If $\xi = a^{-\gamma}t$, then $x = \frac{1}{2}a^{1-\gamma}t \ll 1$, so we expand (2.60) around $x = 0$ to obtain to leading order

$$\int_1^{\frac{1}{2}a^{1-\gamma}t} \sqrt{q_2(s)} ds = -\frac{\pi}{4} + \frac{1}{2}a^{1-\gamma}t.$$

This gives to leading order

$$Z_2 = \sum_{\pm} C_{2\pm} \exp \left[\pm \frac{2}{a} \left(-\frac{\pi}{4} + \frac{1}{2}a^{1-\gamma}t \right) \right].$$

Finally, matching with (2.61) gives $C_{2+} e^{-\frac{\pi}{2a}} = \frac{C_I}{\sqrt{2\pi}}$ and $C_{2-} = 0$. Upon using (2.7), we obtain the following expression for $Q(0)$:

$$Q(0) = \frac{2^{\frac{1-d}{2}} \sqrt{\pi}}{\Gamma\left(\frac{d}{2}\right)} e^{-\frac{\pi}{2a}} C_{2+}. \quad (2.62)$$

2.3.3 Matching Region 2 with the bump region

In this section, we match the solution in the bump region with the WKBJ-solution of Region 2 in the intermediate scale $s = \xi - \frac{\kappa}{a} = a^{-\gamma}t$, where $0 < \gamma < 1$ and $0 > t = O(1)$. From (2.13), we obtain that in the intermediate scale the asymptotic behavior of the solution Q_{bump} in the bump region is to leading order given by

$$Q_{\text{bump}} = 2((\sigma + 1)\alpha)^{\frac{1}{2\sigma}} e^{-\frac{1}{2}ik_0 a^{-\gamma}t} e^{-\sqrt{\alpha}a^{-\gamma}|t|}. \quad (2.63)$$

Using (2.59) and (2.60), we derive an expression for the asymptotic behavior of the WKBJ-solution in the Region 2 in the intermediate scale:

$$\begin{aligned} Q_2\left(\frac{\kappa_0}{a} + \frac{t}{a^\gamma}\right) &= \left(\frac{\kappa_0}{a} + \frac{t}{a^\gamma}\right)^{\frac{1-d}{2}} e^{-\frac{1}{4}ia\left(\frac{\kappa_0}{a} + \frac{t}{a^\gamma}\right)^2} Z_2\left(\frac{\kappa_0}{a} + \frac{t}{a^\gamma}\right) \\ &= \alpha^{-\frac{1}{4}} \sum_{\pm} \left(\frac{a}{\kappa_0}\right)^{\frac{d-1}{2}} e^{-\frac{ik_0^2}{4a}} e^{-\frac{1}{2}ik_0 a^{-\gamma}t} C_{2\pm} e^{\pm \frac{2}{a} \left(\frac{\kappa_0}{4} \sqrt{1 - \frac{\kappa_0^2}{4}} - \frac{1}{2} \arccos \frac{\kappa_0}{2} \right)} e^{\pm \sqrt{1 - \frac{\kappa_0^2}{4}} a^{-\gamma}t}. \end{aligned} \quad (2.64)$$

Upon comparing (2.63) with (2.64), we obtain that

$$\alpha^{-\frac{1}{4}} \left(\frac{a}{\kappa_0}\right)^{\frac{d-1}{2}} e^{-\frac{ik_0^2}{4a}} e^{\pm \frac{2}{a} \left(\frac{\kappa_0}{4} \sqrt{1 - \frac{\kappa_0^2}{4}} - \frac{1}{2} \arccos \frac{\kappa_0}{2} \right)} C_{2\pm} = 2((\sigma + 1)\alpha)^{\frac{1}{2\sigma}}. \quad (2.65)$$

Combining (2.65) with (2.62) and using (2.12) gives

$$Q(0) = 2^{\frac{3-d}{2}} ((\sigma + 1)\alpha)^{\frac{1}{2\sigma}} \alpha^{\frac{1}{4}} \left(\frac{a}{\kappa_0}\right)^{\frac{1-d}{2}} e^{\frac{ik_0^2}{4a}} \frac{\sqrt{\pi}}{\Gamma\left(\frac{d}{2}\right)} e^{-\frac{\kappa_0 \sqrt{a}}{2a} - \frac{1}{a} \arcsin \frac{\kappa_0}{2}}. \quad (2.66)$$

As can be seen, $Q(0)$ is indeed exponentially small in a . Furthermore, we obtain from (2.66) that the phase of $Q(0)$ is to leading order given by

$$\Psi_0 = \frac{ik_0^2}{4a} \pmod{2\pi}. \quad (2.67)$$

2.3.4 A WKBJ analysis in Region 4c

In Region 4c, it will turn out that the solution of equation (2.57) in which we are interested will not be real. Therefore, we do not proceed as in Section 2.3.1. Instead, we construct a solution of equation (2.57) with the complex WKBJ method.

The rescaling $z = \frac{a\xi}{2\sqrt{1-i\varepsilon}}$ transforms equation (2.57) to

$$\partial_z^2 Z_{4c} - \frac{4}{a^2} q_{4c}(z) Z_{4c} = 0, \quad (2.68)$$

with $q_{4c}(z) := 1 - z^2 - \frac{ia}{2} \left(\frac{2}{\sigma} - d \right)$ and where we have written Z_{4c} instead of Z for the solution of (2.68) in Region 4c. This equation has the same functional form as equation (2.58). Therefore, the solution is, similar to (2.59), to leading order given by

$$Z_{4c}(z) = \frac{1}{|q_{4c}(z)|^{1/4}} \left(C_{4c+} e^{\frac{2i}{a} \int_1^z \sqrt{-q_{4c}(s)} ds} + C_{4c-} e^{-\frac{2i}{a} \int_1^z \sqrt{-q_{4c}(s)} ds} \right). \quad (2.69)$$

Note that, contrary to (2.59), expression (2.69) is written in a form with complex exponentials, because in Region 4c, $q_c(z)$ is to leading order real and negative. Finally, we evaluate the integrals in (2.69) at leading order:

$$\begin{aligned} \int_1^z \sqrt{-q_{4c}(s)} \, ds &= \int_1^z \sqrt{s^2 - 1} + \frac{ia(\frac{2}{\sigma} - d)}{4\sqrt{s^2 - 1}} \, ds \\ &= \frac{1}{2}z\sqrt{z^2 - 1} + \left(\frac{ia}{4} \left(\frac{2}{\sigma} - d \right) - \frac{1}{2} \right) \log \left(z + \sqrt{z^2 - 1} \right). \end{aligned} \quad (2.70)$$

2.3.5 Matching the WKBJ solutions with the far field

In this section, we match the WKBJ solution constructed in Region 4c with the solution constructed in the far field. For this purpose, we introduce, based on Definition 1.3.6.c and 1.3.6.g, the intermediate scale $\xi = \frac{2}{a} + \frac{t}{a^\gamma}$, where $1 < \gamma < 2$ and $t = O(1)$. This gives that to leading order $z = \frac{1}{2}a^{1-\gamma}t$ and $\xi = a^{-\gamma}t$.

In Section 2.2.3, we studied the solution in the far field. Upon using (2.54), we obtain that in the above introduced intermediate scale the solution in the far field, denoted by $Q_{\text{far field}}(\xi)$, is to leading order given by

$$Q_{\text{far field}} = \mu \xi^{-\frac{1}{\sigma} - \frac{i}{a}} = \mu (a^{-\gamma}t)^{-\frac{1}{\sigma} - \frac{i}{a}}. \quad (2.71)$$

It can be seen that (2.69) contains an exponentially growing term if $C_{4c-} \neq 0$ which cannot be matched the solution in the far field. Thus, we set

$$C_{4c-} = 0. \quad (2.72)$$

Substitution of the leading order approximation $z = \frac{1}{2}a^{1-\gamma}t$ in (2.69) and (2.70) gives to leading order

$$\begin{aligned} Q_{4c} &= \xi^{\frac{1-d}{2}} e^{-\frac{1}{4}ia\xi^2/(1-i\varepsilon)} Z_{4c}(a^{-\gamma}t) \\ &= C_{4c+} (a^{-\gamma}t)^{\frac{1-d}{2}} e^{-\frac{1}{4}ia(a^{-\gamma}t)^2/(1-i\varepsilon)} \sqrt{\frac{2}{a}} (a^{-\gamma}t)^{-\frac{1}{2}} e^{\frac{1}{4}ia(a^{-\gamma}t)^2/(1-i\varepsilon)} (a^{1-\gamma}t)^{-\frac{1}{\sigma} + \frac{d}{2} - \frac{i}{a}} \\ &= C_{4c+} \sqrt{\frac{2}{a}} a^{\frac{d}{2} - \frac{1}{\sigma}} e^{-\frac{i}{a} \ln a} (a^{-\gamma}t)^{-\frac{1}{\sigma} - \frac{i}{a}}. \end{aligned}$$

This matches with (2.71) if we choose

$$\mu = C_{4c+} \sqrt{\frac{2}{a}} a^{\frac{d}{2} - \frac{1}{\sigma}} e^{-\frac{i}{a} \ln a}. \quad (2.73)$$

2.3.6 The turning point

Equation (2.68) admits a turning point at $z = 1$, i.e. at $\xi = \frac{2}{a}$, where the coefficient q_{4c} of Z_{4c} in equation (2.68) vanishes to leading order. Since $|q_{4c}|$ is no longer $O(1)$ close to the turning point, this results in a different balance in the terms of equation (2.68). And thus, a standard WKBJ solution is no longer valid, see for example [20] Chapter 4.3. In this section, we perform a WKBJ analysis around the turning point.

We zoom in around the turning point by setting $\tilde{z} = a^{-\beta_1}(z-1)$ and denote by $Z(z) = Z_{4b}(\tilde{z})$ the solution around the turning point, i.e. the solution in Region 4b. From equation (2.68),

we obtain to leading order

$$a^{-2\beta_1} \partial_{\tilde{z}}^2 Z_{4b} - \frac{4}{a^2} \left(-2a^{\beta_1} \tilde{z} - \frac{ia}{2} \left(\frac{2}{\sigma} - d \right) \right) Z_{4b} = 0.$$

From balancing the terms containing β_1 , we obtain $\beta_1 = \frac{2}{3}$. It follows

$$\partial_{\tilde{z}}^2 Z_{4b} - \left(-8\tilde{z} - 2ia^{\frac{1}{3}} \left(\frac{2}{\sigma} - d \right) \right) Z_{4b} = 0.$$

The solution of this equation is given by Airy functions:

$$\tilde{Z}_{4b}(y) = C_{Ai} Ai(y) + C_{Bi} Bi(y), \quad (2.74)$$

where $y = -2\tilde{z} - \frac{ia^{\frac{1}{3}}}{2} \left(\frac{2}{\sigma} - d \right)$.

2.3.7 Matching Region 4c with Region 4b

In this section, we match the solution constructed in Region 4b and given by (2.74) with the WKBJ solution constructed in Region 4c and given by (2.69). For this, we use the intermediate scale $z_\eta = \frac{z-1}{a^\eta}$, with $0 < \eta < \frac{2}{3}$ and $z_\eta = O(1)$. In this scale, the asymptotic expansions derived for Region 4b and 4c are both valid. Furthermore, this scale interpolates between the scales of Region 4b and 4c as η ranges from 0 to $\frac{2}{3}$, see Definition 1.3.6. Now, using the asymptotic expansions of $Ai(y)$ and $Bi(y)$, we obtain from (2.74) that if $z_\eta > 0$, then to leading order

$$\begin{aligned} Z_{4b} = & \frac{1}{\sqrt{\pi} \left(2a^{\eta-\frac{2}{3}} z_\eta - \frac{ia^{\frac{1}{3}}}{2} \left(\frac{2}{\sigma} - d \right) \right)^{\frac{1}{4}}} \left[C_{Bi} \cos \left(\frac{2}{3} \left(2a^{\eta-\frac{2}{3}} z_\eta - \frac{ia^{\frac{1}{3}}}{2} \left(\frac{2}{\sigma} - d \right) \right)^{\frac{3}{2}} + \frac{\pi}{4} \right) \right. \\ & \left. + C_{Ai} \sin \left(\frac{2}{3} \left(2a^{\eta-\frac{2}{3}} z_\eta - \frac{ia^{\frac{1}{3}}}{2} \left(\frac{2}{\sigma} - d \right) \right)^{\frac{3}{2}} + \frac{\pi}{4} \right) \right]. \end{aligned} \quad (2.75)$$

From (2.72), it is immediately clear that Z_{4b} should be of the form $Z_{4b} = C_{4b}(Bi + iAi)$, for some constant C_{4b} . Thus, upon choosing $C_{4b} = C_{Bi} = -iC_{Ai}$, expression (2.75) becomes to leading order

$$\begin{aligned} Z_{4b} = & \frac{a^{\frac{1}{6}} C_{4b}}{\sqrt{\pi} 2^{\frac{1}{4}} (a^\eta z_\eta)^{\frac{1}{4}}} \exp \left[\frac{2}{3} i \left(2a^{\eta-\frac{2}{3}} z_\eta - \frac{ia^{\frac{1}{3}}}{2} \left(\frac{2}{\sigma} - d \right) \right)^{\frac{3}{2}} + \frac{\pi}{4} i \right] \\ = & \frac{a^{\frac{1}{6}} e^{i\frac{\pi}{4}} C_{4b}}{\sqrt{\pi} 2^{\frac{1}{4}} (a^\eta z_\eta)^{\frac{1}{4}}} \exp \left[\frac{4\sqrt{2}i}{3a} (a^\eta z_\eta)^{3/2} - \frac{\sqrt{2}}{2} \left(\frac{2}{\sigma} - d \right) \sqrt{a^\eta z_\eta} \right]. \end{aligned} \quad (2.76)$$

We proceed with deriving a leading order expression for the WKBJ solution (2.69) constructed in Region 4c in the intermediate scale. From (2.70), we obtain

$$\frac{2i}{a} \int_1^{1+a^\eta z_\eta} \sqrt{-q_{4c}(s)} \, ds = \frac{4\sqrt{2}i}{3a} (a^\eta z_\eta)^{\frac{3}{2}} - \frac{\sqrt{2}}{2} \left(\frac{2}{\sigma} - d \right) \sqrt{a^\eta z_\eta},$$

substitution in (2.69) gives to leading order

$$Z_{4c} = \frac{C_{4c+}}{2^{\frac{1}{4}} (a^\eta z_\eta)^{\frac{1}{4}}} \exp \left[-\frac{\sqrt{2}}{2} \left(\frac{2}{\sigma} - d \right) \sqrt{a^\eta z_\eta} + \frac{4\sqrt{2}i}{3a} (a^\eta z_\eta)^{\frac{3}{2}} \right]. \quad (2.77)$$

We conclude that (2.77) matches with (2.76) if

$$C_{4c+} = e^{i\frac{\pi}{4}} a^{\frac{1}{6}} \frac{C_{4b}}{\sqrt{\pi}}. \quad (2.78)$$

2.3.8 A WKBJ analysis in Region 4a

On the left hand side of the turning point, we perform, as also done in Region 4c, a WKBJ analysis, see Section 2.3.4. We construct a solution of equation (2.57) with the complex WKBJ method.

Again, the rescaling $z = \frac{a\xi}{2\sqrt{1-i\varepsilon}}$ transforms equation (2.57) to

$$\partial_{zz}^2 Z_{4a} - \frac{4}{a^2} q_{4a}(z) Z_{4a} = 0, \quad (2.79)$$

where $q_{4a}(z) := 1 - z^2 - \frac{ia}{2} \left(\frac{2}{\sigma} - d \right)$. The solution is with the WKBJ method to leading order given by

$$Z_{4a}(z) = \frac{1}{q_{4a}(z)^{1/4}} \left(C_{4a+} e^{\frac{2}{a} \int_1^z \sqrt{q_{4a}(s)} ds} + C_{4a-} e^{-\frac{2}{a} \int_1^z \sqrt{q_{4a}(s)} ds} \right). \quad (2.80)$$

We evaluate to leading order

$$\int_1^z \sqrt{q_{4a}(s)} ds = \frac{1}{2} z \sqrt{1 - z^2} + \left(\frac{1}{2} - \frac{ia}{4} \left(\frac{2}{\sigma} - d \right) \right) \arcsin z - \frac{\pi}{2} \left(\frac{1}{2} - \frac{ia}{4} \left(\frac{2}{\sigma} - d \right) \right). \quad (2.81)$$

2.3.9 Matching Region 4a with Region 4b

In this section, we match the solution (2.74) around the turning point with the WKBJ solution (2.80) of Region 4a. The analysis is similar to the one presented in Section 2.3.7. We use the intermediate scale $z_\eta = \frac{z-1}{a^\eta}$, with $0 < \eta < \frac{2}{3}$ and $t = O(1)$. Using the asymptotics of $\text{Ai}(y)$ and $\text{Bi}(y)$, we obtain for $z_\eta < 0$ that (2.74) implies that to leading order

$$Z_{4b} = \frac{a^{\frac{1}{6}} C_{4b}}{\sqrt{\pi} 2^{\frac{1}{4}} (-a^\eta z_\eta)^{\frac{1}{4}}} \left[e^{\frac{4\sqrt{2}}{3a} (-a^\eta z_\eta)^{3/2} - \frac{i\sqrt{2}}{2} \left(\frac{2}{\sigma} - d \right) \sqrt{-a^\eta z_\eta}} + \frac{i}{2} e^{-\frac{4\sqrt{2}}{3a} (-a^\eta z_\eta)^{3/2} + \frac{i\sqrt{2}}{2} \left(\frac{2}{\sigma} - d \right) \sqrt{-a^\eta z_\eta}} \right]. \quad (2.82)$$

We proceed with deriving a leading order expression for the WKBJ solution (2.80) constructed in Region 4a in the intermediate scale. From (2.81), we obtain

$$\frac{2}{a} \int_1^{1+a^\eta z_\eta} \sqrt{q_{4a}(s)} ds = -\frac{4\sqrt{2}}{3a} (-a^\eta z_\eta)^{\frac{3}{2}} + \frac{\sqrt{2}i}{2} \left(\frac{2}{\sigma} - d \right) \sqrt{-a^\eta z_\eta}.$$

Substitution in (2.80) gives to leading order

$$Z_{4a} = \frac{1}{2^{\frac{1}{4}} (-a^\eta z_\eta)^{\frac{1}{4}}} \sum_{\pm} C_{4a\pm} \exp \left[-\frac{4\sqrt{2}}{3a} (-a^\eta z_\eta)^{\frac{3}{2}} + \frac{\sqrt{2}i}{2} \left(\frac{2}{\sigma} - d \right) \sqrt{-a^\eta z_\eta} \right]. \quad (2.83)$$

We conclude that (2.83) matches with (2.82) if

$$C_{4a+} = \frac{a^{\frac{1}{6}} i C_{4b}}{2\sqrt{\pi}}, \text{ and } C_{4a-} = \frac{a^{\frac{1}{6}} C_{4b}}{\sqrt{\pi}}. \quad (2.84)$$

2.3.10 Matching Region 4a with the bump region

In this section, we match the WKBJ solution (2.80) of Region 4a with the solution from the bump region (2.13). As in Section 2.3.3, we use the intermediate scale $s = \xi - \frac{\kappa}{a} = a^{-\gamma}t$, where $0 < \gamma < 1$ and $0 < t = O(1)$. The asymptotic behavior of the solution in the bump region is to leading order given by (2.63). Using (2.80) and (2.81), we obtain the asymptotic behavior of the WKBJ solution in the Region 4a in the intermediate scale:

$$Q_{4a}\left(\frac{\kappa_0}{a} + \frac{t}{a^\gamma}\right) = a^{-\frac{1}{4}} \sum_{\pm} \left(\frac{a}{\kappa_0}\right)^{\frac{d-1}{2}} e^{-\frac{i\kappa_0^2}{4a}} e^{-\frac{1}{2}i\kappa_0 a^{-\gamma}t} C_{4\pm} e^{\pm \frac{2}{a} \left(\frac{\kappa_0}{4} \sqrt{1 - \frac{\kappa_0^2}{4} - \frac{1}{2} \arccos \frac{\kappa_0}{2}}\right)} e^{\pm \sqrt{1 - \frac{\kappa_0^2}{4}} a^{-\gamma}t}. \quad (2.85)$$

Upon comparing (2.85) with (2.64), we obtain that

$$\alpha^{-\frac{1}{4}} \left(\frac{a}{\kappa_0}\right)^{\frac{d-1}{2}} e^{-\frac{i\kappa_0^2}{4a}} e^{-\frac{2}{a} \left(\frac{\kappa_0}{4} \sqrt{1 - \frac{\kappa_0^2}{4} - \frac{1}{2} \arccos \frac{\kappa_0}{2}}\right)} C_{4a-} = 2((\sigma + 1)\alpha)^{\frac{1}{2\sigma}}. \quad (2.86)$$

In principle, C_{4a+} can be determined from (2.86) and (2.84). Combining (2.86), (2.84), (2.78) and (2.73) gives that at leading order

$$\begin{aligned} \mu &= 2((\sigma + 1)\alpha)^{\frac{1}{2\sigma}} \alpha^{\frac{1}{4}} e^{i\frac{\pi}{4}} \left(\frac{a}{\kappa_0}\right)^{\frac{1-d}{2}} e^{\frac{i\kappa_0^2}{4a}} e^{\frac{2}{a} \left(\frac{\kappa_0}{4} \sqrt{\alpha - \frac{1}{2} \arccos \frac{\kappa_0}{2}}\right)} \sqrt{\frac{2}{a}} a^{\frac{d}{2} - \frac{1}{\sigma}} e^{-\frac{i}{a} \ln a} \\ &= 2\sqrt{2}((\sigma + 1))^{\frac{1}{2\sigma}} \alpha^{\frac{1}{4} + \frac{1}{2\sigma}} e^{i\frac{\pi}{4}} \kappa_0^{\frac{d-1}{2}} e^{\frac{i\kappa_0^2}{4a}} e^{\frac{2}{a} \left(\frac{\kappa_0}{4} \sqrt{\alpha - \frac{1}{2} \arccos \frac{\kappa_0}{2}}\right)} a^{-\frac{1}{\sigma} - \frac{i}{a}}. \end{aligned} \quad (2.87)$$

Since $\frac{\kappa_0}{4} \sqrt{1 - \frac{\kappa_0^2}{4} - \frac{1}{2} \arccos \frac{\kappa_0}{2}}$ attains only negative values for $0 < \kappa_0 < 2$, it follows that μ is exponentially small in a .

2.4 The global estimate

In this section, we present an alternative, but formal approach, to derive an expression for the constant μ as introduced in (2.54) and as given by (2.87). This approach is also presented in [9], but will be presented here in more detail. The analysis in this section leads to three results. First, for the alternative expression for μ to be consistent with (2.87), a condition which is equivalent to (2.26b) must hold. However, in this section, we will only use a leading order analysis to derive this condition, while (2.26b) arose as a solvability condition at higher order. This relation has not been derived [9]. Second, the formal approach presented in this section is valid under milder assumptions than the assumptions under which the asymptotic construction of the m_2^{upper} -solution, see Definition 2.2.8, is presented, see Assumption 2.2.1. As a result, we formally derive an asymptotic expression for the entire $k = 2$ -branch as depicted in Figure 1.3.1b. This second result is also presented in [9] in less detail. And finally, the analysis in this section allows us to make a link with the analysis for the NLS, i.e. the case $\varepsilon = b = 0$, see [7].

The assumptions under which we construct to leading order a solution of Problem (2.3) are as follows:

Assumption 2.4.1. In Section 2.4 we make the following (alternative) assumptions:

2.4.1.a $0 < \varepsilon \ll 1$.

2.4.1.b $0 < a \ll 1$.

2.4.1.c $b > -\frac{W'}{W_1}$, where W_1 and W' are defined by (2.27).

2.4.1.d The valid parameter range for d and σ is as specified in Assumption 1.2.2.

2.4.1.e As in Section 2.2, we assume that the solution we consider in this section has a maximum at $\frac{\kappa}{a}$. This is because we are interested in the m_2^{upper} -solutions that correspond to the $k = 2$ -branch in Figure 1.3.1.

Note that we do not assume a (leading order) linear scaling $\varepsilon = Ka$ between ε and a , see Assumption 2.2.1.c, which is only valid on the lower part of the $k = 2$ -branch. Instead, to keep notation consistent, we set $\varepsilon = K(a)a$, with $K(a) = o\left(\frac{1}{a}\right)$. As stated, one of the objectives is to formally derive an asymptotic expression for the $k = 2$ -branch in Figure 1.3.1. Note that, on the whole branch, ε and a are relatively small, but not extremely small everywhere. Thus, we do not expect that this formal derivation and the resulting formula for $k = 2$ -branch to be accurate for all values of d and b . An example of a case in which both a and ε are small on the upper branch is the limit $d \rightarrow 2$, see Figure 2.4.1a. This example motives the use of this approach.

Note that to leading order the analysis in Sections 2.2 and 2.3 is valid under Assumption 2.4.1. Therefore, in this section, we can readily use the already presented leading order analysis to formally derive an alternative expression for μ and the three results as stated above.

2.4.1 An analytic expression for μ

In this section, we derive an analytic expression for μ . This expression will be evaluated by using a formal leading order analysis in Section 2.4.2. We split Q in an amplitude and a phase by writing $Q(\xi) = A(\xi) \exp\left(i \int_0^\xi \psi(x) dx + i\Psi_0\right)$. Substitution in (2.3a) and splitting in a real and imaginary part leads to

$$A'' = A\psi^2 - \frac{d-1}{\xi}A' + A - A^{2\sigma+1} - \varepsilon\left(2A'\psi + A\psi' + \frac{d-1}{\xi}A\psi\right) + a\xi A\psi, \quad (2.88a)$$

$$\psi' = -2\frac{A'}{A}\psi - \frac{d-1}{\xi}\psi + \frac{\varepsilon}{A}\left(A'' - A\psi^2 + \frac{d-1}{\xi}A'\right) - \frac{a}{A}\left(\frac{A}{\sigma} + \xi A'\right) - b\varepsilon A^{2\sigma}. \quad (2.88b)$$

From (2.88), we derive the following identity:

$$\begin{aligned} \frac{d}{d\xi}\left(\xi^{\alpha_\sigma}A^2\left(\psi + \frac{a\xi}{2}\right)\right) &= (1 + \alpha_\sigma - d)\xi^{\alpha_\sigma-1}A^2\psi + \varepsilon\xi^{\alpha_\sigma}A\left(A'' - A\psi^2 + \frac{d-1}{\xi}A'\right) \\ &\quad - \xi^{\alpha_\sigma}b\varepsilon A^{2\sigma+2}, \end{aligned} \quad (2.89)$$

where $\alpha_\sigma = \frac{2}{\sigma} - 1$. From (2.54), we obtain that in the far field $A = |\mu|\xi^{-\frac{1}{\sigma}}$ and $\psi = -\frac{1}{a\xi}$ to leading order. Thus, for $a\xi \gg 1$, relation (2.89) reduces to

$$|\mu|^2 = \frac{2}{a} \sum_{j=1}^2 M_j, \quad (2.90)$$

where

$$M_1 := \int_0^\infty (1 + \alpha_\sigma - d) \xi^{\alpha_\sigma - 1} A^2 \psi \, d\xi, \quad (2.91a)$$

$$M_2 := \int_0^\infty \varepsilon \xi^{\alpha_\sigma} A \left(A'' - A\psi^2 + \frac{d-1}{\xi} A' \right) - \xi^{\alpha_\sigma} b \varepsilon A^{2\sigma+2} \, d\xi. \quad (2.91b)$$

2.4.2 A global estimate for μ

In this section, we formally evaluate (2.90) by using the leading order expressions for the m_2^{upper} -solution as found in Sections 2.2 and 2.3. Since the resulting expression for μ is based on an integral over the entire domain $\xi = 0 \dots \infty$, we will refer to (2.90) as the global estimate for μ .

Let $M_{j,k}$, where $j = 1, 2$ and $k = 1, 2, 3, 4, 5$ be the contribution from Region k , see Definition 1.3.6, to M_j . Thus:

$$M_j = \sum_{k=1}^5 M_{j,k}.$$

In principle, all regions distinguished in the asymptotic construction, see Definition 1.3.6, contribute when evaluating (2.90). However, apart from the bump region, the m_2^{upper} -solution is exponentially small in a everywhere. Therefore, the only possible leading order contributions to the integral in (2.90) come from the bump region or the far field. The reason for the latter is that the far field region itself, contrary to all other regions, is exponentially large in a , see Definition 1.3.6.c. Thus:

$$M_{j,k} = 0,$$

for $j = 1, 2$ and $k = 1, 2, 4$.

From (2.13) and the analysis in the beginning of Section 2.2.2, we obtain that in the bump region to leading order

$$A = ((\sigma + 1) \alpha)^{\frac{1}{2\sigma}} \operatorname{sech}^{\frac{1}{\sigma}} \left(\sigma \sqrt{\alpha} \left(\xi - \frac{\kappa}{a} \right) \right) \quad (2.92a)$$

$$\psi = -\frac{\kappa}{2}, \quad (2.92b)$$

where $\alpha = 1 - \frac{\kappa^2}{4}$. In the bump region $|\xi - \frac{\kappa}{a}| \ll 1$, thus, upon evaluation of $M_{1,3}$ we obtain to leading order

$$\begin{aligned} M_{1,3} &= \int_{\frac{\kappa}{a} - a^{-\gamma}}^{\frac{\kappa}{a} + a^{-\gamma}} (1 + \alpha_\sigma - d) \xi^{\alpha_\sigma - 1} \psi A^2 \, d\xi = \\ &= -\frac{\kappa}{2} (1 + \alpha_\sigma - d) ((\sigma + 1) \alpha)^{\frac{1}{\sigma}} \int_{-\infty}^{\infty} \left(s + \frac{\kappa}{a} \right)^{\frac{2}{\sigma} - 2} \operatorname{sech}^{\frac{2}{\sigma}} (\sigma \sqrt{\alpha} s) \, ds = O(a^{2 - \frac{2}{\sigma}}), \end{aligned} \quad (2.93)$$

where $0 < \gamma < 1$. For $\sigma = 1$, we have $\alpha_\sigma = 1$ and in that case the integral in (2.93) can be explicitly determined:

$$M_{1,3}|_{\sigma=1} = \int_{\frac{\kappa}{a} - a^{-\gamma}}^{\frac{\kappa}{a} + a^{-\gamma}} (2 - d) \psi A^2 \, d\xi = -2\kappa(2 - d) \sqrt{1 - \frac{\kappa^2}{4}}. \quad (2.94)$$

Similarly, we obtain

$$M_{2,3} = \int_{\frac{\kappa}{a} - a^{-\gamma}}^{\frac{\kappa}{a} + a^{-\gamma}} \varepsilon \xi^{\alpha_\sigma} A \left(A'' - A\psi^2 + \frac{d-1}{\xi} A' - bA^{2\sigma+1} \right) d\xi = \int_{-\infty}^{\infty} K \kappa^{\alpha_\sigma} a^{2-\frac{2}{\sigma}} A \left(A'' - A\psi^2 - bA^{2\sigma+1} \right) ds = O\left(a^{2-\frac{2}{\sigma}}\right). \quad (2.95)$$

And again, for $\sigma = 1$, we get

$$M_{2,3}|_{\sigma=1} = \int_{-\infty}^{\infty} K \kappa A \left(A'' - A\psi^2 - bA^3 \right) ds = -\frac{K\kappa}{3} \sqrt{1 - \frac{\kappa^2}{4}} \left(2 + \kappa^2 + 8b \left(1 - \frac{\kappa^2}{4} \right) \right). \quad (2.96)$$

Both contributions $M_{1,3}$ and $M_{2,3}$ from the bump region are of $O\left(a^{2-\frac{2}{\sigma}}\right)$.

In the far field, a leading order solution is given by (2.54) and we obtain

$$A = |\mu| \xi^{-\frac{1}{\sigma}}, \quad (2.97a)$$

$$\psi = -\frac{1}{a\xi}. \quad (2.97b)$$

This leads to

$$\begin{aligned} |M_{1,5}| &= \left| \int_{\frac{\kappa}{a} + a^{-\gamma}}^{\infty} (1 + \alpha_\sigma - d) \xi^{\alpha_\sigma-1} \psi A^2 d\xi \right| \leq \int_{\frac{\kappa}{a}}^{\infty} \left| -\mu^2 (1 + \alpha_\sigma - d) \xi^{-3} \frac{1}{a} \right| d\xi \\ &= \left| -\frac{1 + \alpha_\sigma - d}{8} \right| |\mu|^2 a = O(a), \end{aligned} \quad (2.98)$$

where $1 < \gamma < 2$. And similarly,

$$\begin{aligned} |M_{1,5}| &= \left| \int_{\frac{\kappa}{a} + a^{-\gamma}}^{\infty} \varepsilon \xi^{\alpha_\sigma} A \left(A'' - A\psi^2 + \frac{d-1}{\xi} A' - bA^{2\sigma+1} \right) d\xi \right| \leq \int_{\frac{\kappa}{a}}^{\infty} \left| -\varepsilon |\mu|^2 \frac{1}{a^2} \xi^{-3} \right| d\xi \\ &= \left| \frac{\varepsilon \mu^2}{8} \right| = O(a), \end{aligned} \quad (2.99)$$

where $1 < \gamma < 2$. We conclude that all the contributions from the far field, i.e. $M_{1,5}$ and $M_{2,5}$, are of $O(a)$.

In the bump region, we have contributions of $O\left(a^{2-\frac{2}{\sigma}}\right)$ and in the far field we have contributions of $O(a)$. If $1 \leq \sigma < 2$, the contributions from the far field are higher order contributions compared to the contributions from the bump region, i.e.

$$|M_{j,5}| \ll |M_{j,3}| \text{ for } j = 1, 2.$$

Thus, (2.90) is to leading order given by

$$|\mu|^2 = \frac{2}{a} (M_{1,3} + M_{2,3}). \quad (2.100)$$

For $\sigma = 1$, substitution of (2.94) and (2.96) into (2.100) leads to

$$|\mu|^2|_{\sigma=1} = -\frac{4\kappa\sqrt{1-\frac{\kappa^4}{4}}}{3a} \left(3(2-d) + K(2+\kappa^2) + 8Kb \left(1 - \frac{\kappa^2}{4} \right) \right). \quad (2.101)$$

This result agrees with expression (6.28) in [9]. Combining this with the expression for μ as given in equation (2.87), we find for $\sigma = 1$ that

$$4\alpha e^{\frac{1}{a}(\kappa\sqrt{\alpha-2}\arccos\frac{\kappa}{2})} = \frac{\kappa^{2-d}}{3} (3(d-2)a - \varepsilon(2+\kappa^2) - 8\varepsilon b\alpha). \quad (2.102)$$

2.4.3 Conclusion

As discussed, the analysis in Section 2.4.2, which is here presented in more detail than in [9], leads to three results. First, equation (2.102) defines a relation between a , ε , d and b . This relation reduces to (2.26b) if we set $\varepsilon = Ka$, where here $K = O(1)$, and take the limit $a \rightarrow 0$. This link with the higher order asymptotic analysis is not discussed in [9]. Note that, contrary to (2.26b), we only used a leading order analysis to obtain (2.102). Furthermore, it is, in principle, possible to derive a second integral equation of this type by substituting one of the equations of the system (2.88) in the integral equation (2.102) for μ . However, if ψ' is present in the integral equation it is not clear how to evaluate the integral equation using only the leading order analysis. A second equation is needed to fix κ . Fortunately, for making the link with the analysis for the NLS, this second equation is not needed.

Second, to make a link with the NLS, we, as in [9], consider the case $b = 0$ and d close to 2. It follows numerically that both ε and a are small along the whole branch in the (ε, a) -plane for which a solution of (2.3) can be constructed under the conditions stated in this section, see Figure 2.4.1a. Therefore, for d close to 2 the analysis at $O(a)$ in Section 2.2.2 is valid. And also, equations (2.26a) and (2.26b) with κ and $K(a)$ instead of κ_0 and K_0 hold. It can be verified that $b = 0$, $\kappa = 1$ and $K(a) = d - 2$ solves this system of equations. For $b = 0$ and $\kappa = 1$ equation (2.87) reduces at leading order to

$$3e^{\frac{1}{a}(\frac{1}{2}\sqrt{3}-\sqrt{2})} = (d-2)a - \varepsilon. \quad (2.103)$$

This result agrees with expression (6.31) in [9] and expression (4.32) in [7] for the NLS.

And finally, for $b = 0$ and $0 < a, \varepsilon \ll 1$, we derived $\kappa = 1$. For various values of d , the curve (2.103) is plotted in the (a, ε) -plane and use $\kappa = 1$, see Figure 2.4.1. If a or ε is not small, it is not guaranteed that the curve is correct. We draw two conclusions upon comparing with numerically computed curves, see [9] for the numerics. Figure 2.4.1 shows that the asymptotically constructed curves are accurate on the lower part of the $k = 2$ -branch for all values of d . Furthermore, the complete $k = 2$ -branch is approximated accurately for d close to 2.

2.5 Rigorous existence results

In this section, we present two rigorous existence results and relate them with the asymptotic construction presented in Section 2.2.

First, in Section 2.5.1 we describe the relation between the asymptotic construction and the rigorous existence result stated in Theorem 2.1.1. From Theorem 2.1.1, we conclude that equation (2.3) has four ring-like solutions. In this section, we explain which of these four solutions we have constructed in Section 2.2.

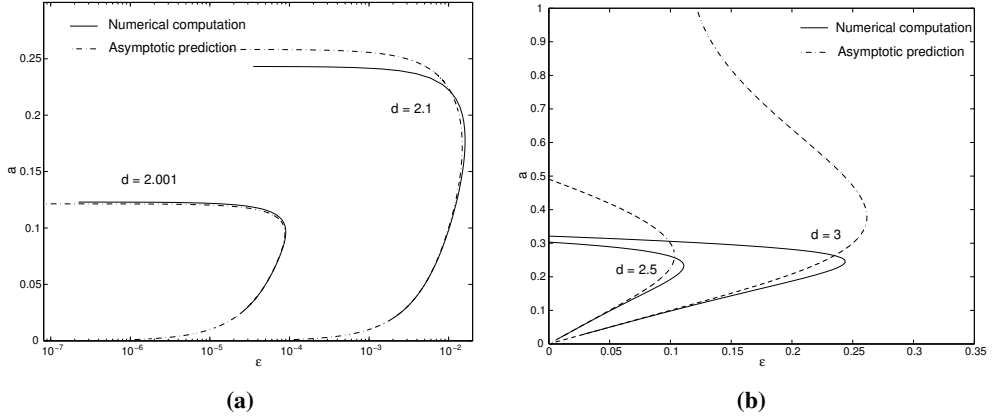


Figure 2.4.1: Comparison of the numerically and asymptotically computed relation between a and ε for $b = 0$. These figures are a reproduction of Figure 6 in [9].

Second, in Section 2.5.2 we formulate a rigorous existence result concerning the asymptotic construction itself.

2.5.1 Connection between the asymptotic construction and the rigorous existence result

The proof of the existence result as stated in Theorem 2.1.1 is schematically summarized in Figure 2.5.1 which is a reproduction of Figure 7 in [40]. The full details of the proof can be found in [40], we restrict ourselves to this geometric summary. The figure shows the phase plane of the non-autonomous 3-dimensional dynamical system (2.3) projected on the $(|Q|, \frac{|Q|_\varepsilon}{|Q|})$ -plane. For convenience, we define $\eta = a\xi$. Furthermore, $\eta_{\min} > 0$ is such that $\eta_{\min} = O(\log \frac{1}{a})$ and $\eta_{\max} = 2 - \sqrt{a}$. Since the system is non-autonomous, the flow changes η is varied from $\eta_{\min}$ to $\eta_{\max}$.

We now explain all the essential elements of Figure 2.5.1. An interval of initial values of length $O(a)$ is tracked forward to the interval $I_{\eta_{\min}}$, see Figure 2.5.1a. W^s and W^u represent the stable and unstable manifold of the perturbed system. The points p_0 and p_1 remain on the stable manifold as they are tracked forward since the stable manifold is tangent to the flow. It is not possible for an orbit to intersect W^s or W^u since they are tangent to the flow. Notice that W^s curls up in a tongue-like way as η is varied from $\eta_{\max}$ to $\eta_{\min}$. Similarly, W^u curls up in a tongue-like way as η is varied from $\eta_{\min}$ to $\eta_{\max}$. The extend to which the tongues curl up depends on a , the smaller the value of a the more the tongues curl up inside themselves. Furthermore, the width of the tongues is of $O(a)$ and the most left piece of the tongues, i.e. the piece in between the points m_2 and m_3 in Figure 2.5.1f and the part on which p_0 lies in Figure 2.5.1a, is exponentially close to the $\frac{|Q|_\varepsilon}{|Q|}$ -axis. Now, as the interval $I_{\eta_{\min}}$ is tracked forward from $\eta_{\min}$ to $\eta_{\max}$, it is also forced into this tongue-like shape, since it cannot intersect the stable and unstable manifold if it does not intersect these manifolds initially. Note that: the end left point of the interval $I_{\eta_{\min}}$ remains close to the $\frac{|Q|_\varepsilon}{|Q|}$ -axis and the points p_0 and p_1 remain on the stable manifold. Therefore, the interval is stretched out as it is forced into this tongue-like shape. The manifold M_∞ in Figure 2.5.1f represents an interval of appropriate

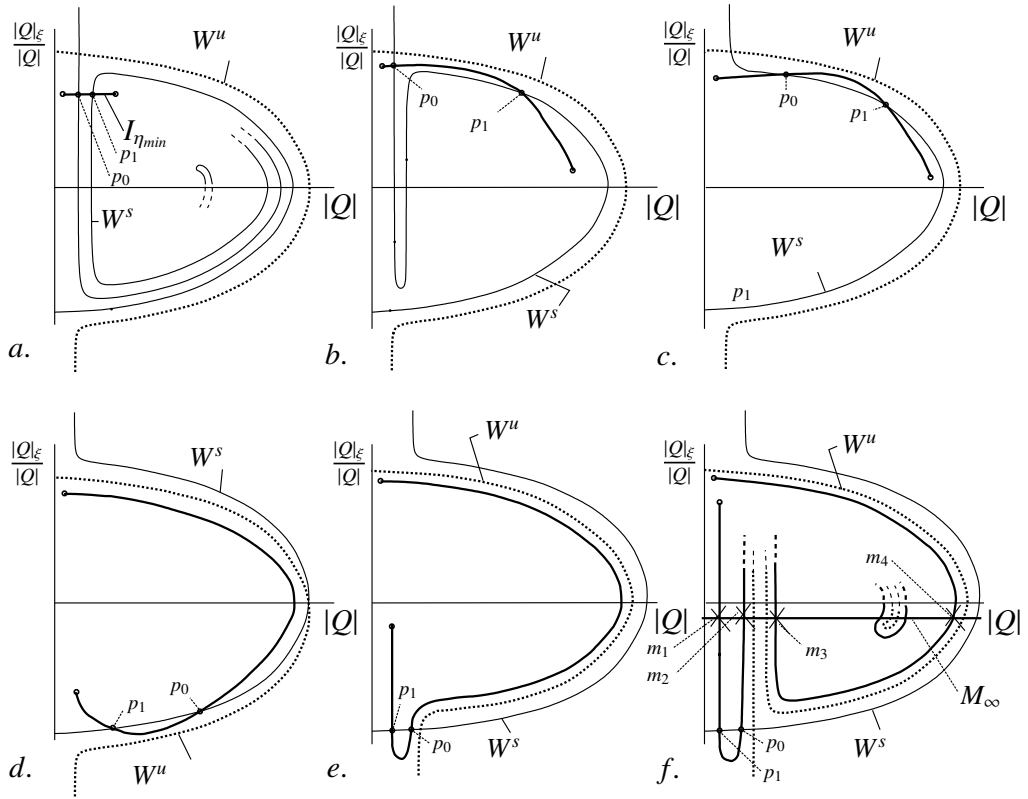


Figure 2.5.1: Phase plane of constant η -slices of the non-autonomous 3-dimensional dynamical system (2.3) projected on the $(|Q|, \frac{|Q|_\xi}{|Q|})$ -plane. From (a) to (f), η is increased from $\eta_{\min}$ to $\eta_{\max}$. We refer to the discussion on page 46 for a full explanation of all the elements in the figure. This figure is a reproduction of Figure 7 in [40].

values tracked back from $\xi = \infty$ to $\eta = \eta_{\max}$. The intersection points m_1, m_2, m_3 and m_4 correspond to the four solutions mentioned in Theorem 2.1.1.

We now explain why the solution labeled with m_2 in Figure 2.5.1 corresponds to the m_2 -solution as defined by Definition 2.2.2. The intersection point m_1 lies at the right hand side of p_1 in Figure 2.5.1a. And, since the width of the tongues are of $O(a)$, its value at $\eta = \eta_{\min}$, i.e. at $\xi = O(\log \frac{1}{a})$, is not exponentially small in a . However, from equation (2.65) and Assumption 2.2.1.a, it follows that the solution we have constructed is exponentially small in a near $\eta = \eta_{\min}$. An asymptotic construction for the solution corresponding to m_1 is presented in [39]. Similarly, since the width of the tongue of W^u between m_2 and m_3 in Figure 2.5.1f is also of $O(a)$, it follows that the value of the solution corresponding to the intersection m_3 is of $O(a)$ close to $\eta_{\max}$. From equation (2.86) and the analysis in Section 2.3.10, it follows that the solution we have constructed is exponentially small in a near $\eta_{\max}$ and thus we discard m_3 . As can be seen in Figure 2.5.1f, the solution corresponding to m_4 attains its maximum near $\eta_{\max}$, this is also not the solution we constructed. We conclude that the m_2 -solution we constructed asymptotically corresponds to the intersection point m_2 .

Finally, we stress again that the Conditions 2.1.1.a and 2.1.1.b under which Theorem 2.1.1 is valid are related to conditions in the asymptotic construction. To be more precise,

these two conditions are exactly the same as the system of equations give by (2.26). Lemma 2.2.6 gives conditions under which (2.26) can be solved uniquely. For example, for $\sigma = 1$ we have shown existence if $(d, b) \in R$, where R is defined by (2.42). This region R also contains, contrary to the conditions under which Theorem 2.1.1 is proved, points for which the b -coordinate is negative. We are convinced that the highly technical proof of Theorem 2.1.1 presented in [40] can at least be extended to the part of the region R in which Lemma 2.2.6 implies that (2.26) can be solved uniquely.

2.5.2 Rigorous existence result for the asymptotic construction

We claim that the asymptotic construction itself can be made rigorous in the sense that the error between the real solution of (2.3) and our (leading order) approximation vanishes as the small parameter a tends to zero. The way to prove this is to consider the difference between our approximation of the m_2 -solution and the exact m_2 -solution of equation (2.3). In order to make the asymptotic analysis rigorous, we must prove that the differential equation for this difference has a bounded solution. This is a technical procedure, that can be done with standard techniques, and will not lead to new mathematical insights. Therefore, we refrain from getting further involved in a purely technical exercise.

Theorem 2.5.1. *Let σ and d be such that Assumption 1.2.2 holds, and let b be such that the system of equations (2.25) has a solution, i.e. the conditions in Lemma 2.2.6 are satisfied. Let Q_0 be the leading order asymptotically constructed solution as constructed in Section 2.2. Now, there exists an m_2 -solution Q_{m_2} of equation (1.15) such that*

$$|Q_{m_2} - Q_0| = O(a) \text{ as } a \rightarrow 0.$$

And also, if $\sigma = 1$ and $Q_0 + aQ_1 + a^2Q_2$ is the higher order asymptotic expansion of the solution in the bump region defined by $|\xi - \frac{\kappa}{a}| \ll a^{-1}$, then in the bump region, we have

$$|Q_{m_2} - (Q_0 + aQ_1 + a^2Q_2)| = O(a^3) \text{ as } a \rightarrow 0.$$

Furthermore, if $b > 0$, then m_2 -solution corresponds to the intersection m_2 as denoted in Figure 2.5.1f.

Chapter 3

Stability with respect to radially symmetric perturbations

In this chapter, we study the spectral stability with respect to radially symmetric perturbations of the stationary m_2 -solution $Q_{m_2}(\xi)$ of equation (2.1) as defined by Definition 2.2.2. The m_2 -solution satisfies the boundary and initial conditions given by (2.2). Furthermore, recall that this solution is constructed asymptotically in Section 2.2, see also [9, 40].

In Section 1.3.1, we described a numerical stability result presented in [9]. For the m_2 -solution, this numerical result suggests that it is stable with respect to radially symmetric perturbations if $d = 3$, $\sigma = 1$, $b = 0$ and $0 < a \ll 1$. Furthermore, the numerical result is in line with our final conclusion of this chapter as formulated in Theorem 3.4.1. In Section 1.4, we described the methods that we will further explain and use in this chapter.

In Section 3.1, the set-up is presented in which we study spectral stability with respect to radially symmetric perturbations. Eventually, this results in an eigenvalue problem which we will study in the remaining sections of this chapter.

Note that there are two obvious approaches to study stability which are not equivalent: studying stability of the m_2 -solution Q_{m_2} in rescaled coordinates, i.e. in the context of equation (2.1), or studying stability in the original coordinates, i.e. studying stability of Φ_{m_2} in context of equation (1.9), where Φ_{m_2} is the m_2 -solution Q_{m_2} scaled back to the original coordinates with rescaling (1.12). The two main differences of these two approaches are the following. The first difference is related to the temporal transformation as given by (1.17b) and implies that exponential growth or decay in τ corresponds to algebraic growth or decay in t . The second difference is related to symmetries of equation (2.1). As will also be illustrated in Sections 3.2.3 and 3.2.4, symmetries lead to families of solutions which in turn lead, by linearizing in a direction tangent to the family, to eigenvalues. The resulting eigenvalues are considered irrelevant for the study of stability if one intends to study the stability of only one particular solution within the complete family. Since transformation (1.12) relates the temporal and spacial coordinates in a nonlinear way, the actual values of the symmetry-related eigenvalues differ in the approaches to studying stability. In Section 3.2.4 (and also Section 4.2.1), we will analyze the symmetry related eigenvalues in more detail. For technical reasons, we restrict ourselves to the study of stability in the context of equation (2.1) in this thesis.

For readers familiar with the study of stability of solutions of partial differential equations and Evans functions techniques (see Section 3.2.2), we sketch the contents of the remaining

sections. First, in Section 3.2 we study solutions of the eigenvalue problem that are exponentially localized in the bump region. For this, we (introduce and) employ Evans function techniques by using and extending the theory developed in [23–25]. Second, in Section 3.3, we study bounded solutions of the eigenvalue problem. From this, we derive conclusions concerning the essential spectrum. Finally, in Section 3.4, we summarize our findings.

The theory in this chapter can be applied for all σ for which the m_2 -solution of equation (2.1) exists, where sufficient existence conditions are provided by Theorem 2.1.1. However, for technical reasons,³ the final result is only derived for $\sigma = 1$. In the text, it is indicated where we have restricted ourselves to the case $\sigma = 1$.

3.1 Spectral stability

In order to study the spectral stability, we analyze a perturbation of the asymptotically constructed solution by substituting

$$Q(\xi, \tau) = Q_{m_2}(\xi) + v(\xi, \tau) \quad (3.1)$$

into equation (2.1), where Q_{m_2} is the m_2 -solution. Here $v(\xi, \tau)$ is a perturbation and $v(\xi, 0)$ is assumed to be small. The linearized equation for v is given by

$$\begin{aligned} v_\tau = & -iv + (i + Ka) \left(v_{\xi\xi} + \frac{d-1}{\xi} v_\xi \right) - a \left(\xi v_\xi + \frac{1}{\sigma} v \right) \\ & + (i - bKa) \left((\sigma + 1) |Q_{m_2}|^{2\sigma} v + \sigma |Q_{m_2}|^{2\sigma-2} Q_{m_2}^2 \bar{v} \right). \end{aligned} \quad (3.2)$$

In order for Q_{m_2} to be linearly stable, we want v to vanish as τ tends to infinity, see Definition 1.4.1. By splitting v in a real part v_1 and an imaginary part v_2 , we can write equation (3.2) as a system:

$$\frac{d}{d\tau} V = L_V(a) V, \quad (3.3)$$

where $V = (v_1, v_2)^\top$ and $L_V(a)$ is a 2×2 matrix containing differential operators up to second order.

In this chapter, we will study spectral stability of Q_{m_2} , see Definition 1.4.2. The spectral stability of Q_{m_2} is determined by the spectrum of the linear operator $L_V(a)$, see Definition 1.4.2. In general, the spectrum can be decomposed into two parts: the discrete spectrum, which are the isolated eigenvalues of finite multiplicity, and the essential spectrum, which is the rest of the spectrum, see [19, 41]. Note that several different definitions of essential spectrum are in use, we adopted the one from [19].

By substitution of $e^{\lambda\tau} V(\xi)$ into equation (3.3), we find the eigenvalue problem

$$L_V(a) V = \lambda V. \quad (3.4)$$

One might be tempted to believe that substitution of $e^{\lambda\tau} v(\xi)$ into (3.2) leads to the same result. But, this ansatz is only correct if equation (3.2) is linear in $v = v_1 + iv_2$. And, equation (3.2) is not linear in $v = v_1 + iv_2$, because it is not scale invariant upon scalar multiplication with complex numbers.

The asymptotic expansion of Q_{m_2} as presented in Section 2.2, was constructed by studying equation (1.15) in several regions among which: the inner region where $a\xi \ll 1$, the bump region where $a\xi = O(1)$ and the far field where $a\xi \gg 1$, see also Figure 1.3.2 and Definition 1.3.6. In a similar fashion, we also analyze eigenvalue problem (3.4) in these regions. This allows us to use the asymptotically constructed solution in the stability analysis.

In the bump region, the existence and eigenvalue problem can be reduced to the existence and stability problem of a perturbed NLS equation, see Section 1.4.1 and equation (1.19). Therefore, it is natural to first investigate the part of the spectrum that can be seen as a perturbation of the spectrum that arises in the study of pulse shaped solutions in the NLS, see for example [23, 25]. Since spectrum of the NLS can be found by studying exponentially decaying solutions and bounded solutions of the eigenvalue problem, we restrict the study of spectral stability to: solutions that decay exponentially in the bump region and also decay outside of the bump region, and solutions that do not decay or grow in the bump region but remain bounded in the bump region and also remain bounded outside the bump region. To be precise, these two types of solutions that are characterized by respectively the following two properties.

Property 3.1.1. *A solution $V(\xi)$ of eigenvalue problem (3.4) is called localized and exponentially localized in the bump region if the following conditions are satisfied:*

- 3.1.1.a *$V(\xi)$ decays exponentially as $s = \xi - \frac{\kappa}{a}$ tends to $-\infty$ in the bump region, i.e. to leading order in the scale $\frac{\kappa}{a} - \frac{1}{a^\gamma}$ with $t > 0$ and $0 < \gamma < 1$, see Definition 1.3.6.b;*
- 3.1.1.b *$V(\xi)$ decays exponentially as $s = \xi - \frac{\kappa}{a}$ tends to ∞ in the bump region, i.e. to leading order in the scale in the scale $\frac{\kappa}{a} + \frac{1}{a^\gamma}$ with $t > 0$ and $0 < \gamma < 1$, see Definition 1.3.6.b;*
- 3.1.1.c *$V(\xi)$ decays as we follow the solution from the bump region into Region 2 and into the inner region, see Definition 1.3.6;*
- 3.1.1.d *$V(\xi)$ decays as ξ tends to ∞ .*

Property 3.1.2. *A solution $V(\xi)$ of eigenvalue problem (3.4) is called bounded and bounded in the bump region if it does not satisfy Property 3.1.1 and the following conditions are satisfied:*

- 3.1.2.a *$\|V(\xi)\|$ remains bounded as $s = \xi - \frac{\kappa}{a}$ tends to $-\infty$ in the bump region, i.e. to leading order in the scale $\frac{\kappa}{a} - \frac{1}{a^\gamma}$ with $t > 0$ and $0 < \gamma < 1$, see Definition 1.3.6.b;*
- 3.1.2.b *$\|V(\xi)\|$ remains bounded as $s = \xi - \frac{\kappa}{a}$ tends to ∞ in the bump region, i.e. to leading order in the scale in the scale $\frac{\kappa}{a} + \frac{1}{a^\gamma}$ with $t > 0$ and $0 < \gamma < 1$, see Definition 1.3.6.b;*
- 3.1.2.c *$\|V(\xi)\|$ remains bounded as we follow the solution from the bump region into Region 2 and into the inner region, see Definition 1.3.6;*
- 3.1.2.d *$\|V(\xi)\|$ remains bounded as ξ tends to ∞ .*

Here $\|\cdot\|$ denotes the standard norm in $\mathbb{C}^2$.

So, by considering solutions that satisfy Property 3.1.1 or 3.1.2, the unperturbed NLS spectrum can be taken as a starting point in Sections 3.2.3 and 3.3.1.

Since the definition of the spectrum of the linear operator $L_V(a)$, as introduced in (3.3), consists, by definition, of all complex-valued λ for which the unbounded operator $L_V(a) - \lambda$ has no bounded inverse (see [19]), it is unclear whether only the two types of solutions of (3.4) that are categorized by Properties 3.1.1 and 3.1.2 yields the complete spectrum. However, we will consider only the part of the spectrum that corresponds these two types solutions for the following two reasons. First, the study of these two types of solutions is already very technical. And second, it is sufficient to draw conclusions concerning non-radially symmetric stability by limiting ourselves to the study of these two types of solutions, see also Chapter 4. First, we consider the localized solutions of eigenvalue problem (3.4) that decay exponentially in the bump region.

3.2 Exponentially localized solutions

In this section, we study localized solutions of eigenvalue problem (3.4) that decay exponentially in the bump region, i.e. they satisfy Property 3.1.1. Since the m_2 -solution is at leading order constant at distances of $O\left(\frac{1}{a^\gamma}\right)$, with $0 < \gamma < 1$, away from $\frac{\kappa}{a}$, at which the m_2 -solution attains its maximum, eigenvalue equation (3.4) has at leading order asymptotically constant coefficients in the bump region, i.e. for $\xi = \frac{\kappa_0}{a} + a^{-\gamma}t$ with $0 < \gamma < 1$. For an eigenvalue problem on the real line, this means that eigenfunctions are exponentially localized, see [41] Theorem 2. In Lemma 3.2.10, we show that provided $|\lambda| \ll 1$, exponential localization in the bump region is a necessary condition for eigenfunctions to decay in the far field.

In the bump region, the spectrum of $L_V(0)$ is known, since for $a = 0$ the eigenvalue problem reduces to the case of the NLS, see equation (1.19). Hence, $\lambda = 0$ is an eigenvalue with algebraic multiplicity 4 of $L_V(0)$ in the bump region and there are no other eigenvalues, see [23, 25, 45]. As a becomes non-zero, an a priori unknown number of these 4 eigenvalues will move away from $\lambda = 0$. If one of these eigenvalues has positive real part for $a \neq 0$, this implies spectral (and linear, see Remark 1.4.3) instability.

In the forthcoming section, we start with considering eigenvalue problem (3.4) in bump region coordinates by using the translation $s = \xi - \frac{\kappa}{a}$. Since we do not pose the restriction $s = o\left(\frac{1}{a}\right)$, this does not imply that the results are only valid in the bump region. The analysis in Sections 3.2.1 to 3.2.4 is completely general and valid as formulated for the entire domain. Section 3.2.5 considers only the bump region and attempts to determine higher order corrections of solutions that only satisfy Properties 3.1.2.a and 3.1.2.b. In Section 3.2.8, we perform a matching procedure to determine whether the solutions that satisfy Properties 3.1.2.a and 3.1.2.b can also satisfy Properties 3.1.1.c and 3.1.1.d.

3.2.1 $L_V(a)$ in the bump region

In a similar way as in the asymptotic analysis in the bump region presented in Section 2.2.2, we zoom in on the bump region by setting $\xi = \frac{\kappa}{a} + s$ and use the expansion for κ as posed in equation (2.4b). We also apply the Liouville transformations $v(\xi) = e^{-ik_0 s/2} w(s)$ and $V(\xi) = e^{-ik_0 s/2} W(s)$ to eliminate the first order derivatives at leading order. From (3.2) and (3.3), we obtain that (3.4) reduces to

$$L(a)W = \lambda W, \quad (3.5)$$

where $L(a)$ is a 2×2 matrix $[L^{ij}]$ containing differential operators. This matrix can be written in the scalar form:

$$\begin{aligned} L(a)(w_1, w_2) := & -iw + (i + \varepsilon) \left(w_{ss} - ik_0 w_s - \frac{\kappa_0^2}{4} w + a \frac{d-1}{\kappa + as} \left(w_s - \frac{ik_0}{2} w \right) \right) \\ & - (\kappa + as) \left(w_s - i \frac{\kappa_0}{2} w \right) - \frac{aw}{\sigma} + (i - b\varepsilon) \left((\sigma + 1)w|S|^{2\sigma} + \sigma S^{\sigma+1} \bar{S}^{\sigma-1} (w_1 - iw_2) \right), \end{aligned} \quad (3.6)$$

where $w = w_1 + iw_2$ and $S(s) = e^{ik_0 s/2} Q(s)$, see Section 2.2.2. The matrix elements L^{ij} can be found by writing this expression in the form: $L(a)w = L^{11}w_1 + L^{12}w_2 + iL^{21}w_1 + iL^{22}w_2$. The explicit expressions for L^{ij} are extensive and only obscure the presentation. Note that although w_1 and w_2 are introduced as the real and imaginary part of w , they are complex functions in the eigenvalue problem. Thus, in the eigenvalue problem formulation they can no longer be thought of as the real and imaginary part of w . This is analogous to complex eigenvectors of a real matrix.

For $a = 0$, the matrix $L(a)$ reduces to

$$L(0) = \begin{bmatrix} 0 & -(\partial_s^2 - \alpha + S_0^{2\sigma}) \\ \partial_s^2 - \alpha + (2\sigma + 1)S_0^{2\sigma} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -L_2 \\ L_1 & 0 \end{bmatrix}, \quad (3.7)$$

where S_0 is given by expression (2.13), α is defined by (2.12) and L_1 and L_2 are the operators given by (2.14) that appear at first order in the construction of the asymptotic expansion of the m_2 -solution. As expected, due to the transformation $v(\xi) = e^{-ik_0 s/2} w(s)$, $L(0)$ is self-adjoint, i.e. it contains no first order derivatives. Note that $L(0)$ is up to a constant the same operator as the one given in equation (5.3) in [23]. This allows us to apply the Evans function techniques described in [23] to eigenvalue problem (3.5).

3.2.2 The Evans function

Now, we consider eigenvalue problem (3.5) with a not necessarily 0 but small. For this, we use Evans function techniques. The Evans function $E(\lambda, a)$ is an analytic function whose zeros correspond to the eigenvalues of $L(a)$, see [1, 15, 23, 35, 41]. Since in the currently existing literature, there is no Evans function construction for eigenvalue problems of the form (3.5), we have to extend the Evans function constructions in the currently existing literature. In order to analyze the behavior of the zeros of the Evans function, i.e. the eigenvalues of $L(a)$, for $0 < a \ll 1$ with Taylor expansion (1.21) of the Evans function, we need to determine derivatives of $E(\lambda, a)$ with respect to λ and a . A set-up for determining derivatives of the Evans function is presented in [23], but it does not readily apply to our setting and, therefore, it needs to be extended. Thus, the currently existing literature has to be extended in two ways in order to use Evans function techniques: the construction has to be extended and derivation for the expressions for the derivatives of the Evans function as presented in [23] has to be extended.

We present an extension of the construction of the Evans function as presented in [23] in such a way that it applies directly to our setting. There exist four linearly independent solutions $Y_1(s, a, \lambda)$, $Y_2(s, a, \lambda)$, $Y_3(s, a, \lambda)$ and $Y_4(s, a, \lambda)$ of the eigenvalue equation (3.5), since this problem forms a 2 dimensional second order system. Now, consider all $\lambda \in \mathbb{C}$ for which these four solutions satisfy the following property.

Property 3.2.1.

- 3.2.1.a $Y_1(s, a, \lambda)$ and $Y_2(s, a, \lambda)$ decay exponentially as s tends to $-\infty$ in the bump region, i.e. to leading order in the scale $\frac{\kappa}{a} - \frac{t}{a^{\gamma}}$ with $t > 0$ and $0 < \gamma < 1$, see Definition 1.3.6.b;
- 3.2.1.b $Y_3(s, a, \lambda)$ and $Y_4(s, a, \lambda)$ decay exponentially as s tends to ∞ in the bump region, i.e. to leading order in the scale in the scale $\frac{\kappa}{a} + \frac{t}{a^{\gamma}}$ with $t > 0$ and $0 < \gamma < 1$, see Definition 1.3.6.b;
- 3.2.1.c $Y_1(s, a, \lambda)$ and $Y_2(s, a, \lambda)$ decay as we follow the solution from the bump region into Region 2 and into the inner region, see Definition 1.3.6;
- 3.2.1.d $Y_3(s, a, \lambda)$ and $Y_4(s, a, \lambda)$ decay as s tends to ∞ .

The type of linearly independent solutions with which the Evans function is constructed here differs from the ones used in the currently existing literature. The currently existing literature mainly deals with linear operators with asymptotically constant coefficients, which is not the case for the linear operator $L(a)$ as defined by (3.6). More details on the existence of solutions that satisfy Property 3.2.1 are given in Section 3.2.8,

The idea is that if there is a linear dependence between the sets $\{Y_1, Y_2\}$ and $\{Y_3, Y_4\}$, then there is a function that decays exponentially for both $s \rightarrow \pm\infty$. Hence, there exists an eigenfunction for that particular λ (and a). This eigenfunction will then satisfy Property 3.1.1.

We now write the 2-dimensional second order system $L(a)Y = \lambda Y$ as a 4-dimensional first order system

$$\mathcal{Y}' = M(s, a, \lambda)\mathcal{Y}, \quad (3.8)$$

where $\mathcal{Y} = \Pi_{24} Y$, Π_{24} is the mapping that maps a two dimensional vector of differentiable functions to a four dimensional vector in the following way

$$\Pi_{24} : (y_1, y_2)^\top \mapsto (y_1, y_1', y_2, y_2')^\top \quad (3.9)$$

and the matrix $M(s, a, \lambda)$ is a real 4×4 matrix such that if $Y = (y_1, y_2)^\top \in \mathbb{C}^2$ solves (3.5), then $\mathcal{Y} = \Pi_{24} Y$ solves the 4-dimensional first order system (3.8). The Evans function is defined for all λ for which Property 3.2.1 can be satisfied in such a way that it vanishes when the functions

$$\mathcal{Y}_1 := \Pi_{24} Y_1, \quad \mathcal{Y}_2 := \Pi_{24} Y_2, \quad \mathcal{Y}_3 := \Pi_{24} Y_3, \quad \mathcal{Y}_4 := \Pi_{24} Y_4 \quad (3.10)$$

are linearly dependent:

$$\begin{aligned} E(\lambda, a) &:= m(s, a, \lambda) \begin{vmatrix} Y_{11} & Y_{31} & Y_{21} & Y_{41} \\ Y'_{11} & Y'_{31} & Y'_{21} & Y'_{41} \\ Y_{12} & Y_{32} & Y_{22} & Y_{42} \\ Y'_{12} & Y'_{32} & Y'_{22} & Y'_{42} \end{vmatrix} \\ &= m(s, a, \lambda) \cdot \Pi_{24} Y_1 \wedge \Pi_{24} Y_3 \wedge \Pi_{24} Y_2 \wedge \Pi_{24} Y_4 \\ &= m(s, a, \lambda) \cdot \mathcal{Y}_1 \wedge \mathcal{Y}_3 \wedge \mathcal{Y}_2 \wedge \mathcal{Y}_4, \end{aligned} \quad (3.11)$$

where $m(s, a, \lambda) := \exp\left(-\int_0^s \text{Trace} M(\tilde{s}, a, \lambda) d\tilde{s}\right)$, Y_{ij} denotes the j -th component of the two dimensional vector Y_i (here $i = 1, 2, 3, 4$ and $j = 1, 2$) and $\cdot$ denotes scalar multiplication. Note that the $\mathcal{Y}_i$'s are not ordered in ascending order. Alternative orderings of the $\mathcal{Y}_i$'s, at most, change the sign of the Evans function and this has no consequences for its zeros.

By applying Liouville's formula (also known as the Abel-Jacobi-Liouville identity, see for example [17] Theorem 1.2), it can be shown that the multiplication by the scalar-valued function $m(s, a, \lambda)$ makes the Evans function independent of s . Due to the construction, the set of zeros of the Evans function coincides with the discrete spectrum. Our goal in this section is to show that a Taylor expansion of the Evans function is given by (1.21). And also, to evaluate all derivatives of the Evans function with respect to a and λ up to fourth order.

From [23], it follows that we can explicitly calculate derivatives of the Evans Function once we can determine the following explicitly. First, we need an explicit expression for the (generalized) eigenfunctions of $L(0)$ and eigenfunctions of the adjoint operator $L(0)^*$ corresponding to $\lambda = 0$. Second, we also need their higher order corrections in terms of the small parameter a if the (generalized) eigenfunction persists. And finally, we need to know the order at which the solvability condition fails if the eigenfunction does not persist. In Section 3.2.3, we determine the (generalized) eigenfunctions of $L(0)$ corresponding to $\lambda = 0$. And, in Section 3.2.5, we determine whether these (generalized) eigenfunctions persist for $a \neq 0$. Section 3.2.6 presents a nontrivial extension of the results in [23]. Finally, in Section 3.2.7, we determine with a Taylor expansion of the Evans function whether eigenvalues perturb into the right half of the complex plane.

3.2.3 Eigenvalues and eigenfunctions of $L(0)$

We denote by

$$\psi^{ji}(s, a) = \begin{bmatrix} \psi_1^{ji}(s, a) \\ \psi_2^{ji}(s, a) \end{bmatrix}$$

the (generalized) eigenfunctions of $L(a)$ corresponding to the eigenvalue $\lambda = 0$:

$$L(a)\psi^{ji} = \psi^{j-1,i}, \quad \psi^{0,i} = 0, \quad (3.12)$$

where i is an integer ranging from 1 up to and including the geometric multiplicity of the eigenvalue $\lambda = 0$ and j is an integer ranging of 0 up to the ascent of the eigenvalue (counted with multiplicity). Moreover, the eigenfunctions of the corresponding adjoint operator, which are contained in $\text{Ker}(L(a)^*)$, are denoted by $v_i(s, a)$. Now, as in [23] Proposition 5.1 and [45], the (generalized) eigenfunctions of $L(0)$ and eigenfunctions of the adjoint operator $L(0)^*$ corresponding to $\lambda = 0$ are given by

$$\begin{aligned} \psi^{1,1}(s, 0) &= \begin{bmatrix} S'_0 \\ 0 \end{bmatrix}, & \psi^{2,1}(s, 0) &= \begin{bmatrix} 0 \\ -\frac{1}{2} s S_0 \end{bmatrix}, \\ \psi^{1,2}(s, 0) &= \begin{bmatrix} 0 \\ S_0 \end{bmatrix}, & \psi^{2,2}(s, 0) &= \begin{bmatrix} \partial_\alpha S_0 \\ 0 \end{bmatrix}, \\ v_1(s) &= 4 \left(\int_{-\infty}^{\infty} S_0^2 d\tilde{s} \right)^{-1} \begin{bmatrix} 0 \\ S'_0 \end{bmatrix}, & v_2(s) &= 2 \left(\partial_\alpha \int_{-\infty}^{\infty} S_0^2 d\tilde{s} \right)^{-1} \begin{bmatrix} S_0 \\ 0 \end{bmatrix}. \end{aligned} \quad (3.13)$$

Here, the v_i are normalized in such a way that $\int_{-\infty}^{\infty} \psi^{2,i} \cdot v_i d\tilde{s} = 1$, where $\cdot$ is the standard inner product on $\mathbb{R}^2$. Note that the $\psi^{i,j}$ and the v_i are real vector-valued functions, since they correspond to a real eigenvalue $\lambda = 0$ and the coefficients of the differential operators in $L(0)$ are also real-valued. As stated in Section 3.1, the unperturbed NLS spectrum will be taken as a starting point for our study and we recognize these eigenvalues as the eigenvalues that also arise in stability analysis of localized solutions in perturbed NLS equations, see [23, 25].

As we will show in the remainder of this section, eigenfunctions corresponding to the eigenvalue $\lambda = 0$ correspond to symmetries of the equation

$$i\partial_t \tilde{S} + \partial_{ss}^2 \tilde{S} - \alpha \tilde{S} + |\tilde{S}|^{2\sigma} \tilde{S} = 0, \quad (3.14)$$

where $\tilde{S}(s, t)$ is a complex-valued function of time $t \geq 0$ and space $s \in \mathbb{R}$. Note that equation (3.14) is the time dependent version of equation (2.11) and, thus, in particular, $S_0(s)$, as given by (2.13), satisfies equation (3.14). By the following arguments, these symmetries lead to the eigenfunctions and generalized eigenfunction given by (3.13).

First, the existence of the eigenfunction $\psi^{1,1}(s, 0)$ can be derived from the fact that equation (3.14) or (2.11) is translation invariant. In order to do so, substitute the translation $S_0(s + \delta)$, where $0 < \delta \ll 1$, of $S_0(s)$ in equation (2.11) and expand in powers of δ . It is then easily seen that

$$L^{21}(0)S'_0(s) = 0,$$

where by $L^{ij}(0)$ we denote the matrix elements of $L(0)$ as given by equation (3.7), see also (3.5). We conclude that $\psi^{1,1}(s, 0) = (S'_0(s), 0)^\top$. Note that we can also obtain this result by differentiation of equation (2.11).

Second, to derive the expression of the generalized eigenfunction $\psi^{2,1}(s, 0)$ corresponding to the translation invariance as given in (3.13), we consider solutions of the form $\tilde{S}_0(s + ct)$

of equation (3.14). Observe that $S_0(s)$ is part of this one parameter family of solutions, since, by definition, $S_0(s) = \tilde{S}_0(s + ct)|_{c=0}$. From (3.14), it follows that $\tilde{S}_0(s')$ satisfies

$$\partial_{s's'}^2 \tilde{S}_0 + ic \partial_s \tilde{S}_0 - \alpha \tilde{S}_0 + |\tilde{S}_0|^{2\sigma} \tilde{S}_0 = 0, \quad (3.15)$$

where $s' = s + ct$. Let $\tilde{S}_0(s', \alpha, c)$ be the solution of (3.15). Differentiation with respect to c and evaluating at $c = 0$ leads to

$$\partial_{s's'}^2 \partial_c \tilde{S}_0 - \alpha \partial_c \tilde{S}_0 + S_0^{2\sigma} \left((\sigma + 1) \partial_c \tilde{S}_0 + \sigma \overline{\partial_c \tilde{S}_0} \right) = -i \partial_s S_0, \quad (3.16)$$

where we implicitly assumed evaluation at $c = 0$ of the partial derivatives with respect to c and we used $\tilde{S}_0(s, \alpha, 0) = S_0(s, \alpha)$ (here in the solution $S_0(s, \alpha)$ of (3.14) the α -dependence is explicitly indicated). From (3.16), it follows that

$$L^{12}(0) \partial_c \tilde{S}_0(s', \alpha, c)|_{c=0} = S'_0. \quad (3.17)$$

Thus, the next step is to derive an explicit expression for $\partial_c \tilde{S}_0$. We solve differential equation (3.15) for $\tilde{S}_0$ by introducing the Liouville transformation

$$\tilde{S}_0(s', \alpha, c) = e^{-\frac{1}{2}ic s'} \hat{S}_0(s', \alpha, c). \quad (3.18)$$

It follows that $\hat{S}_0$ satisfies

$$\partial_{s's'}^2 \hat{S}_0 - \left(\alpha - \frac{c^2}{4} \right) \hat{S}_0 + |\hat{S}_0|^{2\sigma} \hat{S}_0 = 0.$$

We conclude that $\hat{S}(s', \alpha, c) = S_0(s', \alpha - \frac{c^2}{4})$. By using (3.18) and (2.13), it follows that $\partial_c \tilde{S}_0(s', \alpha, c)|_{c=0} = -\frac{1}{2}is' S_0(s', \alpha)$. And, from (3.17), we obtain $L^{12}(0) \left(-\frac{1}{2}s S_0 \right) = S'_0$. Therefore, $\psi^{2,1}(s, 0) = \left(0, -\frac{1}{2}s S_0 \right)^\top$.

Third, the eigenfunction $\psi^{1,2}(s, 0)$ is related to the phase invariance of equation (3.14) and (2.11). To derive an explicit expression for this eigenfunction, we perturb the phase of S_0 by an amount $0 < \delta \ll 1$ and substitute the perturbed solution $S_0(s)e^{i\delta}$ in equation (2.11). Upon expanding in powers of δ , it follows that $L^{12}(0)S_0(s, 0) = 0$, i.e. $L(0)(0, S_0)^\top = 0$. Conclusion, $\psi^{1,2}(s, 0) = (0, S_0)^\top$.

Finally, $S_0(s)$ is also part of a one parameter family of solutions of the form

$$e^{i\tilde{\alpha}t} \tilde{S}_0(s) \quad (3.19)$$

of (3.14). The reason for this is that substitution of (3.19) into (3.14) leads to a translated version of the original equation for $S_0(s)$ (see (2.11)):

$$\partial_{ss}^2 \tilde{S}_0 - (\alpha + \tilde{\alpha}) \tilde{S}_0 + |\tilde{S}_0|^{2\sigma} \tilde{S}_0 = 0.$$

Differentiation with respect to $\tilde{\alpha}$ and evaluating at $\tilde{\alpha} = 0$ leads to the fact that $\partial_{\tilde{\alpha}} S_0(s)$ satisfies $L^{21}(0) \partial_{\tilde{\alpha}} S_0 = S_0$, thus indeed $\psi^{2,2}(s, 0) = (\partial_{\tilde{\alpha}} S_0(s), 0)^\top$.

3.2.4 Symmetries leading to an irrelevant eigenvalue $\lambda = 2a$

Before we will analyze the persistence of the (generalized) eigenfunctions for $a \neq 0$, we discuss two symmetries that lead to the (same) positive eigenvalue $\lambda = 2a$. In Section 3.2.4.3, we will argue that, although this eigenvalue is positive, it does not lead to instability in the

setting in which we study stability of the m_2 -solution. In [36], an explicit symmetry of the Q -equation (1.15) is presented that leads to an eigenvalue $\lambda = 2a > 0$, see Section 3.2.4.1. And also, through perturbing the blowup time T of the constructed blowup solution $\Phi(x, t, T)$ of equation (1.9), we can construct a family of solutions of equation (1.9). In Sections 3.2.4.1 and 3.2.4.2, we will show that this symmetry leads to the same positive eigenvalue $\lambda = 2a$ and also to the same corresponding eigenfunction.

To analytically determine whether there are, apart from this positive eigenvalue, other sources of instability, a set-up based on decomposition of Banach spaces can be used. In [36], such a set-up is suggested and the rigorous analysis of this is posed as an open problem. We do not use a decomposition of Banach spaces, but will show how the Evans function can be used to determine other sources of instability.

3.2.4.1 A symmetry of the Q -equation

We proceed with a symmetry of the Q -equation as discussed in [36]. Define the operator

$$\mathcal{F}(Q, \xi, a, \omega, \varepsilon, b, d, \sigma) := (1 - i\varepsilon) \left(\frac{\partial^2 Q}{\partial \xi^2} + \frac{d-1}{\xi} \frac{\partial Q}{\partial \xi} \right) + ia \left(\xi \frac{\partial Q}{\partial \xi} + \frac{Q}{\sigma} \right) - \omega Q + (1 + ib\varepsilon)|Q|^{2\sigma} Q.$$

It can be seen that a solution $Q(\xi, a, \omega, \varepsilon, b, d, \sigma)$ of (1.15) satisfies

$$\mathcal{F}(Q, \xi, a, \omega, \varepsilon, b, d, \sigma) = 0.$$

Upon rescaling, it follows that if

$$\mathcal{F}(Q(\xi, a, \omega, \varepsilon, b, d, \sigma), \xi, a, \omega, \varepsilon, b, d, \sigma) = 0,$$

then also

$$\mathcal{F}(\eta^{\frac{1}{\sigma} + \frac{i\omega}{a}} Q(\eta\xi, a, \omega, \varepsilon, b, d, \sigma), \xi, \eta^2 a, \eta^2 \omega, \varepsilon, b, d, \sigma) = 0, \quad (3.20)$$

i.e. there is a family of solutions that is part of a family of equations. Linearizing (3.20) around $\eta = 1$, setting $\omega = 1$ and substituting the m_2 -solution Q_{m_2} for Q implies that

$$L_V(a)V = 2aV, \quad (3.21)$$

where

$$V = \begin{bmatrix} \operatorname{Re} \left(\left(\frac{1}{\sigma} + \frac{i}{a} \right) Q_{m_2} + \xi \partial_\xi Q_{m_2} \right) \\ \operatorname{Im} \left(\left(\frac{1}{\sigma} + \frac{i}{a} \right) Q_{m_2} + \xi \partial_\xi Q_{m_2} \right) \end{bmatrix}$$

and $L_V(a)$ is the linear operator in eigenvalue problem (3.4). Upon substituting the transformation $Q = e^{-\frac{1}{2}i\kappa_0 s} S(s)$ into (3.21), we deduce the following.

Lemma 3.2.2. *If the m_2 -solution exists, the operator $L(a)$ defined by (3.6) has an eigenvalue $\lambda = 2a$ with corresponding eigenfunction given by*

$$\begin{bmatrix} \operatorname{Re} \left(\left(\frac{a}{\sigma} + i \right) S(s) + (\kappa + as) \left(-\frac{1}{2} i \kappa_0 S + S' \right) \right) \\ \operatorname{Im} \left(\left(\frac{a}{\sigma} + i \right) S(s) + (\kappa + as) \left(-\frac{1}{2} i \kappa_0 S + S' \right) \right) \end{bmatrix}.$$

It is well known that a family of solutions in general leads to a zero eigenvalue, see for example Section 3.2.3. Note that this situation differs from one where we have a family of solutions to the Q -equation. In this case, however, we have a family of equations.

3.2.4.2 Perturbing the blowup time T

A second way to derive the result obtained in Lemma 3.2.2 is to perturb the blowup time T of the solution $\Phi(x, t, T)$ of equation (1.9). This technique is common in studying self-similar blowup solutions and has, for instance, also been employed in [3].

The rescaled equation for Q does not explicitly depend on the blowup time T , see equation (1.15). Therefore, it follows that we have constructed a blowup solution $\Phi(x, t, T)$ for every $T > 0$, i.e. $\Phi(x, t, T)$ satisfies (1.9) for all $T > 0$. We can thus perturb the blowup time T with δ and write rescaling (1.12), which is also applied for the m_2 -solution, for the (perturbed) blowup time $T + \delta$ as:

$$\Phi_{m_2}(x, t, T + \delta) = L_\delta(t)^{-\frac{1}{\sigma}} e^{i\tau_\delta} Q_{m_2}(\xi_\delta), \quad (3.22a)$$

$$L_\delta(t) := \sqrt{2a(T + \delta - t)} = L_0(t) + \frac{a\delta}{L_0(t)} + O(\delta^2), \quad (3.22b)$$

$$\xi_\delta := \frac{r}{L_\delta(t)} = \xi_0 - \frac{e^{2a\tau_0}\xi_0}{2T}\delta + O(\delta^2), \quad (3.22c)$$

$$\tau_\delta := -\frac{1}{2a} \log \frac{T + \delta - t}{T + \delta} = \tau_0 - \frac{e^{2a\tau_0} - 1}{2aT}\delta + O(\delta^2). \quad (3.22d)$$

In these expressions, we used, through combining (3.22b) and (3.22d), that $L_\delta(t)$ can be expressed as $L_\delta(t) = \sqrt{2aT} e^{-2a\tau_\delta}$. We expand $\Phi_{m_2}(x, t, T + \delta)$ in powers of δ and express the expansions in terms of $L_0(t)$, ξ_0 and τ_0 through using (3.22):

$$\begin{aligned} \Phi_{m_2}(x, t, T + \delta) &= L_\delta(t)^{-\frac{1}{\sigma}} e^{i\tau_\delta} Q_{m_2}(\xi_\delta) \\ &= L_0(t)^{-\frac{1}{\sigma}} e^{i\tau_0} Q_{m_2}(\xi_0) + L_0(t)^{-\frac{1}{\sigma}} e^{i\tau_0} \delta \left(\frac{-Q_{m_2}(\xi_0)}{\sigma L_0(t)} \frac{\partial L_\delta(t)}{\partial \delta} \Big|_{\delta=0} \right. \\ &\quad \left. + iQ_{m_2}(\xi_0) \frac{\partial \tau_\delta(t)}{\partial \delta} \Big|_{\delta=0} + Q'_{m_2}(\xi_0) \frac{\partial \xi_\delta(t)}{\partial \delta} \Big|_{\delta=0} \right) + O(\delta^2) \\ &= L_0(t)^{-\frac{1}{\sigma}} e^{i\tau_0} \tilde{Q}_{m_2}(\xi_0, \tau_0, \delta), \end{aligned} \quad (3.23)$$

where

$$\tilde{Q}_{m_2}(\xi_0, \tau_0, \delta) = \tilde{Q}_{m_2,0}(\xi_0) + \frac{\delta}{2aT} \tilde{Q}_{m_2,1}(\xi_0, \tau_0) + O(\delta^2), \quad (3.24a)$$

$$\tilde{Q}_{m_2,0}(\xi_0) = Q_{m_2}(\xi_0), \quad (3.24b)$$

$$\tilde{Q}_{m_2,1}(\xi_0, \tau_0) = e^{2a\tau_0} \left(\frac{-aQ_{m_2}(\xi_0)}{\sigma} + iQ_{m_2}(\xi_0)(e^{-2a\tau_0} - 1) - a\xi_0 Q'_{m_2}(\xi_0) \right), \quad (3.24c)$$

and where we have also used the relation $\frac{e^{2a\tau_0}}{2aT} = \frac{1}{L_0^2}$. We have thus written $\Phi_{m_2}(x, t, T + \delta)$ in the form $L_0(t)^{-\frac{1}{\sigma}} e^{i\tau_0} \tilde{Q}_{m_2}(\xi_0, \tau_0, \delta)$. Rescaling (3.22) is defined in such a way that $\tilde{Q}_{m_2}(\xi_0, \tau_0, \delta)$ satisfies equation (2.1) for all $\delta > -T$. So, we substitute $\tilde{Q}_{m_2}(\xi, \tau, \delta)$ into equation (2.1) and expand the resulting equation in powers of δ . At $O(1)$, we obtain that $\tilde{Q}_{m_2,0}(\xi_0)$ satisfies, as expected, equation (2.3a). And at $O(\delta)$, we obtain that $\tilde{Q}_{m_2,1}(\xi_0, \tau_0)$ satisfies

$$\begin{aligned} &\left[(1 - i\varepsilon) \left(\partial_{\xi_0}^2 + \frac{d-1}{\xi_0} \partial_{\xi_0} \right) - 1 + \frac{ia}{\sigma} + ia\xi_0 \partial_{\xi_0} \right] \tilde{Q}_{m_2,1}(\xi_0, \tau_0) + i2a\tilde{Q}_{m_2,1}(\xi_0, \tau_0) \\ &+ (1 + i\varepsilon) \left[(\sigma + 1) |Q_{m_2}|^{2\sigma} \tilde{Q}_{m_2,1}(\xi_0, \tau_0) + 2ae^{-2a\tau_0} Q_{m_2}(\xi_0) + \sigma |Q_{m_2}|^{2\sigma-2} Q_{m_2}^2 \overline{\tilde{Q}_{m_2,1}}(\xi_0, \tau_0) \right] = 0. \end{aligned} \quad (3.25)$$

The next step is to substitute (3.24c) into (3.25). All the terms with the factor $e^{-2a\tau_0}$ in front cancel upon using that $Q_{m_2}(\xi_0)$ satisfies equation (2.3a). Upon comparing with equation (3.2), we conclude that $\left(\frac{1}{\sigma} + \frac{1}{a}\right)Q_{m_2} + \xi Q'_{m_2}$ is an eigenfunction of L_V , as defined by (3.3), with eigenvalue $2a$. As in Section 3.2.4.1, we derive the result of Lemma 3.2.2 using the transformation $Q_{m_2} = e^{-\frac{1}{2}ik_0s}S(s)$.

3.2.4.3 Conclusions

Lemma 3.2.2 states that the operator $L(a)$ as defined by (3.6) has a positive eigenvalue $\lambda = 2a$. Whether the perturbations that give rise to this positive eigenvalue are relevant can be determined by looking at the symmetries as discussed in the previous two sections.

Our aim is to study the stability of one particular blowup solution. Therefore, we do not allow for perturbations tangent to a family of blowup solutions. The symmetry discussed in Section 3.2.4.1 is such a symmetry, since we have a one parameter family of blowup solutions inside a family of equations. Similarly, in Section 3.2.4.2, we perturb in the direction tangent to a family of blowup solutions through perturbing the blowup time T .

We formulate the conclusion in a remark.

Remark 3.2.3. *The eigenvalue $\lambda = 2a$ is related to a perturbation tangent to a family of blowup solutions. Since we do not allow for such perturbations, its corresponding eigenvalue does not need to be taken into account in a conclusion about the spectral stability of the m_2 -solution.*

Not allowing for perturbations tangent to a family of blowup solutions is in line with the numerical work on the stability for the case $d = 3$ and $b = 0$ in [9]. In order to determine the stability, a full PDE simulation of equation (1.9) is performed in [9]. As can be seen from Figure 1.3.1, the solution we are studying is numerically stable despite the fact that there is a positive eigenvalue $\lambda = 2a$ when studying stability in the context of equation (2.1).

3.2.5 (Non-)Persistence of (generalized) eigenfunctions

In this section, we perform the following analysis: given the perturbation $L(a) - L(0)$ of $L(0)$ and (generalized) eigenfunctions corresponding to $\lambda = 0$ of $L(0)$, we try to determine the higher corrections of the (generalized) eigenfunctions as if the eigenvalue $\lambda = 0$ would persist for $a > 0$. In order to do so, we will use the higher order asymptotic expansion of the m_2 -solution which is determined in Sections 2.2.2.2, 2.2.2.3 and 2.2.2.4 only for the bump region. The analysis will be completed by considering the other regions in Section 3.2.8.

Since the perturbation $L(a) - L(0)$ of $L(0)$ breaks almost all of the symmetries discussed in Section 3.2.3, it is already clear that at some order of a the attempt to determine the higher corrections of the (generalized) eigenfunctions will fail for cases where the symmetry is broken. This will result in failing solvability conditions, i.e. expressions that should vanish in order for the solution to exist at a particular order. These solvability conditions, and the corresponding expressions that should vanish for the solvability condition to be satisfied, will be of use in Section 3.2.6, where we express derivatives of the Evans function with respect to a and λ in terms of parameters of the eigenvalue problem. The relation between solvability conditions and derivatives of the Evans functions has been discussed in [23]. In Section 3.2.6, we will extend the results presented in [23] such that they apply to the problem at hand.

We will now summarize the conclusion of the remainder of this section, such that the rest of Section 3.2.5 can be skipped upon first reading. The perturbation $L(a) - L(0)$ breaks all the symmetries discussed in Section 3.2.3 except for the phase invariance and, thus, the

eigenfunction related to the phase invariance is (still) given by

$$\psi^{1,2}(s, a) = \begin{bmatrix} 0 \\ S(s) \end{bmatrix}.$$

Despite the fact that the perturbation breaks the translation invariance, it is possible to compute a localized first order correction of the eigenfunction $\psi^{1,1}$, which with abuse of notation will be denoted by $\partial_a \psi^{1,1}(s, 0)$. The analysis does break down at second order. The analysis for both generalized eigenfunctions breaks down at first order. This information is sufficient to continue reading at Section 3.2.6.

3.2.5.1 Persistence of $\psi^{1,1}$ up to $O(a^2)$

We proceed by introducing the expansions for $L(a)$, as given by expression (3.6), and the (generalized) eigenfunctions $\psi^{j,i}$:

$$L(a) = L(0) + a\partial_a L(0) + \frac{a^2}{2}\partial_a^2 L(0) + O(a^3), \quad (3.26a)$$

$$\psi^{j,i}(s, 0) = \psi^{j,i}(s, 0) + a\partial_a \psi^{j,i}(s, 0) + \frac{a^2}{2}\partial_a^2 \psi^{j,i}(s, 0) + O(a^3), \quad (3.26b)$$

where $i, j = 1, 2$. Our goal in this section is to determine the higher order corrections $\partial_a^k \psi^{1,1}(s, 0)$. Upon expanding $L(a)\psi^{1,1}(s, a) = 0$ in powers of a , we obtain

$$\begin{aligned} 0 &= L(0)\psi^{1,1}(s, 0) + a\left(L(0)\partial_a \psi^{1,1}(s, 0) + (\partial_a L(0))\psi^{1,1}(s, 0)\right) \\ &\quad + a^2\left(\frac{1}{2}L(0)\partial_a^2 \psi^{1,1}(s, 0) + (\partial_a L(0))\partial_a \psi^{1,1}(s, 0) + \frac{1}{2}(\partial_a^2 L(0))\psi^{1,1}(s, 0)\right) + O(a^3) \end{aligned} \quad (3.27)$$

From (3.27), we obtain an equation for $\partial_a \psi^{1,1}(s, 0)$:

$$L(0)\partial_a \psi^{1,1}(s, 0) = -(\partial_a L(0))\psi^{1,1}(s, 0). \quad (3.28)$$

From the Fredholm alternative, we conclude that equation (3.28) has a unique solution iff $\int_{-\infty}^{\infty} -(\partial_a L(0))\psi^{1,1}(\tilde{s}, 0) \cdot v_k(\tilde{s}) d\tilde{s} = 0$, where $k = 1, 2$. It can easily be verified that the solvability condition that follows from the Fredholm alternative is equivalent with the condition that follows from Lemmas 2.2.3 and 2.2.4 if we require the solution to be bounded. The next step is to verify whether the solvability condition holds. By (3.6), we find

$$\begin{aligned} \partial_a L(0) &= \begin{bmatrix} -2\sigma S_0^{2\sigma-1} \text{Im } S_1 & -2\sigma S_0^{2\sigma-1} \text{Re } S_1 \\ 2\sigma(2\sigma+1)S_0^{2\sigma-1} \text{Re } S_1 & 2\sigma S_0^{2\sigma-1} \text{Im } S_1 \end{bmatrix} + \\ &\quad \begin{bmatrix} K_0\left(\partial_s^2 - \frac{\kappa_0^2}{4}\right) + \frac{d-1}{2} - \frac{1}{\sigma} - bK_0(2\sigma+1)S_0^{2\sigma} - s\partial_s & K_0\kappa_0\partial_s - \frac{d-1}{\kappa_0}\partial_s - \frac{\kappa_0}{2}s \\ -K_0\kappa_0\partial_s + \frac{d-1}{\kappa_0}\partial_s + \frac{\kappa_0}{2}s & K_0\left(\partial_s^2 - \frac{\kappa_0^2}{4}\right) + \frac{d-1}{2} - \frac{1}{\sigma} - bK_0S_0^{2\sigma} - s\partial_s \end{bmatrix}. \end{aligned} \quad (3.29)$$

Using that $\text{Re } S_1(s)$ is odd and that $\text{Im } S_1(s)$ is even, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} -(\partial_a L(0))\psi^{1,1}(\tilde{s}, 0) \cdot v_1(\tilde{s}) d\tilde{s} &= \int_{-\infty}^{\infty} -\Pi_2(\partial_a L(0))\psi^{1,1}(\tilde{s}, 0) \Pi_2 v_1(\tilde{s}) d\tilde{s} \\ &= \frac{-4}{\int_{-\infty}^{\infty} S_0^2 d\tilde{s}} \int_{-\infty}^{\infty} \underbrace{\left(\left(-K_0\kappa_0\partial_s + \frac{d-1}{\kappa_0}\partial_s + \frac{\kappa_0}{2}s \right) S_0' + 2\sigma(2\sigma+1)S_0^{2\sigma-1} \text{Re } S_1 S_0' \right)}_{\text{even}} \underbrace{S_0'}_{\text{odd}} d\tilde{s} = 0, \end{aligned}$$

where the mapping Π_k is for $k = 1, 2$ the projection operator that projects a two dimensional vector on its k -th component:

$$\Pi_k : (y_1, y_2)^\top \mapsto y_k, \quad k = 1, 2. \quad (3.30)$$

Similarly, we find

$$\begin{aligned} & \int_{-\infty}^{\infty} -(\partial_a L(0)) \psi^{1,1}(\tilde{s}, 0) \cdot v_2(\tilde{s}) \, d\tilde{s} = \int_{-\infty}^{\infty} -\Pi_1(\partial_a L(0)) \psi^{1,1}(\tilde{s}, 0) \Pi_1 v_1(\tilde{s}) \, d\tilde{s} \\ &= \frac{-2}{\partial_a \int_{-\infty}^{\infty} S_0^2 \, d\tilde{s}} \int_{-\infty}^{\infty} \underbrace{\left(\left(K_0 \left(\partial_s^2 - \frac{\kappa_0^2}{4} \right) + \frac{d-1}{2} - \frac{1}{\sigma} - b K_0 (2\sigma+1) S_0^{2\sigma-s\partial_s} \right) S_0' - 2\sigma S_0^{2\sigma-1} \operatorname{Im} S_1 S_0' \right)}_{\text{odd}} \underbrace{S_0}_{\text{even}} \, d\tilde{s} = 0. \end{aligned}$$

This implies that equation (3.28) can be solved uniquely. From (3.28), it follows that $\Pi_1 \partial_a \psi^{1,1}(s, 0)$ satisfies

$$\left(\partial_s^2 - \alpha + (2\sigma + 1) S_0^{2\sigma} \right) \Pi_1 \partial_a \psi^{1,1}(s, 0) = -\Pi_2 (\partial_a L(0)) \psi^{1,1}(s, 0).$$

This equation can be solved in the same way as we solved equation (2.15a), the equation for the real part of S_1 . And thus, the solution is given by

$$\begin{aligned} \Pi_1 \partial_a \psi^{1,1}(s, 0) &= E_1 \psi_1 + E_2 \psi_2 + \psi_1 \int_0^s \psi_2 \Pi_2 (\partial_a L(0)) \psi^{1,1}(\tilde{s}, 0) \, d\tilde{s} \\ &\quad - \psi_2 \int_0^s \psi_1 \Pi_2 (\partial_a L(0)) \psi^{1,1}(\tilde{s}, 0) \, d\tilde{s}, \end{aligned} \quad (3.31)$$

where ψ_1 and ψ_2 are given by (2.16). We want the eigenfunction $\psi^{1,1}$ and hence also $\Pi_1 \partial_a \psi^{1,1}(s, 0)$ to vanish as s tends to infinity. Upon using Lemma 2.2.3, we obtain for $\sigma = 1$ that

$$E_2 = \int_0^{\infty} \psi_1 \Pi_2 (\partial_a L(0)) \psi^{1,1}(s, 0) \, d\tilde{s} = -\frac{2\alpha(\alpha - 1)}{\kappa_0}, \quad (3.32)$$

where α is defined by (2.12). We can choose E_1 freely, since $(\psi_1(s), 0)^\top = \psi^{1,1}(s, 0)$ and we can always add a multiple of an eigenfunction. Thus, we choose $E_1 = 0$. From (3.28), it also follows that $\Pi_2 \partial_a \psi^{1,1}(s, 0)$ satisfies

$$-\left(\partial_s^2 - \alpha + S_0^{2\sigma} \right) \Pi_2 \partial_a \psi^{1,1}(s, 0) = -\Pi_1 (\partial_a L(0)) \psi^{1,1}(s, 0).$$

In a similar way as we obtained the solution of equation (2.15b) for the imaginary part of S_1 , we obtain that the solution of this equation is

$$\begin{aligned} \Pi_2 \partial_a \psi^{1,1}(s, 0) &= D_1 \phi_1 + D_2 \phi_2 - \phi_1 \int_0^s \phi_2 \Pi_1 (\partial_a L(0)) \psi^{1,1}(\tilde{s}, 0) \, d\tilde{s} \\ &\quad + \phi_2 \int_0^s \phi_1 \Pi_1 (\partial_a L(0)) \psi^{1,1}(\tilde{s}, 0) \, d\tilde{s}, \end{aligned} \quad (3.33)$$

where ϕ_1 and ϕ_2 are given by (2.19). Again, we require the eigenfunction $\psi^{1,1}$ and hence also $\Pi_2 \partial_a \psi^{1,1}(s, 0)$ to vanish as s tends to infinity and we obtain from Lemma 2.2.4 that for $\sigma = 1$

$$D_2 = \int_0^\infty \phi_1 \Pi_1 (\partial_a L(0)) \psi^{1,1}(\tilde{s}, 0) d\tilde{s} = -\frac{2}{3} \alpha^2 (3C_1 + 2(b+1)K_0), \quad (3.34)$$

where α is defined by (2.12) and C_1 is as in (2.21), see also the discussion at the end of Section 2.2.2.3. The constant D_1 can be chosen freely, since $(0, \phi_1(s))^\top = \psi^{1,2}(s, 0)$. Therefore, we set $D_1 = 0$.

As in Section 2.2.2.4, which concerned the higher order asymptotic construction of the m_2 -solution, the derivations in this section are involved and cannot be done by hand due to the size of the expressions. And also, for the same reasons described in Section 2.2.2.4, even with a computer algebra system, like Mathematica [47], the evaluation of (3.32) and (3.34) is not straightforward. Mathematica notebooks with calculations can be provided upon request [42].

We now show that, although the eigenfunction $\psi^{1,1}$ corresponding to the translation invariance persists at $O(a)$, it ceases to persist at $O(a^2)$. From (3.27), we obtain

$$L(0) \partial_a^2 \psi^{1,1}(s, 0) = -(\partial_a^2 L(0)) \psi^{1,1}(s, 0) - 2(\partial_a L(0)) \partial_a \psi^{1,1}(s, 0), \quad (3.35)$$

where $\psi^{1,1}(s, 0)$ is given by (3.13), $\partial_a \psi^{1,1}$ is given by (3.31) and (3.33), $\partial_a L(0)$ is given by (3.29) and $\partial_a^2 L(0)$ is given by

$$\partial_a^2 L(0) = \begin{bmatrix} \partial_a^2 L_{11}(0) & \partial_a^2 L_{12}(0) \\ \partial_a^2 L_{21}(0) & \partial_a^2 L_{22}(0) \end{bmatrix}, \quad (3.36)$$

where for $\sigma = 1$:

$$\begin{aligned} \partial_a^2 L_{11}(0) &= 2K_0 \frac{d-1}{\kappa_0} \partial_s - \frac{d-1}{\kappa_0} s - 12bK_0 S_0 \operatorname{Re} S_1 - 2\operatorname{Re} S_1 \operatorname{Im} S_1 - 4S_0 \operatorname{Im} S_2 - 2\kappa_2 \partial_s, \\ \partial_a^2 L_{12}(0) &= (d-1)K_0 + 2\frac{d-1}{\kappa_0^2} s \partial_s - 4bK_0 \operatorname{Im} S_1 S_0 - 4S_0 \operatorname{Re} S_2 - 2(\operatorname{Re} S_1)^2 - 6(\operatorname{Im} S_1)^2 \\ &\quad - \kappa_0 \kappa_2, \\ \partial_a^2 L_{21}(0) &= -(d-1)K_0 - 2\frac{d-1}{\kappa_0^2} s \partial_s - 4bK_0 \operatorname{Im} S_1 S_0 + 12S_0 \operatorname{Re} S_2 + 6(\operatorname{Re} S_1)^2 + 2(\operatorname{Im} S_1)^2 \\ &\quad + \kappa_0 \kappa_2, \\ \partial_a^2 L_{22}(0) &= 2K_0 \frac{d-1}{\kappa_0} \partial_s - \frac{d-1}{\kappa_0} s - 4bK_0 S_0 \operatorname{Re} S_1 + 4\operatorname{Re} S_1 \operatorname{Im} S_1 + 4S_0 \operatorname{Im} S_2 - 2\kappa_2 \partial_s. \end{aligned}$$

From the Fredholm alternative, we conclude that equation (3.35) can only be solved if

$$\int_{-\infty}^{\infty} \left(-(\partial_a^2 L(0)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0)) \partial_a \psi^{1,1}(\tilde{s}, 0) \right) \cdot v_k(\tilde{s}) d\tilde{s} = 0, \quad (3.37)$$

where $k = 1, 2$.

Similar to evaluating (3.32) and (3.34), evaluating (3.37) is also not straightforward and even more involved. Mathematica notebooks with computations can be provided upon request [42]. The final result is, after an involved computation, given by a (surprisingly) short

expression. For $\sigma = 1$, we obtain:

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \left(-(\partial_a^2 L(0)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0)) \partial_a \psi^{1,1}(\tilde{s}, 0) \right) \cdot v_1(\tilde{s}) \, d\tilde{s} \\
 &= \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} \psi_1(\tilde{s}) \Pi_2 \left(-(\partial_a^2 L(0)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0)) \partial_a \psi^{1,1}(\tilde{s}, 0) \right) \, d\tilde{s} \\
 &= -\frac{16(\alpha(\alpha(d+2)-6)+3)}{3\alpha\kappa_0^2},
 \end{aligned} \tag{3.38}$$

and

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \left(-(\partial_a^2 L(0)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0)) \partial_a \psi^{1,1}(\tilde{s}, 0) \right) \cdot v_2(\tilde{s}) \, d\tilde{s} \\
 &= \sqrt{\alpha} \int_{-\infty}^{\infty} \phi_1(\tilde{s}) \Pi_1 \left(-(\partial_a^2 L(0)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0)) \partial_a \psi^{1,1}(\tilde{s}, 0) \right) \, d\tilde{s} \\
 &= \frac{4(\alpha(-2\alpha(d-2)+d-6)+3)}{\alpha\kappa_0}.
 \end{aligned} \tag{3.39}$$

From (3.38) and (3.39), we conclude that, indeed, no bounded order $O(a^2)$ correction for $\psi^{1,1}$ can be determined.

3.2.5.2 Non-persistence of $\psi^{2,1}$ and $\psi^{2,2}$

The goal of this section is trying to determine the higher order corrections $\partial_a^k \psi^{2,i}(s, 0)$ by solving

$$L(a)\psi^{2,j}(s, a) = \psi^{1,j}(s, a). \tag{3.40}$$

We will see that this attempt will immediately fail at $O(a)$. We start with $\psi^{2,1}$. Substituting expansions (3.26a) and (3.26b) into (3.40) leads to

$$0 = L(0)\psi^{2,1}(s, 0) - \psi^{1,1}(s, 0) + a \left(L(0)\partial_a \psi^{2,1}(s, 0) + (\partial_a L(0)) \psi^{2,1}(s, 0) - \partial_a \psi^{1,1}(s, 0) \right) + O(a^2). \tag{3.41}$$

From equation (3.41), we conclude that the first order correction $\partial_a \psi^{2,1}(s, 0)$ satisfies

$$L(0)\partial_a \psi^{2,1}(s, 0) = -(\partial_a L(0)) \psi^{2,1}(s, 0) + \partial_a \psi^{1,1}(s, 0). \tag{3.42}$$

Again, we use the Fredholm alternative to determine whether (3.42) has a unique solution. A non-trivial calculation shows that for $\sigma = 1$

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \left(-(\partial_a L(0)) \psi^{2,1}(\tilde{s}, 0) + \partial_a \psi^{1,1}(\tilde{s}, 0) \right) \cdot v_1(\tilde{s}) \, d\tilde{s} \\
 &= \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} \psi_1(\tilde{s}) \Pi_2 \left(-(\partial_a L(0)) \psi^{2,1}(\tilde{s}, 0) + \partial_a \psi^{1,1}(\tilde{s}, 0) \right) \, d\tilde{s} \\
 &= \frac{4}{3}\alpha K_0,
 \end{aligned} \tag{3.43}$$

and

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \left(-(\partial_a L(0)) \psi^{2,1}(\tilde{s}, 0) + \partial_a \psi^{1,1}(\tilde{s}, 0) \right) \cdot v_2(\tilde{s}) \, d\tilde{s} \\
 &= \sqrt{\alpha} \int_{-\infty}^{\infty} \phi_1(\tilde{s}) \Pi_1 \left(-(\partial_a L(0)) \psi^{2,1}(\tilde{s}, 0) + \partial_a \psi^{1,1}(\tilde{s}, 0) \right) \, d\tilde{s} \\
 &= \frac{4(\alpha - 1)}{\kappa_0}.
 \end{aligned} \tag{3.44}$$

Now, since $0 < \alpha < 1$ and $K_0 > 0$, it follows that both (3.43) and (3.44) do not vanish. One of which is already sufficient to conclude from the Fredholm alternative that equation (3.42) does not have a bounded solution.

We use a similar procedure for the other generalized eigenfunction $\psi^{2,2}$. Substituting expansions (3.26a) and (3.26b) into (3.40) leads at $O(a)$ to

$$L(0) \partial_a \psi^{2,2}(s, 0) = -(\partial_a L(0)) \psi^{2,2}(s, 0) + \partial_a \psi^{1,2}(s, 0). \tag{3.45}$$

We use the Fredholm alternative to determine whether (3.45) has a unique solution and obtain that for $\sigma = 1$

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \left(-(\partial_a L(0)) \psi^{2,2}(\tilde{s}, 0) + \partial_a \psi^{1,2}(\tilde{s}, 0) \right) \cdot v_1(\tilde{s}) \, d\tilde{s} \\
 &= \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} \psi_1(\tilde{s}) \Pi_2 \left(-(\partial_a L(0)) \psi^{2,2}(\tilde{s}, 0) + \partial_a \psi^{1,2}(\tilde{s}, 0) \right) \, d\tilde{s} \\
 &= \frac{4(\alpha - 1)}{\alpha \kappa_0},
 \end{aligned} \tag{3.46}$$

and

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \left(-(\partial_a L(0)) \psi^{2,2}(\tilde{s}, 0) + \partial_a \psi^{1,2}(\tilde{s}, 0) \right) \cdot v_2(\tilde{s}) \, d\tilde{s} \\
 &= \sqrt{\alpha} \int_{-\infty}^{\infty} \phi_1(\tilde{s}) \Pi_1 \left(-(\partial_a L(0)) \psi^{2,2}(\tilde{s}, 0) + \partial_a \psi^{1,2}(\tilde{s}, 0) \right) \, d\tilde{s} \\
 &= 2(2(\alpha - 1)K_0 + d - 2).
 \end{aligned} \tag{3.47}$$

Now, since $\alpha, \kappa_0 > 0$ and $\alpha - 1 = -\frac{\kappa_0^2}{4} < 0$, it follows that (3.46) is always negative. Which is sufficient to conclude from the Fredholm alternative that we cannot construct a bounded solution of (3.45).

We stress again that evaluation of (3.43), (3.44), (3.46) and (3.47) is nontrivial due to reasons also explained at the end of Section 3.2.5.1.

3.2.6 Derivatives of the Evans function

Using [23] as a guide, we extend the expressions for the derivatives of the Evans Function presented therein in a nontrivial way. First, in [23], it is shown (in a different, but general setting) that $Y_1(s, a, \lambda)$, $Y_2(s, a, \lambda)$, $Y_3(s, a, \lambda)$ and $Y_4(s, a, \lambda)$, the solutions of (3.5) as defined

in Section 3.2.2, can be chosen such that:

$$Y_1(., 0, 0) = Y_3(., 0, 0) = \psi^{1,1}(., 0), \quad (3.48a)$$

$$\partial_a Y_1(., 0, 0) = \partial_a Y_3(., 0, 0) = \partial_a \psi^{1,1}(., 0), \quad (3.48b)$$

$$Y_2(., a, 0) = Y_4(., a, 0) = \psi^{1,2}(., a), \quad (3.48c)$$

$$\partial_\lambda Y_1(., 0, 0) = \partial_\lambda Y_3(., 0, 0) = \psi^{2,1}(., 0), \quad (3.48d)$$

$$\partial_\lambda Y_2(., a, 0) = \partial_\lambda Y_4(., a, 0) = \psi^{2,2}(., a). \quad (3.48e)$$

So, in short: for $\lambda = 0$ the Y_i are equal to the eigenfunctions and the $\partial_\lambda Y_i$ are equal to the generalized eigenfunctions (if they persist).

In the setting at hand, it can be seen that these requirements can be satisfied as follows. First, from Properties 3.2.1.a and 3.2.1.b, it follows that both $\psi^{1,1}(s, a = 0)$ and $\psi^{1,2}(s, a = 0)$ are contained in $\text{Span}\{Y_1(s, a = 0, 0), Y_2(s, a = 0, 0)\}$ and also in $\text{Span}\{Y_3(s, a = 0, 0), Y_4(s, a = 0, 0)\}$. Since $\psi^{1,1}(s, a = 0)$ and $\psi^{1,2}(s, a = 0)$ are linearly independent, we can, without loss of generality, redefine Y_k , where $k = 1, 2, 3, 4$, such that (3.48a) and (3.48c) hold. Second, through differentiation of (3.5) with respect to λ and evaluating at $\lambda = 0$, it follows that $\partial_\lambda Y_k$, where $k = 1, 2, 3, 4$, satisfies

$$L(a) \partial_\lambda Y_k|_{\lambda=0} = Y_k|_{\lambda=0}.$$

Combining this result with: (3.48a), (3.48c), (3.12) and the fact that $\psi^{1,2}(s, a = 0)$ and $\psi^{2,2}(s, a = 0)$ are linearly independent, implies that we can, without loss of generality, redefine Y_k , where $k = 1, 2, 3, 4$, such that (3.48d) and (3.48e) hold. And finally, similar arguments combined with the fact that $\partial_a Y_1(., a = 0, 0)$, $\partial_a Y_3(., a = 0, 0)$ and $\partial_a \psi^{1,1}(., a = 0)$ all satisfy the same equation (although with different boundary conditions), implies that $Y_1(., a = 0, 0)$ and $Y_3(., a = 0, 0)$ can be redefined such that (3.48b) holds.

These relations are crucial and will form the link between the solvability conditions in Section 3.2.5 and the Evans function as defined by (3.11). Similar conditions hold for the $\mathcal{Y}_i$'s, which are related to the Y_i 's through (3.10).

In order to determine the derivatives $\partial_a^i \partial_\lambda^j E(a = 0, \lambda = 0)$ with $i + j \leq 4$, that are needed to make a Taylor expansion of the Evans function as defined by (3.11), we use the product rule to obtain from (3.11):

$$\partial_a^i \partial_\lambda^j E(0, 0) = \sum_{\substack{i=i_1+i_2+i_3 \\ j=j_1+j_2+j_3}} \frac{i!}{i_1!i_2!i_3!} \frac{j!}{j_1!j_2!j_3!} \partial_a^{i_1} \partial_\lambda^{j_1} (\mathcal{Y}_1 \wedge \mathcal{Y}_3) \wedge \partial_a^{i_2} \partial_\lambda^{j_2} (\mathcal{Y}_2 \wedge \mathcal{Y}_4) \cdot \partial_a^{i_3} \partial_\lambda^{j_3} m \Big|_{a=0},$$

where by $\cdot$ we denote scalar multiplication. From the relation between the $\mathcal{Y}_i$'s and their derivatives with respect to λ , we can determine which of these terms do and do not vanish. For example, take $i = 0$ and $j > 0$. Since,

$$\begin{aligned} \mathcal{Y}_1 \wedge \mathcal{Y}_3 \Big|_{a=\lambda=0} &= \Pi_{24} \psi^{1,1} \wedge \Pi_{24} \psi^{1,1} \Big|_{a=0} = 0, \\ \partial_\lambda (\mathcal{Y}_1 \wedge \mathcal{Y}_3) \Big|_{a=\lambda=0} &= \partial_\lambda \mathcal{Y}_1 \wedge \mathcal{Y}_3 - \partial_\lambda \mathcal{Y}_3 \wedge \mathcal{Y}_1 \Big|_{a=\lambda=0} \\ &= \Pi_{24} \psi^{2,1} \wedge \Pi_{24} \psi^{1,1} - \Pi_{24} \psi^{2,1} \wedge \Pi_{24} \psi^{1,1} \Big|_{a=0} = 0, \end{aligned}$$

it is necessary that $j_1 \geq 2$ must hold for $\partial_\lambda^{j_1} (\mathcal{Y}_1 \wedge \mathcal{Y}_3) \Big|_{\lambda=0}$ not to vanish. Similarly, it is necessary that $j_2 \geq 2$. Thus, it follows that

$$\partial_\lambda E(0, 0) = \partial_\lambda^2 E(0, 0) = \partial_\lambda^3 E(0, 0) = 0 \quad (3.49)$$

and furthermore that $\partial_\lambda^4 E(0, 0)$ is given by

$$\begin{aligned}\partial_\lambda^4 E(0, 0) &= 6m(s, 0, 0) \cdot \partial_\lambda^2(\mathcal{Y}_1 \wedge \mathcal{Y}_3) \wedge \partial_\lambda^2(\mathcal{Y}_2 \wedge \mathcal{Y}_4) \Big|_{\lambda=a=0} \\ &= 6\partial_\lambda^2(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_\lambda^2(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 \Big|_{\lambda=a=0} \\ &= 6\partial_\lambda^2(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \Pi_{24} \psi^{1,1} \wedge \partial_\lambda^2(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \Pi_{24} \psi^{1,2} \Big|_{\lambda=a=0},\end{aligned}\quad (3.50)$$

where we used $m(s, 0, 0) = 1$, (3.48a) and (3.48c). With a similar reasoning, we obtain that the other only non-zero derivatives with respect to λ and a up to fourth order of the Evans function are given by

$$\begin{aligned}\partial_a \partial_\lambda^3 E(0, 0) &= 3m(s, 0, 0) \partial_a \partial_\lambda(\mathcal{Y}_1 \wedge \mathcal{Y}_3) \wedge \partial_\lambda^2(\mathcal{Y}_2 \wedge \mathcal{Y}_4) \\ &\quad + 3m(s, 0, 0) \partial_\lambda^2(\mathcal{Y}_1 \wedge \mathcal{Y}_3) \wedge \partial_a \partial_\lambda(\mathcal{Y}_2 \wedge \mathcal{Y}_4) \Big|_{\lambda=a=0} \\ &= 3\partial_a \partial_\lambda(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_\lambda^2(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 \\ &\quad + 3\partial_\lambda^2(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_a \partial_\lambda(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 \Big|_{\lambda=a=0} \\ &= 3\partial_a \partial_\lambda(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \Pi_{24} \psi^{1,1} \wedge \partial_\lambda^2(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \Pi_{24} \psi^{1,2} \\ &\quad + 3\partial_\lambda^2(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \Pi_{24} \psi^{1,1} \wedge \partial_a \partial_\lambda(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \Pi_{24} \psi^{1,2} \Big|_{\lambda=a=0},\end{aligned}\quad (3.51)$$

$$\begin{aligned}\partial_a^2 \partial_\lambda^2 E(0, 0) &= m(s, 0, 0) \partial_a^2(\mathcal{Y}_1 \wedge \mathcal{Y}_3) \wedge \partial_\lambda^2(\mathcal{Y}_2 \wedge \mathcal{Y}_4) \\ &\quad + 2m(s, 0, 0) \partial_a \partial_\lambda(\mathcal{Y}_1 \wedge \mathcal{Y}_3) \wedge \partial_a \partial_\lambda(\mathcal{Y}_2 \wedge \mathcal{Y}_4) \Big|_{\lambda=a=0} \\ &= \partial_a^2(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_\lambda^2(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 \\ &\quad + 2\partial_a \partial_\lambda(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_a \partial_\lambda(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 \Big|_{\lambda=a=0} \\ &= \partial_a^2(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \Pi_{24} \psi^{1,1} \wedge \partial_\lambda^2(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \Pi_{24} \psi^{1,2} \\ &\quad + 2\partial_a \partial_\lambda(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \Pi_{24} \psi^{1,1} \wedge \partial_a \partial_\lambda(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \Pi_{24} \psi^{1,2} \Big|_{\lambda=a=0},\end{aligned}\quad (3.52)$$

$$\begin{aligned}\partial_a^3 \partial_\lambda E(0, 0) &= 3m(s, 0, 0) \partial_a^2(\mathcal{Y}_1 \wedge \mathcal{Y}_3) \wedge \partial_a \partial_\lambda(\mathcal{Y}_2 \wedge \mathcal{Y}_4) \Big|_{\lambda=a=0} \\ &= 3\partial_a^2(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_a \partial_\lambda(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 \Big|_{\lambda=a=0} \\ &= 3\partial_a^2(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \Pi_{24} \psi^{1,1} \wedge \partial_a \partial_\lambda(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \Pi_{24} \psi^{1,2} \Big|_{\lambda=a=0}.\end{aligned}\quad (3.53)$$

Note that it follows from (3.49), (3.50), (3.51), (3.52) and (3.53), that a leading order expansion of the Evans function is given by (1.21). The next step is to express

$$\begin{aligned}\partial_\lambda^2(\mathcal{Y}_1 - \mathcal{Y}_3) &= \Pi_{24} \partial_\lambda^2(Y_1 - Y_3), \\ \partial_\lambda^2(\mathcal{Y}_2 - \mathcal{Y}_4) &= \Pi_{24} \partial_\lambda^2(Y_2 - Y_4), \\ \partial_a \partial_\lambda(\mathcal{Y}_1 - \mathcal{Y}_3) &= \Pi_{24} \partial_a \partial_\lambda(Y_1 - Y_3), \\ \partial_a \partial_\lambda(\mathcal{Y}_2 - \mathcal{Y}_4) &= \Pi_{24} \partial_a \partial_\lambda(Y_2 - Y_4)\end{aligned}$$

and

$$\partial_a^2(\mathcal{Y}_1 - \mathcal{Y}_3) = \Pi_{24} \partial_a^2(Y_1 - Y_3),$$

where Π_{24} is the mapping defined by (3.9), in terms of the eigenfunctions, the generalized eigenfunctions and their derivatives with respect to a and λ evaluated at $a = \lambda = 0$. Let the

function Y satisfy $L(a)Y = \lambda Y$. Thus, $L(a)\partial_\lambda^2 Y = \lambda\partial_\lambda^2 Y + 2\partial_\lambda Y$; evaluation at $a = \lambda = 0$ leads to

$$L(0)\partial_\lambda^2 Y|_{a=\lambda=0} = 2\partial_\lambda Y|_{a=\lambda=0}. \quad (3.54)$$

Similarly, we find

$$L(0)\partial_{\lambda a}^2 Y|_{a=\lambda=0} = \partial_a Y|_{a=\lambda=0} - (\partial_a L(0))\partial_\lambda Y|_{a=\lambda=0}$$

and

$$L(0)\partial_a^2 Y|_{a=\lambda=0} = -(\partial_a^2 L(0))Y|_{a=\lambda=0} - 2(\partial_a L(0))\partial_a Y|_{a=\lambda=0}.$$

The derivative $\partial_\lambda^2 Y_1(s, 0, 0)$ also satisfies (3.54) with boundary condition

$$\lim_{s \rightarrow -\infty} \partial_\lambda^2 Y_1(s, 0, 0) = 0.$$

Note that this boundary condition follows from the definition of Y_1 in the beginning of Section 3.2.2. Solving for $\partial_\lambda^2 Y_1(s, 0, 0)$ from this equation is similar to way we solved the equation for S_1 in Section 2.2.2. Upon using (3.48d), it follows that

$$\frac{1}{2}\partial_\lambda^2 Y_1(s, 0, 0) = \mathcal{L}(\partial_\lambda Y_1(s, 0, 0), A_1, A_2, -\infty, s) = \mathcal{L}(\psi^{2,1}(s, 0), A_1, A_2, -\infty, s),$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}\left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, A_1, A_2, a, b\right) := \begin{bmatrix} A_1\psi_1 - \psi_1 \int_a^b \psi_2(\tilde{s})f_2(\tilde{s})d\tilde{s} + \psi_2 \int_a^b \psi_1(\tilde{s})f_2(\tilde{s})d\tilde{s} \\ A_2\phi_1 + \phi_1 \int_a^b \phi_2(\tilde{s})f_1(\tilde{s})d\tilde{s} - \phi_2 \int_a^b \phi_1(\tilde{s})f_1(\tilde{s})d\tilde{s} \end{bmatrix},$$

ψ_1 and ψ_2 are given by (2.16), ϕ_1 and ϕ_2 are given by (2.19), f_1 and f_2 are (real or complex value) functions and A_1 and A_2 are constants. The purpose of this section is to obtain expressions (3.58), (3.59), (3.60) and (3.61). In this section, we will show that these expressions do not depend on the value of the constants A_1 and A_2 and that we can derive these expressions without keeping track of the exact value of these constants. Thus, to simplify the notation, the constants A_1 and A_2 will denote “general” constants in this section and are used in all expressions.

Similarly, we find

$$\begin{aligned} \frac{1}{2}\partial_\lambda^2 Y_3(s, 0, 0) &= \mathcal{L}(\psi^{2,1}(s, 0), A_1, A_2, \infty, s), \\ \frac{1}{2}\partial_\lambda^2 Y_2(s, 0, 0) &= \mathcal{L}(\psi^{2,2}(s, 0), A_1, A_2, -\infty, s), \\ \frac{1}{2}\partial_\lambda^2 Y_4(s, 0, 0) &= \mathcal{L}(\psi^{2,2}(s, 0), A_1, A_2, \infty, s), \\ \partial_{a,\lambda}^2 Y_1(s, 0, 0) &= \mathcal{L}(\partial_a \psi^{1,1}(s, 0) - (\partial_a L(0))\psi^{2,1}(s, 0), A_1, A_2, -\infty, s), \\ \partial_{a,\lambda}^2 Y_3(s, 0, 0) &= \mathcal{L}(\partial_a \psi^{1,1}(s, 0) - (\partial_a L(0))\psi^{2,1}(s, 0), A_1, A_2, \infty, s), \\ \partial_{a,\lambda}^2 Y_2(s, 0, 0) &= \mathcal{L}(\partial_a \psi^{1,2}(s, 0) - (\partial_a L(0))\psi^{2,2}(s, 0), A_1, A_2, -\infty, s), \\ \partial_{a,\lambda}^2 Y_4(s, 0, 0) &= \mathcal{L}(\partial_a \psi^{1,2}(s, 0) - (\partial_a L(0))\psi^{2,2}(s, 0), A_1, A_2, \infty, s), \\ \partial_a^2 Y_1(s, 0, 0) &= \mathcal{L}(-(\partial_a^2 L(0))\psi^{1,1}(s, 0) - 2(\partial_a L(0))\partial_a \psi^{1,1}(s, 0), A_1, A_2, -\infty, s), \\ \partial_a^2 Y_3(s, 0, 0) &= \mathcal{L}(-(\partial_a^2 L(0))\psi^{1,1}(s, 0) - 2(\partial_a L(0))\partial_a \psi^{1,1}(s, 0), A_1, A_2, \infty, s). \end{aligned}$$

By subtracting these expressions, we derive

$$\frac{1}{2}\partial_\lambda^2 (\Upsilon_1 - \Upsilon_3)|_{a=0,\lambda=0} = \Pi_{24} \frac{1}{2}\partial_\lambda^2 (Y_1 - Y_3)|_{a=0,\lambda=0} = \Pi_{24} \mathcal{L}(\psi^{2,1}(s, 0), A_1, A_2, -\infty, \infty), \quad (3.55a)$$

$$\frac{1}{2}\partial_\lambda^2 (\Upsilon_2 - \Upsilon_4)|_{a=0,\lambda=0} = \Pi_{24} \frac{1}{2}\partial_\lambda^2 (Y_2 - Y_4)|_{a=0,\lambda=0} = \Pi_{24} \mathcal{L}(\psi^{2,2}(s, 0), A_1, A_2, -\infty, \infty), \quad (3.55b)$$

$$\begin{aligned} \partial_{a\lambda}^2 (\Upsilon_1 - \Upsilon_3)|_{a=0,\lambda=0} &= \Pi_{24} \partial_{a\lambda}^2 (Y_1 - Y_3)|_{a=0,\lambda=0} \\ &= \Pi_{24} \mathcal{L}(\partial_a \psi^{1,1}(s, 0) - (\partial_a L(0)) \psi^{2,1}(s, 0), A_1, A_2, -\infty, \infty), \end{aligned} \quad (3.55c)$$

$$\begin{aligned} \partial_{a\lambda}^2 (\Upsilon_2 - \Upsilon_4)|_{a=0,\lambda=0} &= \Pi_{24} \partial_{a\lambda}^2 (Y_2 - Y_4)|_{a=0,\lambda=0} \\ &= \Pi_{24} \mathcal{L}(\partial_a \psi^{1,2}(s, 0) - (\partial_a L(0)) \psi^{2,2}(s, 0), A_1, A_2, -\infty, \infty), \end{aligned} \quad (3.55d)$$

$$\begin{aligned} \partial_a^2 (\Upsilon_1 - \Upsilon_3)|_{a=0,\lambda=0} &= \Pi_{24} \partial_a^2 (Y_1 - Y_3)|_{a=0,\lambda=0} \\ &= \Pi_{24} \mathcal{L}(-(\partial_a^2 L(0)) \psi^{1,1}(s, 0) - 2(\partial_a L(0)) \partial_a \psi^{1,1}(s, 0), A_1, A_2, -\infty, \infty). \end{aligned} \quad (3.55e)$$

We are now in a position to derive the expressions for the derivatives of the Evans function we need. First, we start with the fourth order derivative with respect to λ . Upon substitution of (3.55a) and (3.55b) in (3.50), we obtain

$$\begin{aligned} \partial_\lambda^4 E(0, 0) &= 24\Pi_{24} \mathcal{L}(\psi^{2,1}(s, 0), A_1, A_2, -\infty, \infty) \wedge \Pi_{24} \psi^{1,1} \wedge \Pi_{24} \mathcal{L}(\psi^{2,2}(s, 0), A_1, A_2, -\infty, \infty) \wedge \Pi_{24} \psi^{1,2} \\ &= 24\Pi_{24} \left[\begin{array}{c} A_1\psi_1 - \psi_1 \int_{-\infty}^{\infty} \psi_2 \Pi_2 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} + \psi_2 \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \\ A_2\phi_1 + \phi_1 \int_{-\infty}^{\infty} \phi_2 \Pi_1 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} - \phi_2 \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \end{array} \right] \wedge \Pi_{24} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} \\ &\quad \wedge \Pi_{24} \left[\begin{array}{c} A_1\psi_1 - \psi_1 \int_{-\infty}^{\infty} \psi_2 \Pi_2 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} + \psi_2 \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} \\ A_2\phi_1 + \phi_1 \int_{-\infty}^{\infty} \phi_2 \Pi_1 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} - \phi_2 \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} \end{array} \right] \wedge \Pi_{24} \begin{bmatrix} 0 \\ \phi_1 \end{bmatrix} \\ &= 24\Pi_{24} \left[\begin{array}{c} \psi_2 \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \\ -\phi_2 \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \end{array} \right] \wedge \Pi_{24} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} \wedge \Pi_{24} \left[\begin{array}{c} \psi_2 \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} \\ -\phi_2 \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} \end{array} \right] \wedge \Pi_{24} \begin{bmatrix} 0 \\ \phi_1 \end{bmatrix}, \end{aligned} \quad (3.56)$$

where in the last step we used the following (and similar) identities:

$$\begin{aligned} \Pi_{24} \left[\begin{array}{c} A_1\psi_1 - \psi_1 \int_{-\infty}^{\infty} \psi_2 \Pi_2 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} + \psi_2 \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \\ A_2\phi_1 + \phi_1 \int_{-\infty}^{\infty} \phi_2 \Pi_1 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} - \phi_2 \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \end{array} \right] \wedge \Pi_{24} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} \\ = \underbrace{\left(A_1 - \int_{-\infty}^{\infty} \psi_2 \Pi_2 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \right) \Pi_{24} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} \wedge \Pi_{24} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix}}_{=0} \\ + \Pi_{24} \left[\begin{array}{c} +\psi_2 \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \\ A_2\phi_1 + \phi_1 \int_{-\infty}^{\infty} \phi_2 \Pi_1 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} - \phi_2 \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \end{array} \right] \wedge \Pi_{24} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} \\ = \Pi_{24} \left[\begin{array}{c} \psi_2 \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \\ A_2\phi_1 + \phi_1 \int_{-\infty}^{\infty} \phi_2 \Pi_1 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} - \phi_2 \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \end{array} \right] \wedge \Pi_{24} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} \end{aligned}$$

We now use the following equality which can be verified by direct calculation:

$$\Pi_{24} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} \wedge \Pi_{24} \begin{bmatrix} \psi_2 \\ 0 \end{bmatrix} \wedge \Pi_{24} \begin{bmatrix} 0 \\ \phi_1 \end{bmatrix} \wedge \Pi_{24} \begin{bmatrix} 0 \\ \phi_2 \end{bmatrix} = \text{Det} \begin{bmatrix} \psi_1 & \psi_2 & 0 & 0 \\ \psi_1' & \psi_2' & 0 & 0 \\ 0 & 0 & \phi_1 & \phi_2 \\ 0 & 0 & \phi_1' & \phi_2' \end{bmatrix} = 1. \quad (3.57)$$

Upon combining (3.56) and (3.57), it follows that

$$\begin{aligned}
 \partial_\lambda^4 E(0, 0) &= 24 \left(\int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} \underbrace{\Pi_{24} \begin{bmatrix} \psi_2 \\ 0 \end{bmatrix} \wedge \Pi_{24} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} \wedge \Pi_{24} \begin{bmatrix} 0 \\ -\phi_2 \end{bmatrix} \wedge \Pi_{24} \begin{bmatrix} 0 \\ \phi_1 \end{bmatrix}}_{=-1} \right. \\
 &\quad \left. - \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} \underbrace{\Pi_{24} \begin{bmatrix} 0 \\ \phi_2 \end{bmatrix} \wedge \Pi_{24} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} \wedge \Pi_{24} \begin{bmatrix} \psi_2 \\ 0 \end{bmatrix} \wedge \Pi_{24} \begin{bmatrix} 0 \\ \phi_1 \end{bmatrix}}_{=-1} \right) \\
 &= -24 \text{Det} \begin{bmatrix} \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} & \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \\ \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} & \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} \end{bmatrix} \\
 &= -4! \text{Det} \begin{bmatrix} \int_{-\infty}^{\infty} \psi^{2,1}(\tilde{s}, 0) \cdot v_1 d\tilde{s} & \int_{-\infty}^{\infty} \psi^{2,1}(\tilde{s}, 0) \cdot v_2 d\tilde{s} \\ \int_{-\infty}^{\infty} \psi^{2,2}(\tilde{s}, 0) \cdot v_1 d\tilde{s} & \int_{-\infty}^{\infty} \psi^{2,2}(\tilde{s}, 0) \cdot v_2 d\tilde{s} \end{bmatrix}, \tag{3.58}
 \end{aligned}$$

where v_1 and v_2 are the eigenfunctions of the adjoint operator as given by (3.13). This result agrees with Theorem 1.1 in [23]. Second, upon substitution of (3.55c) and (3.55b) in (3.51), we obtain in a similar fashion that

$$\begin{aligned}
 \partial_a \partial_\lambda^3 E(0, 0) &= -6 \text{Det} \begin{bmatrix} \int_{-\infty}^{\infty} \psi_1 \Pi_2 (\partial_a \psi^{1,1}(\tilde{s}, 0) - (\partial_a L(0)) \psi^{2,1}(\tilde{s}, 0)) d\tilde{s} & \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} \\ \int_{-\infty}^{\infty} \phi_1 \Pi_1 (\partial_a \psi^{1,1}(\tilde{s}, 0) - (\partial_a L(0)) \psi^{2,1}(\tilde{s}, 0)) d\tilde{s} & \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} \end{bmatrix} \\
 &\quad - 6 \text{Det} \begin{bmatrix} \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} & \int_{-\infty}^{\infty} \psi_1 \Pi_2 (\partial_a \psi^{1,2}(\tilde{s}, 0) - (\partial_a L(0)) \psi^{2,2}(\tilde{s}, 0)) d\tilde{s} \\ \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} & \int_{-\infty}^{\infty} \phi_1 \Pi_1 (\partial_a \psi^{1,2}(\tilde{s}, 0) - (\partial_a L(0)) \psi^{2,2}(\tilde{s}, 0)) d\tilde{s} \end{bmatrix}. \tag{3.59}
 \end{aligned}$$

Third, upon substitution of (3.55b), (3.55c), (3.55d) and (3.55e) in (3.52), we obtain that

$$\begin{aligned}
 \partial_a^2 \partial_\lambda^2 E(0, 0) &= -2 \text{Det} \begin{bmatrix} \int_{-\infty}^{\infty} \psi_1 \Pi_2 (-\partial_a^2 L(0)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0)) \partial_a \psi^{1,1}(\tilde{s}, 0) d\tilde{s} & \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} \\ \int_{-\infty}^{\infty} \phi_1 \Pi_1 (-\partial_a^2 L(0)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0)) \partial_a \psi^{1,1}(\tilde{s}, 0) d\tilde{s} & \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} \end{bmatrix} \\
 &\quad - 4 \text{Det} \begin{bmatrix} \int_{-\infty}^{\infty} \psi_1 \Pi_2 (\partial_a \psi^{1,1}(\tilde{s}, 0) - (\partial_a L(0)) \psi^{2,1}(\tilde{s}, 0)) d\tilde{s} & \int_{-\infty}^{\infty} \psi_1 \Pi_2 (\partial_a \psi^{1,2}(\tilde{s}, 0) - (\partial_a L(0)) \psi^{2,2}(\tilde{s}, 0)) d\tilde{s} \\ \int_{-\infty}^{\infty} \phi_1 \Pi_1 (\partial_a \psi^{1,1}(\tilde{s}, 0) - (\partial_a L(0)) \psi^{2,1}(\tilde{s}, 0)) d\tilde{s} & \int_{-\infty}^{\infty} \phi_1 \Pi_1 (\partial_a \psi^{1,2}(\tilde{s}, 0) - (\partial_a L(0)) \psi^{2,2}(\tilde{s}, 0)) d\tilde{s} \end{bmatrix}. \tag{3.60}
 \end{aligned}$$

And finally, upon substitution of (3.55d) and (3.55e) in (3.53), we find that

$$\begin{aligned}
 \partial_a^3 \partial_\lambda E(0, 0) &= \\
 &= -3 \text{Det} \begin{bmatrix} \int_{-\infty}^{\infty} \psi_1 \Pi_2 (-\partial_a^2 L(0)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0)) \partial_a \psi^{1,1}(\tilde{s}, 0) d\tilde{s} & \int_{-\infty}^{\infty} \psi_1 \Pi_2 (\partial_a \psi^{1,2}(\tilde{s}, 0) - (\partial_a L(0)) \psi^{2,2}(\tilde{s}, 0)) d\tilde{s} \\ \int_{-\infty}^{\infty} \phi_1 \Pi_1 (-\partial_a^2 L(0)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0)) \partial_a \psi^{1,1}(\tilde{s}, 0) d\tilde{s} & \int_{-\infty}^{\infty} \phi_1 \Pi_1 (\partial_a \psi^{1,2}(\tilde{s}, 0) - (\partial_a L(0)) \psi^{2,2}(\tilde{s}, 0)) d\tilde{s} \end{bmatrix}. \tag{3.61}
 \end{aligned}$$

Note that expressions (3.59), (3.60) and (3.61) depend, through the form of expressions (3.51), (3.52) and (3.53), on the eigenvalue structure for $a \neq 0$. That is, if (generalized)

eigenvalues would persist or would cease to persist at a different order (see Section 3.2.5), then expressions (3.59), (3.60) and (3.61) would take a different form. Since in [23] a different (and simpler) eigenvalue structure is assumed, we, in this section, extended the theory in [23] through applying the basic framework in [23] to the more complicated eigenvalue structure described in Section 3.2.5.

3.2.7 A Taylor expansion of the Evans Function

In this section, we derive a leading order Taylor expansion of the Evans function, i.e. we derive expressions for the coefficients of (1.21) in terms of d , b , κ_0 , K_0 and α . The results in this section are based on the analysis in Section 3.2.5 in which only the bump region is considered. And, they are valid for all λ for which the Evans function can be defined through (3.11), see Section 3.2.8 for more details.

Theorem 3.2.4. *Let $\sigma = 1$ and let $E(\lambda, a)$ be the Evans function associated to eigenvalue problem (3.5) through (3.11). Then, a leading order expansion of the Evans function is given by*

$$E(\lambda, a) = \lambda(\lambda - 2a) \left(E_2 \lambda^2 + E_1 \lambda a + E_0 a^2 + O(3) \right), \quad (3.62)$$

where

$$E_2 := -1, \quad (3.63a)$$

$$E_1 := \frac{4}{3\alpha\kappa_0^2} \left(2\alpha^2(d-7) - 3\alpha(d-8) - 9 \right), \quad (3.63b)$$

$$E_0 := -\frac{4}{3\alpha\kappa_0^2} \left(\alpha^2(d-10)d + 6\alpha(d+2) - 9 \right). \quad (3.63c)$$

Proof. To derive the result, we must evaluate (3.58), (3.59), (3.60) and (3.61). For this, we use the solvability conditions derived in Section 3.2.5. Recall, that for the evaluation of the solvability conditions in Section 3.2.5, the higher order asymptotic analysis as presented in Section 2.2.2 is used.

First, from (3.13), it follows that

$$\int_{-\infty}^{\infty} \psi^{2,1} \cdot v_1 \, d\tilde{s} = 1, \quad \int_{-\infty}^{\infty} \psi^{2,1} \cdot v_2 \, d\tilde{s} = 0, \quad \int_{-\infty}^{\infty} \psi^{2,2} \cdot v_1 \, d\tilde{s} = 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \psi^{2,2} \cdot v_2 \, d\tilde{s} = 1.$$

Substitution into (3.58) leads to

$$\partial_\lambda^4 E(0, 0) = -24. \quad (3.64)$$

Next, substitution of (3.43), (3.44), (3.46) and (3.47) into (3.59) leads to

$$\partial_a \partial_\lambda^3 E(0, 0) = 6 \left(\frac{4}{3} (3 - 4\alpha) K_0 - 2d + 4 \right). \quad (3.65)$$

Similarly, substitution of (3.43), (3.44), (3.46), (3.47), (3.38) and (3.39) into (3.60) leads to

$$\partial_a^2 \partial_\lambda^2 E(0, 0) = -\frac{16(\alpha(\alpha(d^2 - 6d - 28) + 60) - 27)}{3\alpha\kappa_0^2}. \quad (3.66)$$

And finally, substitution of (3.46), (3.47), (3.38) and (3.39) into (3.61) gives

$$\partial_a^3 \partial_\lambda E(0, 0) = \frac{16 \left(\alpha^2(d-10)d + 6\alpha(d+2) - 9 \right)}{\alpha\kappa_0^2}. \quad (3.67)$$

In all these expressions we used the system of equations (2.26) to eliminate b . Substitution of these expressions into (1.21) leads to the desired result. $\square$

The rest of this section is devoted to a discussion of the further implications of Theorem 3.2.4 and its relation with the theory presented so far. Note that the parameters K_0 and κ_0 depend on d and b through the system of nonlinear equations (2.26). Thus, obviously, further analysis is needed to draw conclusions concerning the stability from (3.62).

From Lemma 3.2.2, it follows that $\lambda = 2a$ is an eigenvalue for all $0 < a \ll 1$. And thus, the equality $E(\lambda = 2a, a) = 0$ should hold for all $0 < a \ll 1$. Since $\lambda - 2a$ appears as a factor in (3.62), this is indeed the case. Also, $\lambda = 0$ is an eigenvalue for all $0 < a \ll 1$ due to the persistence of the phase invariance of equation, see Section 3.2.5. Since λ appears as a factor in (3.62), the equality $E(\lambda = 0, a) = 0$ also holds.

From Corollary 2.2.7, it follows that the system (2.3) combined with Assumption 2.2.1 undergoes a saddle-node bifurcation upon (for $\sigma = 1$) passing the curve $D(d, b, \sigma = 1) = 0$ in the strip $2 < d < 3$ of the (d, b) -plane, see also Figure 2.2.1. That is, upon passing this curve, the number of m_2 -solutions of equation (2.3) changes from two to zero. Due to the following formal reasoning, we expect one of the $O(a)$ eigenvalues to vanish at the saddle-node bifurcation. After that, we show that the formal reasoning is in line with the Evans function analysis through proving that the Evans function indeed vanishes at the saddle-node bifurcation.

Now, let $\sigma = 1$ and fix $2 < d < 3$. It follows from Lemma 2.2.6 that there is a curve in the (b, κ) -plane that parametrizes the solution $Q(s, b, \kappa)$ of equation (2.3). Also, define the operator

$$\mathcal{G}(Q, b) := (1 - i\varepsilon) \left(Q_{\xi\xi} + \frac{d-1}{\xi} Q_{\xi} \right) - Q + i \frac{a}{\sigma} Q + ia\xi Q_{\xi} + (1 + ib\varepsilon)|Q|^{2\sigma} Q.$$

The m_2 -solution $Q_{m_2}(\xi, \kappa, b)$, as constructed in Section 2.2, now satisfies $\mathcal{G}(Q_{m_2}(\xi, \kappa, b), b) = 0$. Differentiation with respect to κ leads to

$$\partial_1 \mathcal{G}(Q_{m_2}, b) \frac{\partial Q_{m_2}}{\partial \kappa} + \partial_2 \mathcal{G}(Q_{m_2}, b) \frac{\partial b}{\partial \kappa} = 0,$$

where by $\partial_1 \mathcal{G}$ and $\partial_2 \mathcal{G}$ we denote the linearization of $\mathcal{G}$ with respect to respectively the first and second argument. Since the number of solutions for κ changes from 0 to 2 at the saddle-node bifurcation, it follows that $\frac{\partial b}{\partial \kappa}$ vanishes at the saddle-node bifurcation, see also Figure 3.4.1a. And thus, it follows that at the saddle-node bifurcation

$$\partial_1 \mathcal{G}(Q_{m_2}, b) \frac{\partial Q_{m_2}}{\partial \kappa} = 0.$$

Therefore, we have formally derived that $\frac{\partial Q_{m_2}}{\partial \kappa}$ is an eigenfunction of the linearization $\partial_1 \mathcal{G}(Q_{m_2}, b)$ of $\mathcal{G}$ with eigenvalue 0 at the saddle-node bifurcation. We will show that this formal derivation is in line with our analysis by showing that the Evans function vanishes at the saddle-node bifurcation.

Lemma 3.2.5. *Let $2 < d < 4$ and $\sigma = 1$. Then, the coefficient E_0 in equation (3.62) vanishes iff*

$$D(d, b, \sigma = 1) = d^2 + 24bd - 32b + 16b^2 = 0, \quad (3.68)$$

i.e. at the saddle-node bifurcation, where $D(d, b, \sigma)$ is defined by (2.37), see also (2.43).

Proof. First, we note that from Corollary 2.2.7 it follows that (3.68) holds iff the discriminant of (2.29b) vanishes, see also equation (2.37).

Second, we show that if (3.68) holds, then E_0 also vanishes. If the discriminant of (2.29b) vanishes, we find upon solving the second order polynomial defined by (2.29b) that $\alpha = -\frac{6}{-8+4b+d}$. Substitution of this expression into (3.63c) and using (3.68) to simplify leads to $E_0 = 0$.

Finally, we show that if $E_0 = 0$, then (3.68) holds. We eliminate z from (2.29b) by using $z = 3\frac{1-\alpha}{\alpha}$. If E_0 vanishes, we can use (3.63c) to eliminate α , this leads to

$$(d-2)(d-1)(d^2 + 24bd - 32b + 16b^2) = 0.$$

This clearly implies that (3.68) holds. $\square$

Strictly speaking, our asymptotic analysis is not valid exactly at the saddle-node, see Remark 2.2.12. However, since E_0 in equation (3.62) depends continuously on its parameters, it will vanish at the saddle-node bifurcation. We derive the following corollary.

Corollary 3.2.6. *Let $0 < a \ll 1$ and $\sigma = 1$, then upon passing the curve $D(d, b, \sigma = 1) = 0$, as defined by (2.43), in the strip $2 < d < 3$ of the (d, b) -plane equation, i.e. when (2.3) combined with Assumption 2.2.1 undergoes a saddle-node bifurcation, eigenvalue equation (3.5) has two eigenvalues λ of $o(a)$.*

Proof. Equation (3.62) has one trivial zero $\lambda = 0$ related to the phase invariance. The zero $\lambda = 0$ is obviously of $o(a)$. At the saddle-node bifurcation, it follows from Lemma 3.2.5 that a second $o(a)$ zero arises. Since equation (3.62) presents a leading order expansion of the Evans function, it cannot be concluded from the Evans function analysis that this second $o(a)$ zero is exactly 0. $\square$

So far, we have shown that E_0 only changes sign at the saddle-node bifurcation. We will now prove that E_1 is always negative. Recall the definition of the region R defined by (2.42), i.e. R denotes the part of the strip $2 < d < 4$ of the (d, b) -plane where the m_2 -solution exists, see also Figure 2.2.1.

Lemma 3.2.7. *In the region R defined by (2.42) the inequality $E_1 < 0$ holds.*

Proof. Consider the curve $E_1 = 0$ in the (α, d) -plane. From (3.63b), it follows that on this curve, d is given as a graph over α . From Assumption 2.2.1.b, it follows that the inequality $0 < \alpha < 1$ holds, we determine the maximum value of d on the curve as we vary α from 0 to 1, we find $d_{\max} = 4 - 2\sqrt{2} \approx 1.17157$. This implies that the curve $E_1 = 0$ does not pass through the for our application relevant square $0 \leq \alpha \leq 1, 2 \leq d \leq 4$. Conclusion, the sign of E_1 is fixed and turns out to be negative. $\square$

We proceed with the characterization of the zeros of equation (3.62). This equation has four zeros:

$$\lambda_0 = 0, \quad \lambda_1 = a\tilde{\lambda}_1, \quad \lambda_2 = a\tilde{\lambda}_2, \quad \text{and} \quad \lambda_3 = 2a \quad (3.69)$$

where without loss of generality we assume $\text{Re } \tilde{\lambda}_1 \leq \text{Re } \tilde{\lambda}_2$. Furthermore, $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ solve

$$E_2\tilde{\lambda}^2 + E_1\tilde{\lambda} + E_0 = 0. \quad (3.70)$$

From Lemma 3.2.5 and 3.2.7, we draw the following conclusions.

Corollary 3.2.8. *The following statements hold for the eigenvalues λ_1 and λ_2 of eigenvalue problem (3.5), where λ_1 and λ_2 are as in (3.69).*

3.2.8.a $\operatorname{Re} \lambda_1 < 0$;

3.2.8.b *If an eigenvalue passes the imaginary axis upon varying the parameters of the problem its imaginary part vanishes;*

3.2.8.c *If $E_0 < 0$, then both $\operatorname{Re} \lambda_1 \leq \operatorname{Re} \lambda_2 < 0$;*

3.2.8.d *If $E_0 > 0$, both λ_1 and λ_2 are real-valued. Furthermore, $\lambda_1 < 0$ and $\lambda_2 > 0$.*

Proof. $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ both satisfy (3.70). Since $E_2 = -1$, it follows that $\operatorname{Re} \tilde{\lambda}_1 + \operatorname{Re} \tilde{\lambda}_2 = \operatorname{Re} E_1 < 0$. This proves 3.2.8.a. Now, since the coefficients of (3.70) are real-valued, the zeros are either both real-valued or complex conjugates. We just proved that there must be at least one eigenvalue with negative real part. Therefore, if one eigenvalue crosses the imaginary axis, its imaginary part must vanish. This proves 3.2.8.b. From the equalities $\tilde{\lambda}_1 \tilde{\lambda}_2 = -E_0$ and $\tilde{\lambda}_1 + \tilde{\lambda}_2 = E_1 < 0$, it is straightforward to derive 3.2.8.c and 3.2.8.d. $\square$

3.2.8 Matching

As explained in the beginning of Section 3.2, the aim of Section 3.2 is to study localized solutions of eigenvalue problem (3.4) that decay exponentially in the bump region, i.e. solutions that satisfy Property 3.1.1. For this, we employed Evans function techniques. The exponential localization in the bump region is reflected in the requirement that the functions Y_i (where $i = 1, 2, 3, 4$), which are used in the construction of the Evans function, satisfy Property 3.2.1.a and 3.2.1.b, see also equation (3.11). However, it is currently unclear whether there exist solutions Y_i that can satisfy both Property 3.2.1.a and 3.2.1.c (or 3.2.1.b and 3.2.1.d). Furthermore, the Evans function construction is also linked to eigenfunctions of $L(0)$ which, so far, are only constructed in the bump region, see also the discussion in Section 3.2.6 and equation (3.48).

In this section, we will derive conditions under which exponential decay in the bump region can be matched to (algebraic) decay in the far field and we will also show that exponential decay in the bump region can always be matched with decay in the inner region. That is, we show that only the requirement that Y_3 and Y_4 should satisfy both Property 3.2.1.b and Property 3.2.1.d leads to an extra condition which is formulated in the following proposition.

Proposition 3.2.9. *Let $\sigma = 1$, $2 < d < 4$, R be defined by (2.42), $(d, b) \in R$ and $0 < a \ll 1$, then: the functions Y_3 and Y_4 as used in the construction of the Evans function, see (3.11), that satisfy Property 3.2.1.b can only satisfy Property 3.2.1.d if the inequality*

$$\operatorname{Re} \left(\frac{\lambda}{a} \right) > -\frac{1}{\sigma} \quad (3.71)$$

holds.

Note that this proposition has no implications for any conclusions we might draw concerning the stability of the m_2 -solution, because it does not concern the part of the spectrum in the right half of the complex plane.

Second, we also show that if $|\lambda| = O(a)$, then exponential localization in the bump region is necessary for a solution to be bounded, see Lemma 3.2.10. A main ingredient to derive these results is that, solutions of eigenvalue problem (3.3) either exhibit algebraic behavior or exponential growth. Using this, we will show that exponential localization in the bump region

corresponds to algebraic behavior for large ξ , and, under the conditions stated in Proposition 3.2.9, that this algebraic behavior corresponds algebraic decay.

Note that standard Evans function literature does not always allow for algebraically decaying behavior since this may cause existence problems for the solutions Y_i used in the construction of the Evans function, see [1, 15, 23, 35, 41]. However, since we explicitly construct the solutions Y_i in the definition of the Evans function, the fact that standard Evans function literature does not always allow for algebraically decaying behavior does not pose a restriction in the setting in which we use the Evans function.

The results in this section are obtained through a matching procedure which is similar to the one used in the asymptotic construction of the m_2 -solution in Section 2.3, see also [7]. For the matching procedure, we use the same regions as we used in the asymptotic construction of the solution, see Figure 1.3.2 and Definition 1.3.6.

Finally, through combining Proposition 3.2.9 with equation (3.48), it follows that the (generalized) eigenfunction can also be extended to solutions that satisfy Property 3.1.1.

The analysis in this and forthcoming sections closely resembles the one presented in Section 2.3. Since the details are crucial for the conclusions we draw in Proposition 3.2.9 and Lemma 3.2.10, we still choose to present these sections in considerable detail.

3.2.8.1 Bump region

In this section, we determine the leading order exponential behavior of solutions of eigenvalue equation (3.5) valid in the bump region, Region 3, in which $|s| = \left| \xi - \frac{\kappa}{a} \right| \ll \frac{1}{a}$, see Definition 1.3.6.b. We will not assume that λ is an eigenvalue, but from Theorem 3.2.4, it follows that to study the perturbed NLS spectrum, it is sufficient to restrict the analysis to the case $\lambda = O(a)$. In this case, eigenvalue equation (3.5) is to leading order given by

$$L(0)W = 0,$$

where $L(0)$ is given by (3.7). We are interested in the behavior of solutions to this equation as s tends to $\pm\infty$, more specifically $s = O(a^{-\gamma})$ with $0 < \gamma < 1$. In this scale, $L(0)W = 0$ reduces to

$$\begin{aligned} W_1'' - \alpha W_1 &= 0, \\ W_2'' - \alpha W_2 &= 0, \end{aligned}$$

where $\alpha = 1 - \frac{\kappa_0^2}{4}$ and $W = (W_1, W_2)^\top$. For large $|s|$, the behavior of solutions $W_{\text{bump}} = (W_{\text{bump},1}, W_{\text{bump},2})^\top$ in the bump region is given by

$$W_{\text{bump},1} = C_1 e^{\pm \sqrt{\alpha} s} \text{ and } W_{\text{bump},2} = C_2 e^{\pm \sqrt{\alpha} s}. \quad (3.72)$$

Note that this yields four solutions: two solutions that decay as s tends to $+\infty$ and two solutions that decay as s tends to $-\infty$. These are exactly the leading order approximations of the solutions needed in construction of the Evans function, see equation (3.11) and also Properties 3.2.1.a and 3.2.1.b.

3.2.8.2 Matching with bump region through a WKBJ analysis

In this section, we determine to leading order the solution of eigenvalue equation (3.4) in Regions 2, 4a and 4b, see Figure 1.3.2. From Definition 1.3.6, it follows that in these regions: $|Q| \ll 1$, $\xi \gg 1$ and $\left| \xi - \frac{\kappa}{a} \right| \gg 1$. The analysis in this section closely resembles the one presented in Section 2.3.1.

We will employ a WKBJ analysis to obtain a solution. However, this analysis cannot be extended to regions in which ξ is close to zero (see the assumption used in equation (3.74)), as is the case in the inner region, or regions in which ξ is close to $\frac{2}{a}$ due to the presence of a turning point (see Section 3.2.8.4). Again, we only have to consider the case where λ is $O(a)$.

From (3.2), it follows that the eigenvalue equation in regions where $|Q| \ll a$ is to leading order given by

$$-v + (1 - i\varepsilon) \left(v_{\xi\xi} + \frac{d-1}{\xi} v_{\xi} \right) + ia \left(\xi v_{\xi} + \frac{1}{\sigma} v \right) + ia\tilde{\lambda}v = 0, \quad (3.73)$$

where we introduced $\lambda = a\tilde{\lambda}$ such that $\tilde{\lambda} = O(1)$. With the Liouville transformation

$$v(\xi) = \xi^{\frac{1-d}{2}} e^{-\frac{1}{4}ia\xi^2/(1-i\varepsilon)} Z(\xi),$$

this equation is put in self-adjoint form. For ξ large, we obtain to leading order

$$(1 - i\varepsilon)Z_{\xi\xi} + \left(\frac{a^2\xi^2}{4(1-i\varepsilon)} - 1 + \frac{ia}{2} \left(\frac{2}{\sigma} + 2\tilde{\lambda} - d \right) \right) Z = 0. \quad (3.74)$$

The aim is to match the leading order behavior of solutions of this equation with exponentially decaying solutions in the bump region. From equation (3.72), it follows that this exponential decay takes the form $(C_1 + iC_2)e^{-\sqrt{a}s}$. Note that it is possible to rescale the phase of Z such that in the region where we match with the bump region Z is real-valued. Therefore, we try to find a real-valued solution of equation (3.74). We split Z in a real part Z_1 and an imaginary part Z_2 to obtain to leading order the system of equations

$$\begin{aligned} Z_{1\xi\xi} + Z_1 \left(\frac{a^2\xi^2}{4} - 1 \right) - \frac{a^3\xi^2}{2} K_0 Z_2 &= 0, \\ Z_{2\xi\xi} + Z_2 \left(\frac{a^2\xi^2}{4} - 1 \right) + \frac{a^3\xi^2}{2} K_0 Z_1 &= 0. \end{aligned}$$

From this system, it follows, ignoring for the moment that there is a turning point at $\xi = \frac{2}{a}$, that the solution $Z_2 = 0$ is, at leading order, valid as long as $\xi \ll a^{-\frac{3}{2}}$. Now, Z_1 satisfies $Z_{1\xi\xi} + Z_1 \left(\frac{a^2\xi^2}{4} - 1 \right) = 0$, which can be solved with the WKBJ method. Upon setting $x = \frac{a\xi}{2}$, such that $x = O(1)$ in Regions 2 and 4, it follows that

$$Z_{1xx} - \frac{4}{a^2} q_1(x) Z_1 = 0, \quad (3.75)$$

where $q_1(x) := 1 - x^2$. With a WKBJ analysis, we obtain

$$Z_1 = \frac{1}{q_1(x)^{1/4}} \left(C_- e^{-\frac{2}{a} \int_1^x \sqrt{q_1(\tilde{s})} d\tilde{s}} + C_+ e^{\frac{2}{a} \int_1^x \sqrt{q_1(\tilde{s})} d\tilde{s}} \right). \quad (3.76)$$

Since $q_1(x)$ is zero at $x = 1$, i.e. at $\xi = \frac{2}{a}$, the WKBJ expansion is not valid near this so-called turning point due to secular growth. A leading order approximation of the integrals in (3.76) on the left hand side of the turning point is given by:

$$\int_1^x \sqrt{q_1(\tilde{s})} d\tilde{s} = \int_1^x \sqrt{1 - \tilde{s}^2} d\tilde{s} = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2} \arcsin x + C. \quad (3.77)$$

Since we are only interested in whether matching is possible and not in the values of the constants that result from the matching conditions, we simplify notation and use the same constant C in all expressions. Thus, the constant C will denote a “general” constant and will be used in this and forthcoming sections.

First, we match Z_1 on both the left and the right hand side of the bump region (regions 2 and 4) with the exponential behavior of the bump region (Region 3). From (3.72), we obtain to leading order

$$v_{\text{bump}}(s) := e^{-i\frac{1}{2}\kappa_0 s} \left(W_{\text{bump},1}(s) + iW_{\text{bump},2}(s) \right) = C e^{-\frac{ik_0 s}{2}} e^{\pm \sqrt{1 - \frac{\kappa_0^2}{4}} s}.$$

If $s = a^{-\gamma}t$, where $0 < \gamma < 1$ and $t = O(1)$, both the WKBJ solution and v_{bump} are valid. So, for v_{bump} , we obtain to leading order

$$v_{\text{bump}}(a^{-\gamma}t) = C e^{-\frac{1}{2}ik_0 a^{-\gamma}t} e^{\pm \sqrt{1 - \frac{\kappa_0^2}{4}} a^{-\gamma}t}, \quad (3.78)$$

and, for

$$v_1(\xi) := \xi^{\frac{1-d}{2}} e^{-\frac{1}{4}ia\xi^2/(1-i\varepsilon)} Z_1(\xi),$$

we obtain to leading order

$$v_1\left(\frac{\kappa_0}{a} + \frac{t}{a^\gamma}\right) = \left(\frac{\kappa_0}{a} + \frac{t}{a^\gamma}\right)^{\frac{1-d}{2}} e^{-\frac{1}{4}ia\left(\frac{\kappa_0}{a} + \frac{t}{a^\gamma}\right)^2} Z_1\left(\frac{\kappa_0}{a} + \frac{t}{a^\gamma}\right) = C \left(\frac{a}{\kappa_0}\right)^{\frac{d-1}{2}} e^{-\frac{1}{2}ik_0 a^{-\gamma}t} C_{\pm} e^{\pm \sqrt{1 - \frac{\kappa_0^2}{4}} a^{-\gamma}t}, \quad (3.79)$$

where the $+$ sign should be chosen for $t < 0$ and the $-$ sign should be chosen for $t > 0$. Expressions (3.78) and (3.79) match perfectly. We have now extended the exponentially decaying solutions as s tends to $+\infty$ in the bump region to Region 4a, and the exponentially decaying solutions as s tends to $-\infty$ to Region 2, see Figure 1.3.2 or Definition 1.3.6.

3.2.8.3 Matching the WKBJ solutions with the inner region

In the inner region, Region 1, $|\xi| \ll \frac{1}{a}$, see Definition 1.3.6. Since $|Q| \ll 1$, eigenvalue equation as formulated by (3.73) is also valid in this region. We want to study the solution to this equation for small ξ . The solution is as in Section 2.2.1 given in terms of modified Bessel functions:

$$v_{\text{inner}}(\xi) = C(a)\xi^{\frac{2-d}{2}} I_{\frac{d-2}{2}}(\xi). \quad (3.80)$$

And the asymptotic form for large ξ resembles (2.8). The matching of

$$Z_{\text{inner}}(\xi) := \xi^{\frac{d-1}{2}} e^{\frac{1}{4}ia\xi^2/(1-i\varepsilon)} v_{\text{inner}}(\xi)$$

with $Z_1(\xi)$, as given by (3.76), is completely similar as the matching procedure presented in Section 2.3.2. The matching can always be done and there are no conditions arising from the matching procedure.

3.2.8.4 The turning point

The analysis in this section closely resembles the one presented in Section 2.3.6. However, in order to derive Lemma 3.2.10 in Section 3.2.8.7, we choose to represent a considerable part of the analysis. Our aim in this section is to construct a solution to equation (3.74) in Region 4b, see Figure 1.3.2 and Definition 1.3.6. Following the reasoning presented in Section 2.3.6, we conclude that equation (3.74) admits a turning point at $\xi = \frac{2}{a}$, i.e. at $x = 1$, where the

coefficient of Z in this equation vanishes at leading order. Therefore, a WKBJ solution is not valid.

We again consider equation (3.74), this time, however, we are not looking for a real-valued solution. Thus, we set $z = \frac{a\xi}{2\sqrt{1-i\varepsilon}}$, zoom in around the turning point by setting $\tilde{z} = a^{-\beta_1}(z - 1)$ and pose the leading order expansion

$$Z_{\text{turning}}(z) = a^{\beta_2} \tilde{Z}(\tilde{z}), \quad (3.81)$$

where by $Z_{\text{turning}}(z)$ we denote the solution of equation (3.74) in Region 4b. From equation (3.74), we obtain to leading order

$$\tilde{Z}_{\tilde{z}\tilde{z}} - \left(-8\tilde{z} - 2ia^{2\beta_1-1} \left(\frac{2}{\sigma} + 2\tilde{\lambda} - d \right) \right) \tilde{Z} = 0$$

and from balancing terms, we obtain $\beta_1 = \frac{2}{3}$. The solution of this equation is given by Airy functions:

$$\tilde{Z}(y) = C_3 \text{Ai}(y) + C_4 \text{Bi}(y),$$

where $y = -2\tilde{z} - \frac{ia^{\frac{2}{3}}}{2} \left(\frac{2}{\sigma} + 2\tilde{\lambda} - d \right)$.

3.2.8.5 A WKBJ analysis in Region 4c

In this section, we construct a solution to equation (3.74) in Region 4c, see Definition 1.3.6. The analysis in this section closely resembles the one presented in Section 2.3.4. Since, as also stated in Section 2.3.4, the solution of equation (3.74) in which we are interested will not be real-valued in this region, we cannot proceed as in Section 3.2.8.2. We construct a solution of equation (3.74) with the complex WKBJ method.

With the rescaling $z = \frac{a\xi}{2\sqrt{1-i\varepsilon}}$, we transform equation (3.74) to

$$Z_{rzr} - \frac{4}{a^2} q_2(x) Z_r = 0, \quad (3.82)$$

where $q_2(x) := 1 - z^2 - \frac{ia}{2} \left(\frac{2}{\sigma} + 2\tilde{\lambda} - d \right)$ and by Z_r we denote the solution of equation (3.74) in Region 4c, i.e. on the right hand side of the turning point. This equation has the same form as equation (3.75). Therefore, the solution is, similarly to (3.76), to leading order given by

$$Z_r = \frac{1}{q_2(z)^{1/4}} \left(C_{r+} e^{\frac{2i}{a} \int_1^z \sqrt{-q_2(\tilde{s})} d\tilde{s}} + C_{r-} e^{-\frac{2i}{a} \int_1^z \sqrt{-q_2(\tilde{s})} d\tilde{s}} \right). \quad (3.83)$$

Note that $-q_2(s)$ is at leading order real-valued and positive in Region 4b. We evaluate to leading order

$$\begin{aligned} \int_1^z \sqrt{-q_2(\tilde{s})} d\tilde{s} &= \int_1^z \sqrt{\tilde{s}^2 - 1} + \frac{ia \left(\frac{2}{\sigma} + 2\tilde{\lambda} - d \right)}{4\sqrt{\tilde{s}^2 - 1}} d\tilde{s} \\ &= \frac{1}{2} z \sqrt{z^2 - 1} + \left(\frac{ia}{4} \left(\frac{2}{\sigma} + 2\tilde{\lambda} - d \right) - \frac{1}{2} \right) \log \left(z + \sqrt{z^2 - 1} \right) + C. \end{aligned} \quad (3.84)$$

We now match with (3.81), i.e. with the solution around the turning point, by introducing the intermediate scale $z_\eta = \frac{z - \sqrt{\frac{1}{1-i\varepsilon}}}{a^\eta}$, with $0 < \eta < \frac{2}{3}$. Using the asymptotics of $\text{Ai}(y)$ and $\text{Bi}(y)$,

(3.81) becomes to leading order

$$Z_{\text{turning}} = a^{\beta_2} \frac{C}{(a^\eta z_\eta)^{\frac{1}{4}} a^{-\frac{1}{6}}} \left[C_3 \cos \left(\sqrt{2} \frac{4}{3a} (a^\eta z_\eta)^{\frac{3}{2}} - \frac{\pi}{4} \right) + C_4 \sin \left(\sqrt{2} \frac{4}{3a} (a^\eta z_\eta)^{\frac{3}{2}} - \frac{\pi}{4} \right) \right], \quad (3.85)$$

which should match with Z_r . We evaluate to leading order

$$\begin{aligned} \int_1^{1+a^\eta z_\eta} \sqrt{q_2(\tilde{s})} d\tilde{s} &= \int_1^{1+a^\eta z_\eta} \sqrt{1 - \tilde{s}^2} - \frac{ia \left(\frac{2}{\sigma} + 2\tilde{\lambda} - d \right)}{4 \sqrt{1 - \tilde{s}^2}} d\tilde{s} \\ &= \int_1^{1+a^\eta z_\eta} \sqrt{-2(\tilde{s} - 1)} - \frac{ia \left(\frac{2}{\sigma} + 2\tilde{\lambda} - d \right)}{4 \sqrt{-2(\tilde{s} - 1)}} d\tilde{s} \\ &= \frac{2}{3} i \sqrt{2} (a^\eta z_\eta)^{\frac{3}{2}} - \frac{a \left(\frac{2}{\sigma} + 2\tilde{\lambda} - d \right)}{2 \sqrt{2}} \sqrt{a^\eta z_\eta} \\ &= \frac{2}{3} i \sqrt{2} (a^\eta z_\eta)^{\frac{3}{2}}, \end{aligned}$$

which gives

$$Z_r = \frac{1}{(a^\eta z_\eta)^{1/4}} \sum_{\pm} C_{r\pm} \exp \left[\pm \frac{4}{3a} i \sqrt{2} (a^\eta z_\eta)^{\frac{3}{2}} \right]. \quad (3.86)$$

We conclude that (3.85) matches with (3.86) if $\beta_2 = -\frac{1}{6}$.

The matching of Z_{turning} with Z_1 , the solution in Region 4a, can be done in a completely similar fashion.

3.2.8.6 Matching the WKB solutions with the far field

In the far field, Region 5, $\xi \gg \frac{1}{a}$ and it is located far right of the turning point, see Definition 1.3.6.c. Therefore, we introduce the intermediate scale $\xi = \frac{2}{a} + a^{-\gamma} t$, where $1 < \gamma < 2$. This gives at leading order $z = \frac{1}{2} a^{1-\gamma} t$ and $\xi = a^{-\gamma} t$.

As in Section 2.2.3, the far field behavior of the solution of equation (3.73) is to leading order given by

$$v_\infty = C \xi^{-\frac{1}{\sigma} - \tilde{\lambda} - \frac{i}{a}} = C (a^{-\gamma} t)^{-\frac{1}{\sigma} - \tilde{\lambda} - \frac{i}{a}}. \quad (3.87)$$

Alternatively, one could use the far field behavior of solutions of the parabolic cylinder equation (3.82) which is, for example, given by expression (12.9.1) in [33], this leads to the same expression. If we choose $C_{r-} = 0$ and use the leading order approximation $z = \frac{1}{2} a^{1-\gamma} t$ in (3.83) and (3.84), we obtain to leading order

$$\begin{aligned} v_r &:= \xi^{\frac{1-d}{2}} e^{-\frac{1}{4} i a \xi^2} Z_r = \xi^{\frac{1-d}{2}} e^{-\frac{1}{4} i a \xi^2 / (1-i\varepsilon)} Z_r \\ &= C (a^{-\gamma} t)^{\frac{1-d}{2}} \exp \left(-\frac{ia^{1-2\gamma} t^2}{4(1-i\varepsilon)} \right) (a^{-\gamma} t)^{-\frac{1}{2} - \frac{1}{\sigma} + \frac{d}{2}} \exp \left(\frac{ia^{1-2\gamma} t^2}{4(1-i\varepsilon)} \right) \exp \left(-\frac{i \ln a^{-\gamma} t}{a} + o \left(\frac{\ln a^{-\gamma}}{a} \right) \right) \\ &= C (a^{-\gamma})^{-\frac{1}{\sigma} - \tilde{\lambda} - \frac{i}{a}}. \end{aligned} \quad (3.88)$$

If we would not choose $C_{r-} = 0$, then we would introduce an exponentially growing term in equation (3.88). Thus, the WKB solution (3.83) matches with the far field behavior given by (3.87) iff $C_{r-} = 0$.

The function v_r , as defined by (3.88), only decays (and is only bounded) if $-\frac{1}{\sigma} - \text{Re}(\tilde{\lambda}) < 0$ holds. Thus, from the far field analysis, we conclude that the analysis in Section 3.2 is valid only if

$$-\frac{1}{\sigma} - \text{Re}(\tilde{\lambda}) < 0.$$

This concludes the proof of Proposition 3.2.9.

3.2.8.7 Exponential localization of eigenfunctions in the bump region

As stated in Section 3.2.8.6, we can only match with a decaying (or bounded) solution in the far field if $C_{r-} = 0$. In this section, we show that, if $|\lambda| \ll 1$, only solutions that are exponentially localized in the bump region match with algebraic solutions in the far field. We formulate the result in a lemma.

Lemma 3.2.10. *Let $v = e^{\lambda\tau}v_0(\xi)$ solve (3.2) with v_0 bounded and $\lambda = O(a)$, then $|v_0(s)|$, where $s = \xi - \frac{\kappa_0}{a}$, is at leading order exponentially decaying as $|s|$ tends to ∞ , i.e. in the scale $|s| = a^{-\gamma}t$ with $0 < \gamma < 1$ and $0 < t = O(1)$.*

Proof. As stated, to match with algebraic behavior in the far field, it is required that $C_{r-} = 0$. This implies that (3.85) matches with (3.86) iff $iC_3 = C_4$. In the intermediate scale in between Region 4a and 4b, on the left hand side of the turning point, the Airy $\text{Ai}(\tilde{z})$ function grows exponentially as $\tilde{z}$ tends to $-\infty$ and the Bairy $\text{Bi}(\tilde{z})$ function decays exponentially as $\tilde{z}$ tends to $-\infty$. By continuing further through Region 4a, we conclude that this can only be matched with exponential decay from the bump region. A similar argument hold on the left hand side of the bump region (Regions 1 and 2). $\square$

3.3 Bounded solutions in the bump region

In this section, we analyze solutions of eigenvalue problem (3.4) that satisfy Property 3.1.2. Due to the reasoning in the following paragraphs, we expect solutions that satisfy Property 3.1.2 to correspond to the essential spectrum of the linear operator $L_V(a)$ in eigenvalue problem (3.4).

We first consider the case of a linear second order differential operator $L'(x)$, where $x \in \mathbb{R}$, with asymptotically constant coefficients as also considered in the appendix of Chapter 5 in [19]. For such an operator, the essential spectrum consists of those λ for which there is a solution u in the kernel of $L' - \lambda$ such that: u is bounded, but does not decay for either x tending to ∞ or x tending to $-\infty$. These bounded solutions in the kernel of $L' - \lambda$ appear in two-parameter families, since imaginary eigenvalues appear in complex conjugate pairs. Thus, this means that there is a one-parameter family of bounded solutions. This parametrizes the essential spectrum, see [19]. Thus, we conclude that if a linear operator has asymptotically constant coefficients, then the essential spectrum can be found by analyzing solutions of $L' - \lambda$ that are bounded for large x .

The described method to determine the essential spectrum is only derived for systems with asymptotically constant coefficients. The linear operator $L_V(a)$ as defined by (3.3), which we are studying in this chapter, does not have asymptotically constant coefficients. However, as explained in Section 3.1, this operator can, in the bump region, be seen as a perturbation of the linear operator that also arises in the NLS. And, as is the case in the NLS, the linear operator $L_V(a)$ does, in the bump region, have asymptotically constant coefficients, i.e. in the scale $s = a^{-\gamma}t$ with $0 < \gamma < 1$. Consequently, in Section 3.3.1, we show that, at leading order, the spectrum that arises by studying solutions that satisfy Property 3.1.2 indeed corresponds to the essential spectrum that arises in the NLS.

The family of, in the bump region, bounded solutions that parametrizes this part of the spectrum will be taken as a starting point. There are now two questions that remain unanswered. First, can this family of bounded solutions be extended (with a leading order matching procedure) to the whole domain in such a way that Property 3.1.2 is satisfied? If this is possible, we have constructed a family, parametrized by λ of bounded solutions in the kernel of $L_V(0) - \lambda$. Since, discrete spectrum is by definition isolated (see [19]), this corresponds to essential spectrum. And second, can we through a perturbation analysis determine the perturbed (essential) spectrum?

Answering the latter question in the previous paragraph appears to be enough to conclude that potential essential spectrum does not lead to instability: we will show that if any essential spectrum would exist, then it must be contained in the subset $\{z \in \mathbb{C} : \text{Re } z < 0\}$. This then results in Theorem 3.3.3 which states that the essential spectrum does not lead to spectral instability.

We now write eigenvalue problem (3.4) in a form that is convenient for the analysis performed in this section. Consider equation (3.4) and write $v = v_1 + iv_2$ and $\tilde{v} = v_1 - iv_2$. Note that $\tilde{v}$ is not the complex conjugate of v in the eigenvalue problem, since in the eigenvalue problem the components v_1 and v_2 of V can be complex! We obtain the system:

$$-v + (1 - i\varepsilon) \left(v_{\xi\xi} + \frac{d-1}{\xi} v_{\xi} \right) + ia \left(\xi v_{\xi} + \frac{1}{\sigma} v \right) + (1 + ibKa) \left((\sigma + 1)|Q|^{2\sigma} v + \sigma|Q|^{2\sigma-2} Q^2 \tilde{v} \right) = -i\lambda v \quad (3.89a)$$

$$-\tilde{v} + (1 + i\varepsilon) \left(\tilde{v}_{\xi\xi} + \frac{d-1}{\xi} \tilde{v}_{\xi} \right) - ia \left(\xi \tilde{v}_{\xi} + \frac{1}{\sigma} \tilde{v} \right) + (1 - ibKa) \left((\sigma + 1)|Q|^{2\sigma} \tilde{v} + \sigma|Q|^{2\sigma-2} \bar{Q}^2 v \right) = i\lambda \tilde{v} \quad (3.89b)$$

Remark 3.3.1. Note, that if $(v, \tilde{v}, \lambda)$ solves this system, then $(\bar{\tilde{v}}, \bar{v}, \bar{\lambda})$ solves this system as well. Thus, the spectrum is symmetric in the real axis.

Remark 3.3.2. In this section, we do not restrict ourselves to the case $\sigma = 1$.

3.3.1 Bump region

In this section, we study solutions to (3.89) in the bump region far away from the bump, i.e. for $\xi = \frac{\kappa}{a} + s$, $s = a^{-\gamma} t \ll \frac{1}{a}$ with $t = O(1)$ and $0 < \gamma < 1$, that satisfy Property 3.1.2.a or 3.1.2.b. In this scale, differential equation (3.89) has at leading order constant coefficients. Moreover, since Q is exponentially small in this scale, see Section 2.2.2.1, system (3.89) decouples in this scale. We only study the first equation of the system in the bump region, since the second equation is the complex conjugate. The full spectrum can be found by reflection in the real axis. We pose the following expansions:

$$v(s) = v_0(s) + av_1(s) + O(a^2), \quad (3.90a)$$

$$\lambda = \lambda_0 + a\lambda_1 + O(a^2). \quad (3.90b)$$

At leading order, we obtain

$$v_0'' + ik_0 v_0' + (i\lambda_0 - 1)v_0 = 0,$$

where by ' we denote differentiation with respect to s . As stated, we look for bounded solutions. Taking

$$v_0 = C_0 e^{i\zeta_0 s} \quad (3.91)$$

leads to

$$i\lambda_0 = 1 + \zeta_0^2 + \kappa_0 \zeta_0. \quad (3.92)$$

This set, parametrized by ζ_0 , corresponds to the imaginary axis from which the segment $i[-\alpha, \alpha]$ is removed, here as before $\alpha = 1 - \frac{\kappa_0^2}{4} > 0$. As stated in Section 3.1, the unperturbed NLS spectrum will be taken as a starting point for our study and we recognize this as the essential spectrum of the NLS, see [25].

We proceed with the analysis at next order. The equation for v_1 is given by

$$v_1'' + i\kappa_0 v_1' + (i\lambda_0 - 1)v_1 = -\frac{d-1}{\kappa_0} v_0' + iK_0 v_0'' - isv_0' - \frac{i}{\sigma} v_0 - i\lambda_1 v_0. \quad (3.93)$$

Our goal is to determine all λ_1 for which this equation has a bounded solution. Furthermore, this may also yield restrictions on ζ_0 . Equation (3.93) can be solved using variation of constants. It follows that in order to prevent secular growth in v_1 , the $-isv_0'$ term in the right hand side of (3.93) has to vanish. The $-isv_0'$ term in the right hand side of (3.93) also distinguishes this case from the one discussed in [19]. Using (3.91) leads to the following conclusions:

$$\zeta_0 = 0, \quad (3.94a)$$

$$v_0(s) = C_0, \quad (3.94b)$$

$$\lambda = \lambda_0 + a\lambda_1 + o(a) = -i - \frac{a}{\sigma} + o(a). \quad (3.94c)$$

Whether essential spectrum exists or not, we can already conclude that it will not lead to spectral instability since $\text{Re } \lambda < 0$.

In problems where the linear operator has asymptotically constant coefficients, the methods in [19] can also be used to determine the perturbed essential spectrum. Note that the above perturbation analysis would in those cases lead to the same result.

3.3.2 Outside the bump region

So far, we have constructed bounded solutions in the bump region, but far away from the bump. Since the solution is defined on an interval, this information enables us to find a unique solution of equation (3.89), i.e. we can extend it to a solution on the whole domain. From $\zeta_0 = 0$, we concluded that v_0 is constant. Upon inspection, it can be verified that the constant solution indeed solves equation (3.89) at leading order in all regions away from the bump in the bump region if $\lambda = -i + \frac{a}{\sigma}$.

Note that a construction in the bump region itself is hard, since then the nonlinear terms in $\mathcal{Q}$ will play a role, we will not go into those details.

3.3.3 Conclusion

We formulate the main conclusion of this section in a theorem.

Theorem 3.3.3. *Let $0 < a \ll 1$ and let σ and d be such that Assumption (1.2.2). Then, the spectrum of the linear operator $L_V(a)$ defined by eigenvalue problem (3.4) that arises from studying solutions of (3.4) that satisfy Property 3.1.2 is either empty or contained in the set $\{z \in \mathbb{C} : \text{Re } z < 0\}$.*

Note that, although there is no essential spectrum that leads to instability, there is a possibility that eigenvalues pop out of the essential spectrum, see [25]. Since this type of analysis will be technical, we, currently, will not investigate this possibility for two reasons. First, as mentioned in Section 1.3.1, numerically we do not expect any spectral instability arising from radially symmetric perturbations for the m_2^{upper} -solution. Second, in Chapter 4, we will show that the m_2^{upper} -solution is unstable anyway with respect to non-radially symmetric perturbations.

3.4 Conclusions

We briefly summarize the findings of the previous sections. We analyzed the stability of the radially symmetric m_2 -solution $Q_{m_2}(\xi)$ of equation (2.3) with a bump at $\xi = \frac{\kappa}{a}$, see Figure 1.3.1 and Definition 2.2.2. An asymptotic construction in the small parameter a of the m_2 -solution is presented in [9] and is extended to higher order in Chapter 2. Upon analyzing the stability, we arrived at eigenvalue problem (3.4). We investigated two types of solutions of this eigenvalue problem: solutions that decay in the bump region and solutions that remain bounded (but do not decay) in the bump region, see also Properties 3.1.1 and 3.1.2.

In Section 3.3, we showed that spectrum that arises from solutions that satisfy Property 3.1.2, i.e. the perturbed NLS essential spectrum, is contained in the subset $\{z \in \mathbb{C} : \text{Im}(z) \leq 0\}$ and thus does not lead to spectral instability, see Theorem 3.3.3. Furthermore, potential eigenvalues popping out of the essential spectrum are not investigated, see Section 3.3.3 and [25].

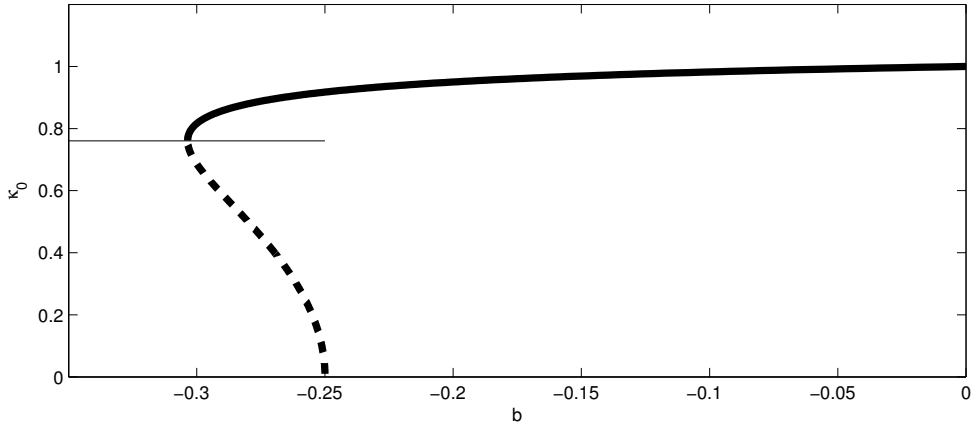
In Section 3.2, we analyzed exponentially decaying solutions in the bump region, i.e. solutions that satisfy Property 3.1.1. For $a = 0$, there is an eigenvalue $\lambda = 0$ with algebraic multiplicity 4 corresponding to certain symmetries among which a phase and a translation invariance, see Section 3.2.3. These eigenvalues correspond to the unperturbed NLS point spectrum. The eigenfunctions decay exponentially in the bump region. The perturbation breaks the translation invariance and three of these eigenvalues perturb and become $O(a)$ close to 0. One of the eigenvalues becomes positive, but corresponds to an irrelevant symmetry, see Remark 3.2.3.

In Section 3.2.8, we matched solutions that are exponentially localized in the bump region to the far field and inner region in such a way that Property 3.1.1 is satisfied, see also Figure 1.3.2. We showed that this is only possible if the inequality $\text{Re} \frac{\lambda}{a} > -\frac{1}{\sigma}$ holds, see Proposition 3.2.9. Since this result does not influence the right half of the complex plane, it has no impact on any conclusions drawn concerning the stability of the m_2 -solution.

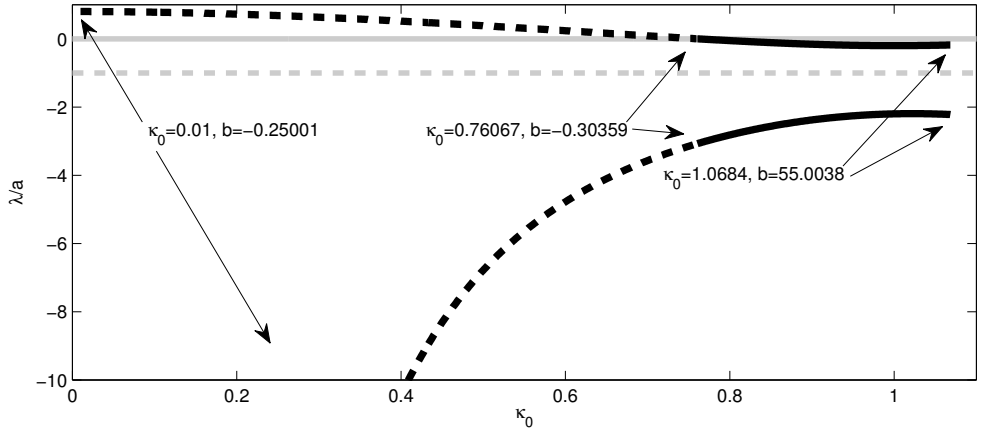
From Corollary 2.2.7, it follows that the system (2.3) combined with Assumption 2.2.1 undergoes a saddle-node bifurcation upon (for $\sigma = 1$) passing the curve $D(d, b, \sigma = 1) = 0$ in the strip $2 < d < 3$ of the (d, b) -plane, where D is defined by (2.43), see also Figure 2.2.1.

When only the perturbed NLS spectrum is considered (for $\sigma = 1$), it follows from Lemma 3.68, Lemma 3.2.7 and Corollary 3.2.8 that only at the saddle-node bifurcation the stability of the m_2 -solution can change. From this, it can be verified that the m_2^{upper} -solution is spectrally stable and that the m_2^{lower} -solution is spectrally unstable, see Figure 3.4.1a.

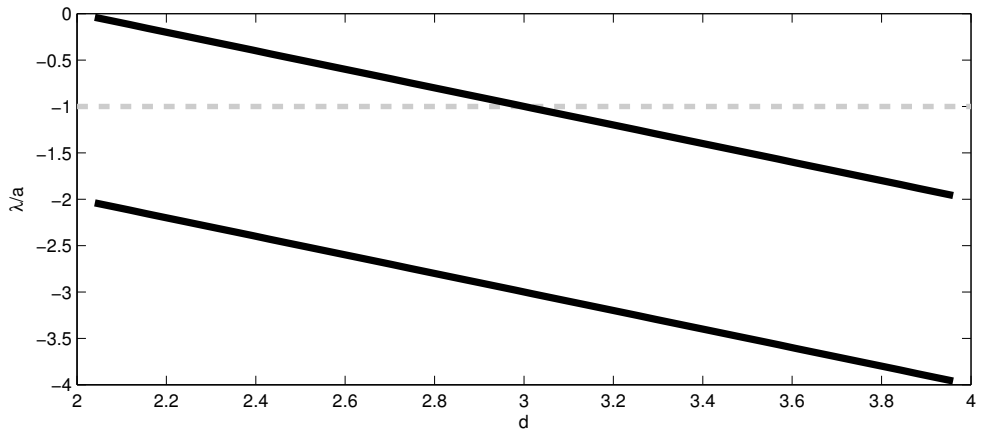
Finally, through expressions (3.62), (3.63) and (2.26), the location of the two non-trivial eigenvalues ($\lambda = 0$ and $\lambda = 2a$ are considered trivial) can be computed through numerically solving a system of equations. For two specific examples, we have constructed a graphical representation of this, see Figures 3.4.1b and 3.4.1c. Figure 3.4.1b is directly related to Figure 3.4.1a. This figure shows for each point on the curve displayed in Figure 3.4.1a and parametrized by κ_0 the location of the two nontrivial real-valued zeros ($\lambda = 0$ and $\lambda = 2a$



(a) The saddle-node bifurcation of the m_2 -solution for $\sigma = 1$ and $d = 2.2$, see Corollary 2.2.7. When only the perturbed NLS spectrum is considered, the upper branch corresponding to the m_2^{upper} -solution is spectrally stable, and the lower branch corresponding to the m_2^{lower} -solution is spectrally unstable.



(b) The location of the two nontrivial real-valued zeros ($\lambda = 0$ and $\lambda = 2a$ are trivial) of the Evans functions as given by (3.62) for each point on the curve displayed in Figure 3.4.1a and parametrized by κ_0 .



(c) The location of the two nontrivial zeros ($\lambda = 0$ and $\lambda = 2a$ are trivial) of the Evans functions as given by (3.62) for $\sigma = 1$, $b = 0$ and d ranging from 2 to 4.

Figure 3.4.1: Several stability results. Solid lines indicate spectral stability, dashed lines indicate spectral instability and the gray dashed lines indicate the threshold at $\text{Re } \lambda = -\frac{a}{\sigma}$, see Proposition 3.2.9.

are trivial) of the Evans functions as given by (3.62). The solid lines correspond to spectral stability and the dashed lines correspond to spectral instability. Exactly at the saddle-node bifurcation, one of the eigenvalues crosses the imaginary axis. The gray dashed lines indicate the threshold at $\operatorname{Re} \lambda = -\frac{a}{\sigma}|_{\sigma=1} = -a$, see Proposition 3.2.9. This proposition implies that everything below the gray dashed line should be “ignored”. Since although for these values of λ there exists a solution V of eigenvalue problem (3.4) that satisfies Property 3.1.1.a and 3.1.1.b, this solution does not satisfy Property 3.1.1.c and 3.1.1.d. Figure 3.4.1c is constructed and can be interpreted in a similar way as Figure 3.4.1b. In Figure 3.4.1c, the parameter $b = 0$, $\sigma = 1$ and d ranges from 2 to 4. For $b = 0$, only the m_2^{upper} -solution exists (see Lemma 2.2.6 and Figure 2.2.1) and, as can be seen in Figure 3.4.1c, the non-trivial eigenvalues are negative.

We formulate the main result of this chapter in a theorem.

Theorem 3.4.1. *Let $0 < a \ll 1$, let $\sigma = 1$, consider the region R in the (d, b) -plane defined by (2.42) and let $(d, b) \in R$. If $d < 3$, then the spectrum obtained by studying solutions of eigenvalue problem (3.4) that satisfy either Property 3.1.1 or 3.1.2 does not lead to instability for the m_2^{upper} -solution of (2.3) as defined by Definition 2.2.8 and does lead to instability for the m_2^{lower} -solution of (2.3) as defined by Definition 2.2.8. Furthermore, if $d \geq 3$, then the spectrum obtained by studying solutions of eigenvalue problem (3.4) that satisfy either Property 3.1.1 or 3.1.2 does not lead to instability for the m_2 -solution as defined by Definition 2.2.2.*

Proof. As in Theorem 2.5.1, we argue that the stability analysis in Chapter 3 using asymptotic methods can be made rigorous. However, we will refrain from such a purely technical exercise.

From Corollary 2.2.7, it now follows that only in the region R this solution can be constructed. Combining Lemma 3.2.5, Corollary 3.2.8, Proposition 3.2.9 and Theorem 3.3.3 leads to the desired result. $\square$

Corollary 3.4.2. *Under the conditions stated in Theorem 3.4.1, the m_2^{lower} -solution is linearly unstable with respect to radially symmetric perturbations.*

Proof. Spectral instability caused by an eigenvalue implies linear instability, see Remark 1.4.3. $\square$

Finally, we note that this result is in line with the numerical simulation in [9]. These simulations imply that the m_2 -solution is stable with respect to radially symmetric perturbations for $d = 3$ and $b = 0$.

Chapter 4

Stability with respect to non-radially symmetric perturbations

In this chapter, we study the spectral stability (see Definition 1.4.2) with respect to non-radially symmetric perturbations of the stationary m_2^{upper} -solution of equation (2.1) as defined by Definition 2.2.8. Recall that the m_2^{upper} -solution is constructed asymptotically in Section 2.2, see also [9, 40]. In Chapter 3, we showed that, when only the perturbed NLS spectrum is considered, the m_2^{upper} -solution is spectrally stable with respect to radially symmetric perturbations, see Theorem 3.4.1. We also showed in Chapter 3 that the m_2^{lower} -solution as defined by Definition 2.2.8 is spectrally (and linearly) unstable with respect radially symmetric perturbations, again see Theorem 3.4.1. Therefore, we do not study stability with respect to non-radially symmetric perturbation of the m_2^{lower} -solution.

In this chapter, we set ourselves the following goal. We want to investigate to what extent we can use the developed machinery in Chapters 2 and 3 to study spectral stability with respect to non-radially symmetric perturbations of the m_2^{upper} -solution.

The theory in this chapter is valid for all σ for which the m_2^{upper} -solution of equation (2.1) exists, where sufficient existence conditions are provided by Theorem 2.1.1. However, for technical reasons, the final result is only derived for $\sigma = 1$. In the text, it is indicated where such a restriction is made.

4.1 Set-up

We now extend the set-up in which we study spectral stability as presented in Sections 1.2.4 and 3.1 to a non-radially symmetric setting. We proceed as in Section 1.2.4 and start with equation (1.9) with boundary and initial conditions given by (1.10). After transforming to spherical coordinates, equation (1.9) becomes

$$i\Phi_t + (1 - i\varepsilon) \left(\Phi_{rr} + \frac{d-1}{r} \Phi_r + \frac{1}{r^2} \Delta_{S^{d-1}} \Phi \right) + (1 + ib\varepsilon) |\Phi|^{2\sigma} \Phi = 0, \quad (4.1)$$

where $x \in \mathbb{R}^d$, $r = |x| \geq 0$ is the radial coordinate and $\Delta_{S^{d-1}}$ is the Laplace-Beltrami operator on the unit sphere S^{d-1} in $\mathbb{R}^d$. For $d = 3$, the Laplace-Beltrami operator takes the form

$$\Delta_{S^{d-1}} = \partial_{\phi_1}^2 + \cot \phi_1 \partial_{\phi_1} + \frac{1}{\sin^2 \phi_1} \partial_{\phi_2}^2,$$

where $0 \leq \phi_1 \leq \pi$ is the polar angle and $0 \leq \phi_2 < 2\pi$ is the azimuthal angle. As in Chapter 3, our aim is to study spectral stability for non-integer values of the dimension d . We limit ourselves to the parameter range for d and σ as specified in Assumption 1.2.2. In Section 4.1.2, we explain how to interpret the Laplace-Beltrami operator for non-integer values of d .

4.1.1 The dynamical rescaling

Following the analysis in Section 1.2.4, we pose the transformation

$$\Phi(x, t) = \Phi(r, \phi, t) = (2a(T - t))^{-\frac{1}{2}(\frac{1}{\sigma} + i\frac{\omega}{a})} Q(\xi, \phi, \tau), \quad (4.2)$$

where: $\phi \in S^{d-1}$, $\xi = \frac{r}{\sqrt{2a(T-t)}}$ is the rescaled radial coordinate, $\tau = -\frac{1}{2a} \log \frac{T-t}{T}$ is the rescaled temporal coordinate, $\omega > 0$ and $0 < a \ll 1$. As in Section 1.2.4, we can, upon rescaling, without loss of generality restrict ourselves to the case $\omega = 1$, see also Assumption 1.3.2. Now, Q satisfies

$$iQ_\tau - Q + (1 - i\varepsilon) \left(Q_{\xi\xi} + \frac{d-1}{\xi} Q_\xi + \frac{1}{\xi^2} \Delta_{S^{d-1}} Q \right) + ia \left(\frac{1}{\sigma} Q + \xi Q_\xi \right) + (1 + ib\varepsilon) |Q|^{2\sigma} Q = 0. \quad (4.3)$$

From (1.10), it follows that the boundary condition is given by

$$|Q(\xi, \phi, \tau)| \rightarrow 0 \text{ as } \xi \rightarrow \infty. \quad (4.4)$$

4.1.2 Spectral stability analysis

In this section, we set up a framework that allows us to study spectral stability with respect to non-radially symmetric perturbations for non-integer values of d . Although we restricted the parameter range for d as specified in Assumption 1.2.2, the approach we present to generalize the study of non-radially symmetric spectral stability to non-integer values of d is general.

Consider equation (4.3) and assume for the moment that d is an integer. We are interested in stability of the radially symmetric m_2^{upper} -solution as defined by Definition 2.2.8. Thus, as in Section 3.1, we add a perturbation v to the stationary radially symmetric m_2^{upper} -solution $Q_{m_2^{\text{upper}}}(\xi)$ in rescaled variables:

$$Q(\xi, \phi, \tau) = Q_{m_2^{\text{upper}}}(\xi) + v(\xi, \phi, \tau). \quad (4.5)$$

Note that equation (4.5) can only be interpreted for integer values of d . Substitution into equation (4.3) and linearizing around the initially small perturbation $v(\xi, \phi, \tau)$ leads to

$$iv_\tau - v + (1 - i\varepsilon) \left(v_{\xi\xi} + \frac{d-1}{\xi} v_\xi + \frac{1}{\xi^2} \Delta_{S^{d-1}} v \right) + ia \left(\xi v_\xi + \frac{1}{\sigma} v \right) + (1 + b\varepsilon) \left((\sigma + 1) |Q|^{2\sigma} v + \sigma |Q|^{2\sigma-2} Q^2 \bar{v} \right) = 0. \quad (4.6)$$

We split equation (4.6) and also v in a real and imaginary part and denote the real and imaginary part of v by v_1 and v_2 respectively. Equation (4.6) can now be written as the system

$$\frac{d}{d\tau} V = L_V(a) V, \quad (4.7)$$

where V is the vector $(v_1, v_2)^\top$ and $L(a)$ is a 2×2 matrix containing differential operators. The next step is to expand both the real and imaginary part of v in terms of spherical harmonics. We refer to [46] Section 18.31, [22] and [30] for more information on spherical harmonics. This expansion takes the form

$$v_j(\xi, \phi, \tau) = \sum_{\ell=0}^{\infty} \sum_{m \in I(\ell, d)} v_j^{\ell, m}(\xi, \tau) Y_\ell^m(\phi),$$

where $j = 1, 2$, Y_ℓ^m is a spherical harmonic of degree ℓ and order m and $I(\ell, d)$ is a set of integers consisting of all valid orders for degree ℓ in dimension d . For example for $d = 3$, this set is given by

$$I(\ell, 3) = \{m \in \mathbb{Z} : -\ell \leq m \leq \ell\}.$$

Note that, although v_j is real-valued, the modes $v_j^{\ell, m}(\xi, \tau)$ can be complex-valued. This is not surprising and is completely analogous to the situation where a real-valued function is expanding in Fourier modes $e^{im\theta}$: there the coefficients of individual modes can also be complex. For every dimension d , the d -dimensional spherical harmonic $Y_\ell^m(\phi)$ satisfies

$$\Delta_{S^{d-1}} Y_\ell^m(\phi) = -\ell(\ell + d - 2) Y_\ell^m(\phi). \quad (4.8)$$

Note that the coefficient of $Y_\ell^m(\phi)$ on the right hand side of this expression does not depend on m . Furthermore, the spherical harmonics are orthogonal with respect to the inner product

$$\int_{S^{d-1}} f \bar{g} d\Omega_{d-1} = 0,$$

where Ω_{d-1} is the standard measure on S^{d-1} . Substitution of the expansions of both v_1 and v_2 into (4.7) and using that the spherical harmonics are orthogonal leads to

$$\frac{d}{d\tau} V = L_V(a, \ell) V, \quad (4.9)$$

where we slightly abuse notation and have dropped the ℓ, m -index for V , and $L_V(a, \ell)$ is a 2×2 matrix $[L_V^{ij}]$ containing differential operators. This matrix can be written in the scalar form:

$$\begin{aligned} L_V(a, \ell)(v_1, v_2)v_\tau = & -iv + (i + Ka) \left(v_{\xi\xi} + \frac{d-1}{\xi} v_\xi - \frac{\ell(\ell+d-2)}{\xi^2} v \right) - a \left(\xi v_\xi + \frac{1}{\sigma} v \right) \\ & + (i - bKa) \left((\sigma + 1)|Q|^{2\sigma} v + \sigma|Q|^{2\sigma-2} Q^2(v_1 - iv_2) \right), \end{aligned} \quad (4.10)$$

where $v = v_1 + iv_2$. The matrix elements L_V^{ij} can be found by writing this expression in the form: $L_V(a, \ell)w = L_V^{11}v_1 + L_V^{12}v_2 + iL_V^{21}v_1 + iL_V^{22}v_2$. We stress again that, for reasons explained above, v_1 and v_2 are not (necessarily) real-valued!

By substituting $e^{i\tau} V(\xi)$ for V into equation (4.9), we find the eigenvalue problem

$$L_V(a, \ell)V = \lambda V. \quad (4.11)$$

Note that, contrary to (4.5), eigenvalue problem (4.11) can be interpreted for non-integer values of d and ℓ . Thus, through formulating the eigenvalue problem in the form of equation (4.11), the non-radially symmetric spectral stability problem can be interpreted for non-integer dimensions.

The spectral stability of Q is determined by solutions of (4.11). We proceed as in Chapter 3 and study solutions of (4.11) that as function of the radial coordinate satisfy Property 3.1.1. In this way, we can again use the different regions used in the asymptotic construction of the solution, see Figure 1.3.2 and Definition 1.3.6. We start with an analysis in the bump region, since eigenvalue problem (4.11) has at leading order asymptotically constant coefficients in the bump region, i.e. for $\xi = \frac{\kappa_0}{a} + a^{-\gamma}t$ with $0 < \gamma < 1$. We are now in the same setting as described in Chapter 3. Therefore, as in Sections 3.2 and 3.3, we study exponentially localized solutions in to the bump region to find the discrete spectrum that arise from the perturbed NLS discrete spectrum. This study will be the topic of the forthcoming sections.

4.2 Localized solutions of (4.11) in the bump region

In this section, we study solutions of equation (4.11) that are localized in the bump region and that as a function of the radial coordinate satisfy Property 3.1.1. We will follow the approach presented in Section 3.2. In a similar way as in the asymptotic construction of the m_2 -solution, we zoom in on the bump region by setting $\xi = \frac{\kappa}{a} + s$ and set $v(\xi) = e^{-i\kappa_0 s/2} w(s)$ and $V(\xi) = e^{-i\kappa_0 s/2} W(s)$. From (4.6) and (4.9), we obtain that (4.11) reduces to

$$L(a, \ell)W = \lambda W. \quad (4.12)$$

Here $L(a, \ell)$ is a 2×2 matrix $[L^{ij}]$ which is most conveniently written in the form:

$$\begin{aligned} L(a, \ell)(w_1, w_2) := & (i + \varepsilon) \left(w_{ss} - i\kappa_0 w_s - \frac{\kappa_0^2}{4} w + a \frac{d-1}{\kappa + as} \left(w_s - \frac{i\kappa_0}{2} w \right) - a^2 \frac{\ell(\ell + d - 2)}{(\kappa + as)^2} w \right) \\ & - iw + -(\kappa + as) \left(w_s - i \frac{\kappa_0}{2} w \right) - \frac{aw}{\sigma} + (i - b\varepsilon) \left((\sigma + 1) |S|^{2\sigma} w + \sigma S^{\sigma+1} \bar{S}^{\sigma-1} (w_1 - iw_2) \right), \end{aligned} \quad (4.13)$$

where $S(s) = e^{i\kappa_0 s/2} Q(s)$ and $w = w_1 + iw_2$, see Section 2.2.2. Analogous to the notation used in (4.10), the matrix elements L^{ij} can be found by writing this expression in the form: $L(a)w = L^{11}w_1 + L^{12}w_2 + iL^{21}w_1 + iL^{22}w_2$.

If $\ell \ll \frac{1}{a}$, the term $-a^2 \frac{\ell(\ell+d-2)}{(\kappa+as)^2} w$ in equation (4.13) is of $o(1)$ and, therefore, the leading order of eigenvalue problem (4.12) is independent of ℓ . Thus, as in Section 3.2.1, the matrix $L(a, \ell)$ reduces for $a = 0$ to (3.7) and $L(0, \ell)$ is independent of ℓ . Furthermore, if $\ell \ll \frac{1}{\sqrt{a}}$, the term $-a^2 \frac{\ell(\ell+d-2)}{(\kappa+as)^2} w$ in equation (4.13) is of $o(a)$ and, therefore, $\partial_a L(0, \ell)$ is also independent of ℓ . From considering different orders of magnitude of ℓ and balancing terms in (4.13), we conclude that three different orders of magnitude for ℓ that have to be considered: $\ell = O(1)$, $\ell = O\left(\frac{1}{\sqrt{a}}\right)$ and $\ell = O\left(\frac{1}{a}\right)$.

Remark 4.2.1. *In order to remain in the setting of Chapter 3, we do not consider the case $\ell = O\left(\frac{1}{a}\right)$. Furthermore, in Section 4.2.2.1, we explain why it is not possible to apply the methods developed here and in Chapter 3 directly to the case $\ell = O\left(\frac{1}{\sqrt{a}}\right)$. Thus, we will restrict ourselves to the case $\ell = O(1)$ in this chapter.*

As in the radially symmetric case, the study of exponentially localized solutions in the bump region, i.e. solutions that satisfy Property 3.1.1, consists of several steps.

First, the spectrum of the leading order linear operator, which is $L(0, \ell)$ in the non-radially symmetric case, must be determined. Since $L(0, \ell)$ is independent of ℓ , this is completely similar to the radially symmetric case, i.e. the case $\ell = 0$. Thus, it follows that, as in Section

3.2.3, $\lambda = 0$ is an eigenvalue of $L(0, \ell)$ with algebraic multiplicity 4 and the (generalized) eigenfunctions are given by (3.13). This spectrum coincides with the unperturbed NLS point spectrum and it will be taken as a starting point for our study, see also Section 3.2.3.

The second step consists of a discussion of several symmetries, see Section 3.2.3. These symmetries are related to (generalized) eigenfunctions and eigenvalues of the leading order linear operator. Since, the leading order operator $L(0, \ell)$ is independent of ℓ , Section 3.2.3 holds for all ℓ of $O(1)$, i.e. not only for $\ell = 0$ which corresponds to the radially symmetric case.

The third step is to discuss a symmetry which is related to invariance under perturbing the blowup point itself and will be discussed in Section 4.2.1. We will show that the operator $L(a, \ell = 1)$ must have an eigenvalue $\lambda = a$. This symmetry discussion is similar to the symmetry that gave rise to the eigenvalue $\lambda = 2a$ of $L(a, \ell = 0)$ and which is related to invariance under perturbing the blowup time, see Section 3.2.4 and Theorem 3.2.4.

The first three steps can be seen as obtaining a priori information on the spectrum through using invariances under symmetries. The fourth step is the introduction of Evans function techniques, this is covered in Section 4.2.2. The Evans function is an analytic function associated to eigenvalues problem (4.12) and its zeros correspond to the eigenvalues of (4.12). The results derived by using the Evans function must of course be in line with the information about the spectrum obtained by using symmetry arguments.

The fifth step is to determine whether the eigenvalues and (generalized) eigenfunctions persist for the perturbed linear operator $L(a, \ell)$. Cases in which the eigenvalues do not persist correspond to a breaking of the symmetries that gave rise to the corresponding eigenvalue in the unperturbed linear operator $L(0, \ell)$. This analysis leads, similar to the one in Section 3.2.5, to solvability conditions that can be used to evaluate derivatives of the Evans function. These derivatives of the Evans function can be used to determine the perturbed eigenvalues by means of a Taylor expansion of the Evans function.

The analysis will end with a detailed discussion of the cases $\ell = 1$ and $\ell = 2$. We will show that both of these modes will lead to spectral instability. Note that this also makes the study of the higher order modes less relevant, see Remark 4.2.1.

4.2.1 Symmetries and an irrelevant positive eigenvalue $\lambda = a$

In Sections 3.2.3 and 3.2.4, we related symmetries of equations (1.15) and (3.14) to eigenvalues and corresponding eigenfunctions. For this, we used the blowup time T : a free parameter of rescaling (1.12) which was introduced in the construction presented in Section 1.2.4. In this section, we will use another free parameter: the position of the blowup point x_0 , i.e. the point to which the blowup contracts, to find a positive eigenvalue $\lambda = a > 0$ of $L(a, \ell = 1)$. This possibility to perturb x_0 is the second freedom introduced through rescaling (1.12). Note that in rescaling (1.12), we have implicitly chosen $x_0 = 0$. But, it can be seen that equation (2.1) for Q is independent of x_0 and thus x_0 is indeed a free parameters introduced through rescaling (1.12).

It is desirable to express the from the symmetry resulting eigenfunction in terms of the spherical modes Y_ℓ^m , since, then, it is easier to relate the results to the forthcoming Evans function analysis presented in Section 4.2.2. Therefore, we will first present an intuitive approach in Section 4.2.1.1 through which we can find the positive eigenvalue $\lambda = a$ and corresponding eigenfunctions by inspection. However, we still need a symmetry argument, as in Section 3.2.4, to argue that the corresponding positive eigenvalue is irrelevant.

4.2.1.1 Intuitive approach

In this section, we present an intuitive approach through which we derive that the operator $L(a, \ell = 1)$ has a positive eigenvalue $\lambda = a$. Through differentiation of equation (2.3a) with respect to ξ , it follows from (4.13) that $\partial_\xi Q(\xi)$ satisfies

$$\begin{aligned} -iL_V(a, \ell = 0)(\operatorname{Re} \partial_\xi Q, \operatorname{Im} \partial_\xi Q) &= \left[(1 - i\varepsilon) \left(\partial_{\xi\xi}^2 + \frac{d-1}{\xi} \partial_\xi \right) - Q + i \frac{a}{\sigma} Q + ia\xi \partial_\xi \right] \partial_\xi Q \\ &+ (1 + b\varepsilon) \left((\sigma + 1)|Q|^{2\sigma} \partial_\xi Q + \sigma|Q|^{2\sigma-2} Q^2 \overline{\partial_\xi Q} \right) = (1 - i\varepsilon) \frac{d-1}{\xi^2} \partial_\xi Q - ia \partial_\xi Q. \end{aligned}$$

If the term $(1 - i\varepsilon) \frac{d-1}{\xi^2} \partial_\xi Q$ would not be present on the right hand side, then $\partial_\xi Q$ would be an eigenfunction of $L(a, \ell)$ with eigenvalue $\lambda = a$. Equation (4.8) inspires to try the educated guess $(Y_\ell^m \operatorname{Re} \partial_\xi Q, Y_\ell^m \operatorname{Im} \partial_\xi Q)^\top$ as eigenfunction for $L(a, \ell)$. Note that the components of this vector are complex. A direct computation shows

$$L_V(a, \ell) \begin{bmatrix} Y_\ell^m \operatorname{Re} \partial_\xi Q \\ Y_\ell^m \operatorname{Im} \partial_\xi Q \end{bmatrix} = \left((i + \varepsilon) \frac{d-1}{\xi^2} - (i + \varepsilon) \frac{\ell(\ell + d - 2)}{\xi^2} + a \right) \begin{bmatrix} Y_\ell^m \operatorname{Re} \partial_\xi Q \\ Y_\ell^m \operatorname{Im} \partial_\xi Q \end{bmatrix}.$$

We conclude that choosing $\ell = 1$ leads for the m_2 -solution to the following lemma.

Lemma 4.2.2. *The operator $L(a, \ell)$ defined by (4.13) has for $\ell = 1$ and $m \in \{-1, 0, 1\}$ an eigenvalue $\lambda = a$ with corresponding eigenfunction given by*

$$\begin{bmatrix} Y_{\ell=1}^m \operatorname{Re} (S' - \frac{1}{2} i \kappa_0 S) \\ Y_{\ell=1}^m \operatorname{Im} (S' - \frac{1}{2} i \kappa_0 S) \end{bmatrix}.$$

4.2.1.2 Perturbing the blowup point x_0

A second way to derive the result obtained in Lemma 4.2.2 is to perturb the blowup point x_0 of the solution $\Phi(x, t, x_0)$ of equation (1.9). This section follows the same approach as the one presented in Section 3.2.4.2. As in Section 3.2.4.2, we will argue that eigenfunctions related to a symmetry that gives rise to a family of blowup solutions are not relevant in the present spectral stability analysis, since they correspond to perturbations in the direction of another blowup solution.

Note that symmetry arguments related to perturbing the blowup point itself in a non-radially symmetric setting can only be interpreted for integer values of d . We will generalize the result to non-integer values of d by not allowing for eigenfunctions that for integer values of d correspond to a perturbation in the direction of a different blowup solution. So, in this section, we will assume d to be an integer at first and then we will generalize the result to non-integer values of d . For existence of solutions, we restrict the range of the valid values for d as described in Assumption 1.2.2, although the arguments in this section are general and would hold for all integer-valued $d \geq 2$ if a valid m_2 -solution exists.

As in Section 3.2.4.2, we write rescaling (1.12), which is also used in the non-radially symmetric case (see (4.2)), for a perturbed blowup point

$$x_\delta := x_0 + \delta \Delta x_0,$$

where $0 < \delta \ll 1$ and $\Delta x_0 \in \mathbb{R}^d$ with $|\Delta x_0| = 1$, as:

$$\xi_\delta := \frac{|x - x_\delta|}{L(t)}, \quad \tau := -\frac{1}{2a} \log \frac{T - t}{T}, \quad \Phi(x, t, x_\delta) = L(t)^{-\frac{1}{\sigma}} e^{i\tau} Q(\xi_\delta), \quad (4.14)$$

where

$$L(t) := \sqrt{2a(T-t)}.$$

Since the rescaled equation for Q does not explicitly depend on the blowup point x_0 , see equation (1.15), it follows that we have constructed a blowup solution $\Phi(x, t, x_0)$ for every $x_0 \in \mathbb{R}^d$, i.e. the blowup profile is translation invariant and $\Phi(x, t, x_0)$ satisfies (1.9) for all $x_0 \in \mathbb{R}^d$. We expand $\Phi(x, t, x_\delta)$ in powers of δ and obtain

$$\begin{aligned} \Phi(x, t, x_\delta) &= L(t)^{-\frac{1}{\sigma}} e^{i\tau} Q(\xi_\delta) = L(t)^{-\frac{1}{\sigma}} e^{i\tau} \left(Q(\xi_0) + \delta Q'(\xi_0) \frac{\partial \xi_\delta}{\partial \delta} \Big|_{\delta=0} + O(\delta^2) \right) \\ &= L(t)^{-\frac{1}{\sigma}} e^{i\tau} \tilde{Q}(\xi_0, \tau, \delta), \end{aligned}$$

where

$$\begin{aligned} \tilde{Q}(\xi_0, \tau, \delta) &= \tilde{Q}_0(\xi_0) - \frac{\delta}{\sqrt{2aT}} \tilde{Q}_1(\xi_0, \tau) + O(\delta^2), \\ \tilde{Q}_0(\xi_0) &= Q(\xi_0), \\ \tilde{Q}_1(\xi_0, \tau) &= e^{a\tau} \frac{(x - x_0) \cdot \Delta x_0}{|x - x_0|} Q'(\xi_0) = e^{a\tau} Q'(\xi_0) \cos \angle(x - x_0, \Delta x_0), \end{aligned}$$

and where we have also used the relation $\frac{e^{a\tau}}{\sqrt{2aT}} = \frac{1}{L(t)}$. Here, we denoted by $\angle(x - x_0, \Delta x_0)$ the angle between the vectors $x - x_0$ and Δx_0 . Rescaling (4.14) is defined in such a way that $\tilde{Q}(\xi_0, \tau_0, \delta)$ satisfies equation (4.3) for all $\delta \ll 1$. Thus, we substitute $\tilde{Q}(\xi, \tau, \delta)$ into equation (4.3) and expand the resulting equation in powers of δ . At $O(1)$, we obtain that $\tilde{Q}_0(\xi_0)$ satisfies, as expected, equation (2.3a). At $O(\delta)$, we obtain that $\tilde{Q}_1(\xi_0, \tau)$ satisfies

$$\begin{aligned} &\left[(1 - i\varepsilon) \left(\partial_{\xi_0}^2 + \frac{d-1}{\xi_0} \partial_{\xi_0} + \frac{1}{\xi_0^2} \Delta_{0,S^{d-1}} \right) - 1 + \frac{ia}{\sigma} + ia\xi_0 \partial_{\xi_0} \right] \tilde{Q}_1(\xi_0, \tau) \\ &+ (1 + ib\varepsilon) \left[(\sigma + 1) |Q|^{2\sigma} \tilde{Q}_1(\xi_0, \tau) + \sigma |Q|^{2\sigma-2} Q^2 \overline{\tilde{Q}_1(\xi_0, \tau)} \right] + ia \tilde{Q}_1(\xi_0, \tau) = 0. \end{aligned} \quad (4.15)$$

Upon comparing this expression with equation (4.10), we conclude that

$$\begin{bmatrix} \operatorname{Re} Q'(\xi_0) \cos \angle(x - x_0, \Delta x_0) \\ \operatorname{Im} Q'(\xi_0) \cos \angle(x - x_0, \Delta x_0) \end{bmatrix}$$

is an eigenfunction of $L_V(a, \ell)$ corresponding to eigenvalue a . However, it is desirable to express this eigenfunction in terms of the spherical modes Y_ℓ^m , since then we can relate this result to the forthcoming Evans function analysis. In order to do so, we use the addition theorem for the spherical harmonic functions (see [22]):

$$\cos \angle(x, y) = \frac{x \cdot y}{|x||y|} = \frac{\sigma_d}{\#I(\ell = 1, d)} \sum_{m \in I(\ell=1, d)} \overline{Y_{\ell=1}^m(\phi_x)} Y_{\ell=1}^m(\phi_y), \quad (4.16)$$

where ϕ_x and ϕ_y are $d-1$ dimensional vectors and denote the angular coordinates of respectively the points x and y in $\mathbb{R}^d$, σ_d is the surface area of the unit sphere in dimension d (in particular $\sigma_3 = 4\pi$) and $\#I(\ell, d)$ is the number of spherical harmonics modes of degree ℓ in dimension d (in particular $\#I(\ell = 1, d = 3) = 3$). We conclude that for all angles $0 \leq \phi_0 < 2\pi$ the relation

$$Q'(\xi_0) \cos \angle(x - x_0, \Delta x_0) = Q'(\xi_0) \frac{\sigma_d}{\#I(\ell = 1, d)} \sum_{m \in I(\ell=1, d)} \overline{Y_{\ell=1}^m(\phi_{\Delta x_0})} Y_{\ell=1}^m(\phi_0), \quad (4.17)$$

holds, where (ξ_0, ϕ_0) are spherical coordinates centered around $\frac{x_0}{L(t)}$. Note that the right hand side of (4.16) is always real-valued. Through varying the angle $\phi_{\Delta x_0}$, it can be seen that (4.15) and (4.17) are equivalent with the statement that

$$\begin{bmatrix} Y_{\ell=1}^m \operatorname{Re} Q' \\ Y_{\ell=1}^m \operatorname{Im} Q' \end{bmatrix}$$

is an eigenfunction of $L_V(a, \ell)$ corresponding to eigenvalue $\lambda = a$. This concludes the second proof of Lemma 4.2.2.

4.2.1.3 Conclusion

Lemma 4.2.2 shows that the operator $L(a, \ell = 1)$ as defined by (4.13) has a positive eigenvalue $\lambda = a$. This eigenvalue is related to a translation invariance of the blowup profile itself and corresponds for each $m \in \{-1, 0, 1\}$ and $\ell = 1$ to an eigenfunction of $L(a, \ell)$. Analogous to the conclusion reached in Remark 3.2.3, we again argue that this positive eigenvalue is irrelevant since we do not allow for perturbations in the direction of a different blowup solution. The symmetry discussed in Section 4.2.1.2 shows that the corresponding eigenfunction is indeed a perturbation in the direction of a different blowup solution. As already noted, we will consider the positive eigenvalue to be irrelevant for both integer and non-integer values of d . We formulate the conclusion in a remark.

Remark 4.2.3. *The eigenvalue $\lambda = a$ of operator $L(a, \ell)$ as defined by (4.13) is related to a perturbation in a direction tangent to a family of blowup solutions. Since we do not allow for such perturbations, it is irrelevant in the setting in which we study spectral stability.*

4.2.2 Evans function techniques

We will now define the Evans function $E(\lambda, a, \ell)$ related to eigenvalue problem (4.12). As explained in Section 3.2.2, the Evans function $E(\lambda, a, \ell)$ is an analytic function whose zeros correspond to the eigenvalues of the related eigenvalue problem.

The Evans function is constructed in a similar way as in Section 3.2.2: consider the linearly independent solutions $Y_1(s, a, \lambda, \ell)$, $Y_2(s, a, \lambda, \ell)$, $Y_3(s, a, \lambda, \ell)$, $Y_4(s, a, \lambda, \ell)$ of the eigenvalue problem (4.12) for which Property 3.2.1 can be satisfied, then the Evans function $E(\lambda, a, \ell)$ is defined as in (3.11).

Since, as is also the case in Section 3.2.5, the upcoming Evans function analysis is very technical, we will, for simplicity, again make the following assumption:

Assumption 4.2.4. *Unless stated otherwise, in Sections 4.2.2, 4.2.3, 4.2.5 and 4.2.6 we assume that $\sigma = 1$.*

As discussed in Section 4.2, for $\ell \ll \frac{1}{a}$, the eigenfunctions of $L(a = 0, \ell)$ are given by (3.13). And thus, as in (3.62), we have

$$E(\lambda, a = 0, \ell) = -\lambda^4.$$

These four zero eigenvalues may be perturbed into the right half of the complex plane for $a \neq 0$, which will imply spectral instability, see Definition 1.4.2. The location of the perturbed eigenvalues can be determined by constructing a Taylor expansion of the Evans function. Theorem 3.2.4 presents such a Taylor expansion for $\ell = 0$ and can be used to investigate spectral (in)stability with respect to radially symmetric perturbations. In this section, we will construct such an expansion for $0 \neq \ell = O(1)$. From Lemma 4.2.2, it follows that for $\ell = 1$ we expect the Taylor expansion of the Evans function to have the following form:

$$E(\lambda, a, \ell = 1) = (\lambda - a) \left(E_3 \lambda^3 + E_2 \lambda^2 a + E_1 \lambda a^2 + E_0 a^3 + o(3) \right). \quad (4.18)$$

For $\ell \geq 2$, the leading order Taylor expansion of the Evans function will have the following general form:

$$E(\lambda, a, \ell) = \frac{\partial_\lambda^4 E(0, 0, \ell)}{4!} \lambda^4 + \frac{\partial_a \partial_\lambda^3 E(0, 0, \ell)}{3!1!} \lambda^3 a + \frac{\partial_a^2 \partial_\lambda^2 E(0, 0, \ell)}{2!2!} \lambda^2 a^2 + \frac{\partial_a^3 \partial_\lambda E(0, 0, \ell)}{1!3!} \lambda a^3 + \frac{\partial_a^4 E(0, 0, \ell)}{4!} a^4 + o(4), \quad (4.19)$$

where we expect no additional structure arising from symmetries.

Initially, we will perform the Evans function construction (and analysis) with functions Y_i , where $i = 1, 2, 3, 4$, that satisfy only Property 3.2.1.a or Property 3.2.1.b. In Section 4.2.4, we will determine whether Property 3.2.1.c or 3.2.1.d can also be satisfied. This will lead to the condition formulated in Proposition 4.2.6. This proposition is similar to Proposition 3.2.9 as derived in Section 3.2.8.

The rest of this Section is devoted to deriving expressions for the coefficients in (4.18) and (4.19). In order to do so, we first proceed as in Section 3.2.5 with verifying solvability conditions and computing the higher order corrections of the (generalized) eigenfunctions. And second, we express derivatives of the Evans function, analogous to the analysis in Section 3.2.6, in terms of these solvability conditions.

4.2.2.1 The case $\ell = O\left(\frac{1}{\sqrt{a}}\right)$

In this section, we will show, for $\sigma = 1$, that one cannot directly extend the Evans function techniques presented in Section 3.2.6 if $\ell = O\left(\frac{1}{\sqrt{a}}\right)$. The forthcoming arguments rely heavily on the analysis presented in Sections 3.2.5 and 3.2.6.

First, we set

$$\ell = \frac{\bar{\ell}}{\sqrt{a}}.$$

Although, we hereby introduce fractional powers in the expansion of $L(a, \ell)$ in orders of a , they only appear at $o(a)$. Thus, a leading order expansion of $L(a, \ell)$ is given by:

$$L(a, \ell) = L(a, \ell) + \partial_a L\left(a, \frac{\bar{\ell}}{\sqrt{a}}\right)\Big|_{a=0} a + o(a).$$

Now, from (4.13), we obtain

$$\begin{aligned} \partial_a L\left(a, \frac{\bar{\ell}}{\sqrt{a}}\right)\Big|_{a=0} &= \begin{bmatrix} -2\sigma S_0^{2\sigma-1} \text{Im} S_1 & -2\sigma S_0^{2\sigma-1} \text{Re} S_1 \\ 2\sigma(2\sigma+1) S_0^{2\sigma-1} \text{Re} S_1 & 2\sigma S_0^{2\sigma-1} \text{Im} S_1 \end{bmatrix} \\ &+ \begin{bmatrix} K_0 \left(\partial_s^2 - \frac{\kappa_0^2}{4} \right) + \frac{d-1}{2} - \frac{1}{\sigma} - b K_0 (2\sigma+1) S_0^{2\sigma} - s \partial_s & K_0 \kappa_0 \partial_s - \frac{d-1}{\kappa_0} \partial_s - \frac{\kappa_0}{2} s + \frac{\bar{\ell}^2}{\kappa_0^2} \\ -K_0 \kappa_0 \partial_s + \frac{d-1}{\kappa_0} \partial_s + \frac{\kappa_0}{2} s - \frac{\bar{\ell}^2}{\kappa_0^2} & K_0 \left(\partial_s^2 - \frac{\kappa_0^2}{4} \right) + \frac{d-1}{2} - \frac{1}{\sigma} - b K_0 S_0^{2\sigma} - s \partial_s \end{bmatrix}. \end{aligned} \quad (4.20)$$

We expand the (generalized) eigenfunctions in powers of a , see equation (3.26b) and proceed as in Section 3.2.5. The persistence of $\psi^{1,1}$ breaks down at $O(a)$, and, for one of the solvability conditions, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} -\left(\partial_a L\left(a, \frac{\bar{\ell}}{\sqrt{a}}\right)\Big|_{a=0} \right) \psi^{1,1}(\tilde{s}, 0) \cdot v_1 \, d\tilde{s} \\ = \int_{-\infty}^{\infty} -\Pi_2 \left(\partial_a L\left(a, \frac{\bar{\ell}}{\sqrt{a}}\right)\Big|_{a=0} \right) \psi^{1,1}(\tilde{s}, 0) \Pi_2 v_1 \, d\tilde{s} = -\frac{4\alpha \bar{\ell}^2}{3\kappa_0^2} \end{aligned} \quad (4.21)$$

where Π_2 is defined by (3.30). From Assumption 2.2.1.b and equation (2.12), it follows that (4.21) only vanishes for $\bar{\ell} = 0$. Thus, the persistence of $\psi^{1,1}$ breaks down at $O(a)$ in the non-radially symmetric case if $\ell = O\left(\frac{1}{\sqrt{a}}\right)$. This leads to the following complications in the Evans function analysis. As we have seen in Section 3.2.6, in the expression for $\partial_a^2(Y_1 - Y_3)$, the solvability condition of $\partial_a^2\psi^{1,1}$ appears, see equation (3.55e). However, we have just shown that the persistence of $\psi^{1,1}$ breaks down at $O(a)$. So, it is unclear how to proceed along the lines of Section 3.2.6 and thus how derive an expansion for the Evans function. Moreover, in the expression for $\partial_a\partial_\lambda(Y_1 - Y_3)$, the solvability condition for $\partial_a\psi^{2,1}$ appears, see equation (3.55c). To evaluate this solvability condition, an explicit expression for $\partial_a\psi^{1,1}$ is required.

A similar problem arises for the persistence of $\psi^{2,1}$. In that case the non-zero solvability condition is, equivalent to (4.21), given by

$$\begin{aligned} \int_{-\infty}^{\infty} -\left(\partial_a L\left(a, \frac{\bar{\ell}}{\sqrt{a}}\right)\right)\bigg|_{a=0} \psi^{1,2}(\tilde{s}, 0) \cdot v_2 \, d\tilde{s} \\ = \int_{-\infty}^{\infty} -\Pi_2\left(\partial_a L\left(a, \frac{\bar{\ell}}{\sqrt{a}}\right)\right)\bigg|_{a=0} \psi^{1,1}(\tilde{s}, 0) \Pi_2 v_1 \, d\tilde{s} = \frac{4\alpha\bar{\ell}^2}{\kappa_0^2}, \end{aligned}$$

where α is defined by (2.12)

Thus, the Evans function analysis cannot be extended straightforwardly if $\ell = O\left(\frac{1}{\sqrt{a}}\right)$ with the techniques presented in Chapter 3. We will not consider this case any further. Note that we will show that the $\ell = 1$ -mode and $\ell = 2$ -mode will lead to spectral instability. Hence, it is also not necessary to analyze this case to draw conclusion concerning spectral stability with respect to non-radially symmetric perturbations.

4.2.2.2 The case $\ell = O(1)$

In this section, we extend the Evans function techniques in the case $\ell = O(1)$ and $\sigma = 1$. The analysis in this section strongly relies on the analysis presented in Sections 3.2.5 and 3.2.6.

We proceed as in Section 3.2.5 and pose expansions for both $L(a, \ell)$ and the (generalized) eigenfunctions, see equation (3.26). From (4.13) and as discussed in Section 4.2, it follows that the derivative $\partial_a L(0, \ell)$ is independent of ℓ and is given by (3.29). We also obtain that $\partial_a^2 L(0, \ell)$ takes the form:

$$\partial_a^2 L(0, \ell) = \begin{bmatrix} \partial_a^2 L_{11}(0, \ell) & \partial_a^2 L_{12}(0, \ell) \\ \partial_a^2 L_{21}(0, \ell) & \partial_a^2 L_{22}(0, \ell) \end{bmatrix}, \quad (4.22)$$

where

$$\partial_a^2 L_{11}(0, \ell) = 2K_0 \frac{d-1}{\kappa_0} \partial_s - \frac{d-1}{\kappa_0} s - 12bK_0 S_0 \operatorname{Re} S_1 - 2\operatorname{Re} S_1 \operatorname{Im} S_1 - 4S_0 \operatorname{Im} S_2 - 2\kappa_2 \partial_s, \quad (4.23a)$$

$$\begin{aligned} \partial_a^2 L_{12}(0, \ell) = (d-1)K_0 + 2\frac{d-1}{\kappa_0^2} s \partial_s - 4bK_0 \operatorname{Im} S_1 S_0 - 4S_0 \operatorname{Re} S_2 - 2(\operatorname{Re} S_1)^2 \\ - 6(\operatorname{Im} S_1)^2 - \kappa_0 \kappa_2 + 2\frac{\ell(\ell+d-2)}{\kappa_0^2}, \end{aligned} \quad (4.23b)$$

$$\begin{aligned} \partial_a^2 L_{21}(0, \ell) = -(d-1)K_0 - 2\frac{d-1}{\kappa_0^2} s \partial_s - 4bK_0 \operatorname{Im} S_1 S_0 + 12S_0 \operatorname{Re} S_2 + 6(\operatorname{Re} S_1)^2 \\ + 2(\operatorname{Im} S_1)^2 + \kappa_0 \kappa_2 - 2\frac{\ell(\ell+d-2)}{\kappa_0^2}, \end{aligned} \quad (4.23c)$$

$$\partial_a^2 L_{22}(0, \ell) = 2K_0 \frac{d-1}{\kappa_0} \partial_s - \frac{d-1}{\kappa_0} s - 4bK_0 S_0 \operatorname{Re} S_1 + 4\operatorname{Re} S_1 \operatorname{Im} S_1 + 4S_0 \operatorname{Im} S_2 - 2\kappa_2 \partial_s. \quad (4.23d)$$

Analogous to the analysis presented in Section 3.2.5, we evaluate the solvability conditions for (non-)persistence of the higher orders in a regular expansion of the (generalized) eigenfunction. The analysis is completely similar, but the results are different, because (4.23b) and (4.23c) are now ℓ -dependent.

The persistence of the eigenfunction $\psi^{1,1}$ breaks down at $O(a^2)$ and the solvability conditions are given by:

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(-(\partial_a^2 L(0, \ell)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,1}(\tilde{s}, 0) \right) \cdot v_1 \, d\tilde{s} \\ &= \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} \psi_1(\tilde{s}) \Pi_2 \left(-(\partial_a^2 L(0, \ell)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,1}(\tilde{s}, 0) \right) \, d\tilde{s} \\ &= \frac{8(\alpha(\alpha(d(\ell-2) + \ell(\ell-2) - 4) + 12) - 6)}{3\alpha\kappa_0^2}, \end{aligned} \quad (4.24)$$

and

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(-(\partial_a^2 L(0, \ell)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,1}(\tilde{s}, 0) \right) \cdot v_2 \, d\tilde{s} \\ &= \sqrt{\alpha} \int_{-\infty}^{\infty} \phi_1(\tilde{s}) \Pi_1 \left(-(\partial_a^2 L(0, \ell)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,1}(\tilde{s}, 0) \right) \, d\tilde{s} \\ &= \frac{4(\alpha(-2\alpha(d-2) + d-6) + 3)}{\alpha\kappa_0}, \end{aligned} \quad (4.25)$$

where Π_1 and Π_2 are defined by (3.30). The persistence of the generalized eigenfunction $\psi^{2,1}$ breaks down at $O(a)$ and the solvability conditions are given by (3.43) and (3.44). The persistence of the eigenfunction $\psi^{1,2}$ breaks down at $O(a^2)$ and the solvability conditions are given by:

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(-(\partial_a^2 L(0, \ell)) \psi^{1,2}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,2}(\tilde{s}, 0) \right) \cdot v_1 \, d\tilde{s} \\ &= \frac{1}{\sqrt{\alpha}} \int_{-\infty}^{\infty} \psi_1(\tilde{s}) \Pi_2 \left(-(\partial_a^2 L(0, \ell)) \psi^{1,2}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,2}(\tilde{s}, 0) \right) \, d\tilde{s} \\ &= 0, \end{aligned} \quad (4.26)$$

and

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(-(\partial_a^2 L(0, \ell)) \psi^{1,2}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,2}(\tilde{s}, 0) \right) \cdot v_2 \, d\tilde{s} \\ &= \sqrt{\alpha} \int_{-\infty}^{\infty} \phi_1(\tilde{s}) \Pi_1 \left(-(\partial_a^2 L(0, \ell)) \psi^{1,2}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,2}(\tilde{s}, 0) \right) \, d\tilde{s} \\ &= -\frac{8\alpha\ell(d+\ell-2)}{\kappa_0^2}. \end{aligned} \quad (4.27)$$

The persistence of the generalized eigenfunction $\psi^{2,2}$ breaks down at $O(a)$ and the solvability conditions are given by (3.46) and (3.47). We want to stress that all the computations presented above are non-trivial due to the same reasons explained in Section 3.2.5.

Based upon this analysis, we derive the following expressions for the derivatives of the Evans Function. The computation of $\partial_\lambda^4 E(0, 0, \ell)$ and $\partial_a \partial_\lambda^3 E(0, 0, \ell)$ is completely similar as the one presented in Section 3.2.6 and thus from (3.50), (3.51), (3.58), (3.59) and the proof of Theorem 3.2.4, we obtain:

$$\partial_\lambda^4 E(0, 0, \ell) = 6\partial_\lambda^2(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_\lambda^2(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 \Big|_{\lambda=a=0} = -24, \quad (4.28)$$

where $\mathcal{Y}_i$ is as in (3.10). Also,

$$\begin{aligned} \partial_a \partial_\lambda^3 E(0, 0, \ell) &= 3 \left(\partial_a \partial_\lambda(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_\lambda^2(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 + \partial_\lambda^2(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_a \partial_\lambda(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 \right) \Big|_{\lambda=a=0} \\ &= 6 \left(\frac{4}{3}(3 - 4\alpha)K_0 - 2d + 4 \right). \end{aligned} \quad (4.29)$$

Due to the fact that the eigenfunction $\psi^{1,2}$ does not persist for $\ell \neq 0$, it follows that $\partial_a^2(\mathcal{Y}_2 - \mathcal{Y}_4) \Big|_{\lambda=a=0}$ does not vanish. Therefore, the expressions for $\partial_a^2 \partial_\lambda^2 E(0, 0, \ell)$, $\partial_a^3 \partial_\lambda E(0, 0, \ell)$ will differ from their $\ell = 0$ equivalents, as given by (3.60), (3.61) and (3.61), and will depend on ℓ :

$$\begin{aligned} \partial_a^2 \partial_\lambda^2 E(0, 0, \ell) &= \partial_a^2(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_\lambda^2(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 + \partial_\lambda^2(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_a^2(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 \\ &\quad + 2\partial_a \partial_\lambda(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_a \partial_\lambda(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 \Big|_{\lambda=a=0} \\ &= -2 \text{Det} \begin{bmatrix} \int_{-\infty}^{\infty} \psi_1 \Pi_2 (-(\partial_a^2 L(0, \ell)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,1}(\tilde{s}, 0)) d\tilde{s} & \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} \\ \int_{-\infty}^{\infty} \phi_1 \Pi_1 (-(\partial_a^2 L(0, \ell)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,1}(\tilde{s}, 0)) d\tilde{s} & \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,2}(\tilde{s}, 0) d\tilde{s} \end{bmatrix} \\ &\quad + 2 \text{Det} \begin{bmatrix} \int_{-\infty}^{\infty} \psi_1 \Pi_2 (-(\partial_a^2 L(0, \ell)) \psi^{1,2}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,2}(\tilde{s}, 0)) d\tilde{s} & \int_{-\infty}^{\infty} \psi_1 \Pi_2 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \\ \int_{-\infty}^{\infty} \phi_1 \Pi_1 (-(\partial_a^2 L(0, \ell)) \psi^{1,2}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,2}(\tilde{s}, 0)) d\tilde{s} & \int_{-\infty}^{\infty} \phi_1 \Pi_1 \psi^{2,1}(\tilde{s}, 0) d\tilde{s} \end{bmatrix} \\ &\quad - 4 \text{Det} \begin{bmatrix} \int_{-\infty}^{\infty} \psi_1 \Pi_2 (\partial_a \psi^{1,1}(\tilde{s}, 0) - (\partial_a L(0, \ell)) \psi^{2,1}(\tilde{s}, 0)) d\tilde{s} & \int_{-\infty}^{\infty} \psi_1 \Pi_2 (\partial_a \psi^{1,2}(\tilde{s}, 0) - (\partial_a L(0, \ell)) \psi^{2,2}(\tilde{s}, 0)) d\tilde{s} \\ \int_{-\infty}^{\infty} \phi_1 \Pi_1 (\partial_a \psi^{1,1}(\tilde{s}, 0) - (\partial_a L(0, \ell)) \psi^{2,1}(\tilde{s}, 0)) d\tilde{s} & \int_{-\infty}^{\infty} \phi_1 \Pi_1 (\partial_a \psi^{1,2}(\tilde{s}, 0) - (\partial_a L(0, \ell)) \psi^{2,2}(\tilde{s}, 0)) d\tilde{s} \end{bmatrix} \\ &= -16 \frac{\alpha^2(d^2 - 6d - 28) + 60\alpha - 27 - 2\alpha^2\ell(d + \ell - 2)}{3\alpha\kappa_0^2}, \end{aligned} \quad (4.30)$$

$$\begin{aligned} \partial_a^3 \partial_\lambda E(0, 0, \ell) &= 3\partial_a^2(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_a \partial_\lambda(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 + 3\partial_a \partial_\lambda(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_a^2(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 \Big|_{\lambda=a=0} \\ &= -3 \text{Det} \begin{bmatrix} \int_{-\infty}^{\infty} \psi_1 \Pi_2 (-(\partial_a^2 L(0, \ell)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,1}(\tilde{s}, 0)) d\tilde{s} & \int_{-\infty}^{\infty} \psi_1 \Pi_2 (\partial_a \psi^{1,2}(\tilde{s}, 0) - (\partial_a L(0, \ell)) \psi^{2,2}(\tilde{s}, 0)) d\tilde{s} \\ \int_{-\infty}^{\infty} \phi_1 \Pi_1 (-(\partial_a^2 L(0, \ell)) \psi^{1,1}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,1}(\tilde{s}, 0)) d\tilde{s} & \int_{-\infty}^{\infty} \phi_1 \Pi_1 (\partial_a \psi^{1,2}(\tilde{s}, 0) - (\partial_a L(0, \ell)) \psi^{2,2}(\tilde{s}, 0)) d\tilde{s} \end{bmatrix} \\ &\quad + 3 \text{Det} \begin{bmatrix} \int_{-\infty}^{\infty} \psi_1 \Pi_2 (-(\partial_a^2 L(0, \ell)) \psi^{1,2}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,2}(\tilde{s}, 0)) d\tilde{s} & \int_{-\infty}^{\infty} \psi_1 \Pi_2 (\partial_a \psi^{1,1}(\tilde{s}, 0) - (\partial_a L(0, \ell)) \psi^{2,1}(\tilde{s}, 0)) d\tilde{s} \\ \int_{-\infty}^{\infty} \phi_1 \Pi_1 (-(\partial_a^2 L(0, \ell)) \psi^{1,2}(\tilde{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,2}(\tilde{s}, 0)) d\tilde{s} & \int_{-\infty}^{\infty} \phi_1 \Pi_1 (\partial_a \psi^{1,1}(\tilde{s}, 0) - (\partial_a L(0, \ell)) \psi^{2,1}(\tilde{s}, 0)) d\tilde{s} \end{bmatrix} \\ &= \frac{16(\alpha^2(d - 10)d + 6\alpha(d + 2) - \alpha^2\ell(d - 2(K_0 + 1))(d + \ell - 2) - 9)}{\alpha\kappa_0^2}, \end{aligned} \quad (4.31)$$

and

$$\partial_a^4 E(0, 0, \ell) = 6\partial_a^2(\mathcal{Y}_1 - \mathcal{Y}_3) \wedge \mathcal{Y}_1 \wedge \partial_a^2(\mathcal{Y}_2 - \mathcal{Y}_4) \wedge \mathcal{Y}_2 \Big|_{\lambda=a=0}$$

$$\begin{aligned}
&= -6 \operatorname{Det} \begin{bmatrix} \int_{-\infty}^{\infty} \psi_1 \Pi_2(-(\partial_a^2 L(0, \ell)) \psi^{1,1}(\bar{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,1}(\bar{s}, 0)) d\bar{s} & \int_{-\infty}^{\infty} \phi_1 \Pi_1(-(\partial_a^2 L(0, \ell)) \psi^{1,1}(\bar{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,1}(\bar{s}, 0)) d\bar{s} \\ \int_{-\infty}^{\infty} \psi_1 \Pi_2(-(\partial_a^2 L(0, \ell)) \psi^{1,2}(\bar{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,2}(\bar{s}, 0)) d\bar{s} & \int_{-\infty}^{\infty} \phi_1 \Pi_1(-(\partial_a^2 L(0, \ell)) \psi^{1,2}(\bar{s}, 0) - 2(\partial_a L(0, \ell)) \partial_a \psi^{1,2}(\bar{s}, 0)) d\bar{s} \end{bmatrix} \\
&= \frac{128\ell(d + \ell - 2)(\alpha(\alpha(d\ell - 2) + \ell(\ell - 2) - 4) + 12) - 6}{\kappa_0^4}. \tag{4.32}
\end{aligned}$$

Note the above expressions agree with the radially symmetric case, i.e. $\ell = 0$, since for $\ell = 0$ (4.30) reduces to (3.66), (4.31) reduces to (3.67) and (4.32) vanishes.

4.2.3 A Taylor expansion of the Evans function

In this section, we use the expressions for the derivatives of the Evans function, as presented in Section 4.2.2.2, to derive a leading order expansion of the Evans Function for $\sigma = 1$.

Theorem 4.2.5. *Let $\sigma = 1$ and let $E(\lambda, a, \ell)$ be the Evans function associated to the eigenvalue problem $L(a, \ell)W = \lambda W$ through its definition in Section 4.2.2, where $L(a, \ell)$ is given by (4.13). Then if $\ell = O(1)$, a leading order expansion of the Evans function is given by*

$$\begin{aligned}
E(\lambda, a, \ell) &= \frac{\partial_\lambda^4 E(0, 0, \ell)}{4!} \lambda^4 + \frac{\partial_\lambda^3 \partial_a E(0, 0, \ell)}{3!} \lambda^3 a + \frac{\partial_\lambda^2 \partial_a^2 E(0, 0, \ell)}{2!2!} \lambda^2 a^2 + \frac{\partial_\lambda \partial_a^3 E(0, 0, \ell)}{3!} \lambda a^3 \\
&\quad + \frac{\partial_a^4 E(0, 0, \ell)}{4!} a^4 + o(4), \tag{4.33}
\end{aligned}$$

where $\partial_\lambda^4 E(0, 0, \ell)$ is given by (4.28), $\partial_\lambda^3 \partial_a E(0, 0, \ell)$ is given by (4.29), $\partial_\lambda^2 \partial_a^2 E(0, 0, \ell)$ is given by (4.30), $\partial_\lambda \partial_a^3 E(0, 0, \ell)$ is given by (4.31) and $\partial_a^4 E(0, 0, \ell)$ is given by (4.32).

Furthermore, for $\ell = 1$ the Evans function can be written as

$$E(\lambda, a, \ell = 1) = (\lambda - a) \left(-\lambda^3 + E_2 \lambda^2 a + E_1 \lambda a^2 + E_0 a^3 + o(3) \right), \tag{4.34}$$

where

$$E_2 = -\frac{4(\alpha^2(17 - 2d) + 3\alpha(d - 9) + 9)}{3\alpha\kappa_0^2} \tag{4.35a}$$

$$E_1 = \frac{\alpha^2((d - 10)d - 9) + 3\alpha(d + 11) - 18}{3(\alpha - 1)\alpha} \tag{4.35b}$$

$$E_0 = \frac{16(d - 1)((d + 5)\alpha^2 - 12\alpha + 6)}{3\kappa_0^4} \tag{4.35c}$$

As explained in the beginning of Section 4.2.2, the Evans function is, so far, constructed with functions Y_i , where $i = 1, 2, 3, 4$, that satisfy only Property 3.2.1.a or Property 3.2.1.b. To complete the Evans function analysis, it must be determined whether Property 3.2.1.c or 3.2.1.d can also be satisfied. This will lead to the condition formulated in Proposition 4.2.6 and will be the topic of Section 4.2.4.

The remaining sections of this chapter are devoted to a discussion of the further implications of Theorem 4.2.5 and its relation with the already presented theory. Note that the parameters K_0 and κ_0 depend on d and b through the system of nonlinear equations (2.26). Thus, as in Section 3.2.7, further analysis is needed to draw conclusions concerning the spectral stability from the Taylor expansions presented in Theorem 4.2.5.

From Lemma 4.2.2, it follows that $\lambda = a$ is an eigenvalue of the Evans function if $\ell = 1$ for all $0 < a \ll 1$. Note that this eigenvalue indeed appears as a factor in (4.34).

4.2.4 Matching

This section is the equivalent of Section 3.2.8 in a non-radially symmetric setting. As was the case in Section 3.2.8, we will, in this section, not restrict ourselves to the case $\sigma = 1$.

In this section, we will determine whether the functions Y_i , where $i = 1, 2, 3, 4$, that are used in the Evans function construction and satisfy Property 3.2.1.a or Property 3.2.1.b, can also satisfy Property 3.2.1.c or 3.2.1.d. Only the requirement that Y_3 and Y_4 should satisfy both Property 3.2.1.b and Property 3.2.1.d leads an extra condition which is formulated in the following proposition. This proposition is the equivalent of Proposition 3.2.9.

Proposition 4.2.6. *Let $\ell = O(1)$, R be defined by (2.42), $(d, b) \in R$ and $0 < a \ll 1$, then: the functions Y_3 and Y_4 as used in the construction of the Evans function, see Section 4.2.2, that satisfy Property 3.2.1.b can only satisfy Property 3.2.1.d if the inequality*

$$\operatorname{Re} \left(\frac{\lambda}{a} \right) > -\frac{1}{\sigma} \quad (4.36)$$

holds.

To derive Proposition 4.2.6, we can proceed exactly as in Section 3.2.8. Therefore, we here assume that the reader is familiar with the analysis in Section 3.2.8 and will only indicate where the analysis differs.

The main difference is the analysis in the inner region. From (4.11), it follows that the eigenvalue problem in the inner region reduces at leading order to

$$v'' + \frac{d-1}{\xi} v' - \frac{\ell(\ell+d-2)}{\xi^2} v - v = 0.$$

As in Sections 2.2.1 and 3.2.8.3, the solution to this equation is given in terms of modified Bessel functions:

$$v_{\text{inner}}(\xi) = C_I \xi^{\frac{2-d}{2}} I_{\frac{d-2+2\ell}{2}}(\xi) + C_K \xi^{\frac{2-d}{2}} K_{\frac{d-2+2\ell}{2}}(\xi).$$

Since $\xi^{\frac{2-d}{2}} K_{\frac{d-2+2\ell}{2}}(\xi)$ blows up in $\xi = 0$, we require $C_K = 0$.

The rest of matching procedure is completely similar to the one performed in Section 3.2.8, only now the constants determined in the matching also depend on ℓ .

4.2.5 Analysis of the case $\ell = 1$

In this section, we prove, for $\sigma = 1$, that the Evans function $E(\lambda, a, \ell = 1)$ has next to the positive eigenvalue $\lambda = a$ always a second positive eigenvalue. In order to do so, we first prove two statements about the coefficients in leading order expansion of $E(\lambda, a, \ell = 1)$ as given by (4.34).

Lemma 4.2.7. *Let $\sigma = 1$, then in the region R as defined by (2.42) the inequality $E_0 > 0$ holds, where E_0 is given by (4.35c).*

Proof. From (4.35c) and Assumption 1.2.2.a, it follows that E_0 has the same sign as $\tilde{E}_0(d, \alpha)$, where

$$\tilde{E}_0(d, \alpha) := (d+5)\alpha^2 - 12\alpha + 6.$$

From Assumption 1.2.2.b, it follows that $2 < d < 4$ and from Assumption 2.2.1.b and (2.12), it follows that $0 < \alpha < 1$. Thus, (d, α) takes values in the rectangle

$$\{(d, \alpha) : 2 < d < 4 \text{ and } 0 < \alpha < 1\}.$$

The sign of $\tilde{E}_0(d, \alpha)$, and thus also the sign of E_0 , can only change in this rectangle if the curve $\tilde{E}_0(d, \alpha) = 0$ intersects this rectangle. In order to determine whether there is such an intersection, we analyze the curve $\tilde{E}_0(d, \alpha) = 0$. On this curve, d is given as a graph over α . Solving for d gives

$$d_0(\alpha) = \frac{-5\alpha^2 + 12\alpha - 6}{\alpha^2}.$$

The function $d_0(\alpha)$ attains on the interval $0 \leq \alpha \leq 1$ its maximum value $d_{0,\max} = 1$ at $\alpha_{0,\max} = 1$. We, thus, find that the curve $\tilde{E}_0(d, \alpha) = 0$ has no intersection with the defined rectangle and that the sign of $\tilde{E}_0(d, \alpha)$ is fixed. Evaluation at, for example, $\alpha = 0.001$ and $d = 3$ shows the sign of $\tilde{E}_0(d, \alpha)$ is positive. $\square$

Lemma 4.2.8. *Let $\sigma = 1$, then in the region R as defined by (2.42) the inequality $E_2 < 0$ holds, where E_2 is given by (4.35a).*

Proof. We proceed as in the proof of Lemma 4.2.7. Thus again, for $\sigma = 1$, the inequalities $2 < d < 4$ and $0 < \alpha < 1$ hold. We define

$$\tilde{E}_2(d, \alpha) := \alpha(-17\alpha + 2\alpha d - 3d + 27) - 9.$$

Now, from (4.35a), it follows that E_2 has the same sign as $\tilde{E}_2(d, \alpha)$. We investigate whether the curve defined by $\tilde{E}_2(d, \alpha) = 0$ can have an intersection with the rectangle

$$\{(d, \alpha) : 2 < d < 4 \text{ and } 0 < \alpha < 1\}.$$

On the curve $\tilde{E}_2(d, \alpha) = 0$, d is given as a graph over α . Solving for d gives

$$d_2(\alpha) = \frac{17\alpha^2 - 27\alpha + 9}{\alpha(2\alpha - 3)}.$$

The function $d_2(\alpha)$ attains on the interval $0 \leq \alpha \leq 1$ its maximum value $d_{2,\max} = 5 - 2\sqrt{3}$ at $\alpha_{2,\max} = 6 - 3\sqrt{3}$. We, thus, find that the curve $\tilde{E}_2(d, \alpha) = 0$ has no intersection with the defined rectangle and that the sign of $\tilde{E}_2(d, \alpha)$ is fixed. Evaluation at, for example, $\alpha = 0.001$ and $d = 3$ shows the sign of $\tilde{E}_2(d, \alpha) = 0$ is negative. $\square$

Lemma 4.2.9. *Let $\sigma = 1$, then in the region R as defined by (2.42), the Evans function $E(\lambda, a, \ell = 1)$ has for $a \ll 1$ exactly two zeros with positive real part.*

Proof. Theorem 4.2.5 states that a leading order expansion of $E(\lambda, a, \ell = 1)$ is given by (4.34). Thus $\lambda = a$ is one of the zeros and we prove that there is always one other zero with positive real part. From (4.34), it follows that it remains to show that the equation

$$-\lambda^3 + E_2\lambda^2a + E_1\lambda a^2 + E_0a^3 = 0, \quad (4.37)$$

where the coefficients are given by (4.35), always has one zero with positive real part. Let λ_1 , λ_2 and λ_3 be the three zeros of (4.37). Equation (4.37) can now be written as

$$-(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0. \quad (4.38)$$

Combining (4.37) and (4.38) with Lemma 4.2.7 implies

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = E_0a^3 > 0. \quad (4.39)$$

And similarly, combining (4.37) and (4.38) with Lemma 4.2.8 implies

$$\lambda_1 + \lambda_2 + \lambda_3 = E_2 a < 0. \quad (4.40)$$

From the fact that zeros of (4.38) come in complex conjugate pairs, it follows that, up to equivalence under permutation of the zeros, there are six possibilities:

1. λ_1 and λ_2 have non-zero imaginary part, are complex conjugate zeros and $\lambda_3 > 0$;
2. λ_1 and λ_2 have non-zero imaginary part, are complex conjugate zeros and $\lambda_3 < 0$;
3. λ_1, λ_2 and λ_3 are real-valued and are all positive;
4. λ_1, λ_2 and λ_3 are real-valued, $\lambda_1, \lambda_2 > 0$, and $\lambda_3 < 0$;
5. λ_1, λ_2 and λ_3 are real-valued, $\lambda_1 > 0$, and $\lambda_2, \lambda_3 < 0$;
6. λ_1, λ_2 and λ_3 are real-valued and are all negative.

Now, inequality (4.39) rules out possibility 2, 4 and 6. And, inequality (4.40) rules out possibility number 3. So only possibility 1 and 5 remain. Furthermore, inequality (4.40) implies that in possibility 1, the two complex conjugate eigenvalues have negative real part. Thus, we conclude that (4.37) has exactly one zero with positive real part. $\square$

Note that Lemma 4.2.9 implies that the $\ell = 1$ is unstable and, thus, that the m_2^{upper} -solution is spectrally (and linearly, see Remark 1.4.3) unstable with respect to non-radially symmetric perturbations. Nevertheless, in Section 4.2.6, the case $\ell = 2$ will also be analyzed as an illustration of how to analyze any higher order mode with $\ell = O(1)$,

4.2.6 Analysis of the case $\ell = 2$

In this section, we show (with numerical evaluation) that the Evans function $E(\lambda, a, \ell)$ has for $\ell = 2$ (and $\sigma = 1$) at least one zero with positive real part for all valid parameter values. In order to do so, we derive two implicit equations that define two curves in the (d, b) -plane. These two curves divide the (d, b) -plane in several regions in which the number of eigenvalues with positive real part is constant. The procedure is based on the following Lemma concerning the coefficients in the leading order expansion of $E(\lambda, a, \ell)$ as given by (4.33). The lemma is valid for general ℓ of $O(1)$, but will, as an illustration, only be applied to the case $\ell = 2$. We first prove two statements about the coefficients in leading order expansion of $E(\lambda, a, \ell)$ as given by (4.33).

Lemma 4.2.10. *Let $\sigma = 1$, R be defined by (2.42), $(d, b) \in R$, $E(\lambda, a, \ell)$ be given by (4.33) and $\lambda_0 = a\tilde{\lambda}_0$ be a zero of $E(\lambda, a, \ell)$. Now, the following two statements hold.*

4.2.10.a *If $\tilde{\lambda}_0 = 0$, the equality*

$$\partial_a^4 E(0, 0, \ell) = 0 \quad (4.41)$$

holds, where $\partial_a^4 E(0, 0, \ell)$ is given by (4.32).

4.2.10.b *If $\text{Re } \tilde{\lambda}_0 = 0$, but $\text{Im } \tilde{\lambda}_0 \neq 0$, then the following equality holds:*

$$-24 \left(\partial_\lambda \partial_a^3 E \right)^2 - 6 \partial_\lambda \partial_a^3 E \partial_\lambda^2 \partial_a^2 E \partial_\lambda^3 \partial_a E + \partial_a^4 E \left(\partial_\lambda^3 \partial_a E \right)^2 \Big|_{a=\lambda=0} = 0. \quad (4.42)$$

Proof. The Evans function $E(\lambda, a, \ell)$ as given by (4.33) can be written as a fourth order polynomial in $\tilde{\lambda} := \frac{\lambda}{a}$ of the form:

$$-a^4 (\tilde{\lambda} - \tilde{\lambda}_1)(\tilde{\lambda} - \tilde{\lambda}_2)(\tilde{\lambda} - \tilde{\lambda}_3)(\tilde{\lambda} - \tilde{\lambda}_4),$$

where the $\tilde{\lambda}_i$ are the four zero's. Comparing coefficients of the powers of $\tilde{\lambda}$ in this polynomial with (4.33) leads to

$$\frac{\partial_a^4 E(0, 0, \ell)}{4!} = -\tilde{\lambda}_1 \tilde{\lambda}_2 \tilde{\lambda}_3 \tilde{\lambda}_4, \quad (4.43a)$$

$$\frac{\partial_\lambda \partial_a^3 E(0, 0, \ell)}{3!} = \tilde{\lambda}_1 \tilde{\lambda}_2 \tilde{\lambda}_3 + \tilde{\lambda}_1 \tilde{\lambda}_2 \tilde{\lambda}_4 + \tilde{\lambda}_1 \tilde{\lambda}_3 \tilde{\lambda}_4 + \tilde{\lambda}_2 \tilde{\lambda}_3 \tilde{\lambda}_4, \quad (4.43b)$$

$$\frac{\partial_\lambda^2 \partial_a^2 E(0, 0, \ell)}{2!2!} = -\tilde{\lambda}_1 \tilde{\lambda}_2 - \tilde{\lambda}_1 \tilde{\lambda}_3 - \tilde{\lambda}_1 \tilde{\lambda}_4 - \tilde{\lambda}_2 \tilde{\lambda}_3 - \tilde{\lambda}_2 \tilde{\lambda}_4 - \tilde{\lambda}_3 \tilde{\lambda}_4, \quad (4.43c)$$

$$\frac{\partial_\lambda^3 \partial_a E(0, 0, \ell)}{3!} = \tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3 + \tilde{\lambda}_4. \quad (4.43d)$$

Obtaining Statement 4.2.10.a is straightforward: suppose that, without loss of generality,

$$\tilde{\lambda}_1 = 0,$$

then, from (4.43a), it follows that $\partial_a^4 E(0, 0, \ell) = 0$.

For Statement 4.2.10.b, we proceed as follows. Suppose that, without loss of generality,

$$\tilde{\lambda}_1 = i\tilde{\lambda}_{12}, \quad (4.44)$$

where λ_{12} is real-valued. Note that complex zeros of $E(\lambda, a, \ell)$ always come in pairs since the coefficients of λ in (4.33) are real-valued. So, we also suppose that, without loss of generality,

$$\tilde{\lambda}_2 = -i\tilde{\lambda}_{12}. \quad (4.45)$$

Substitution of (4.44) and (4.45) into (4.43a) and then using (4.43c) leads to

$$\lambda_{12}^2 = \frac{-\partial_a^4 E(0, 0, \ell)}{4! \tilde{\lambda}_3 \tilde{\lambda}_4} = \frac{\frac{\partial_a^4 E(0, 0, \ell)}{4!}}{\lambda_{12}^2 + \frac{\partial_\lambda^2 \partial_a^2 E(0, 0, \ell)}{2!2!}}. \quad (4.46)$$

Moreover, substitution of (4.44) and (4.45) into (4.43b), and then using (4.43d) leads to

$$\lambda_{12}^2 = \frac{\partial_\lambda^3 \partial_a E(0, 0, \ell)}{3! (\tilde{\lambda}_3 + \tilde{\lambda}_4)} = \frac{\partial_\lambda^3 \partial_a E(0, 0, \ell)}{\partial_\lambda \partial_a^3 E(0, 0, \ell)}. \quad (4.47)$$

Through combining (4.46) and (4.47), we derive 4.2.10.b. $\square$

Since, when zeros of $E(\lambda, a, \ell = 2)$ cross the imaginary axis either (4.41) or (4.42) holds, we can use these curves to divide the subset R as defined by (2.42) of the (d, b) -plane, i.e. the part of the (d, b) -plane in which the m_2^{upper} -solution exists, into regions in which the number of zeros of $E(\lambda, a, \ell = 2)$ is constant. In this way, Figure 4.2.1 is constructed. This figure shows the number of zeros of $E(\lambda, a, \ell = 2)$ with positive real part in Regions 1–3. The union of Regions 1–3 corresponds to the existence region R . In this figure, we assumed $\sigma = 1$, $\ell = 2$ and, in line with Assumption 1.2.2.b, the figure is restricted to the strip $2 < d < 4$ of the (d, b) -plane. On the curve separating Region 1 from Region 3, (4.42) holds. And similarly, on the curve separating Region 1 from Region 2 and also separating Region 2 from Region 3, (4.41) holds. At the point where the line separating Region 1 from Region 3 (the dashed line) and the line separating Region 1 from Region 2 (the dash dotted line) intersect, $\lambda = 0$ is a zero of $E(\lambda, a, \ell = 2)$ with algebraic multiplicity 2.

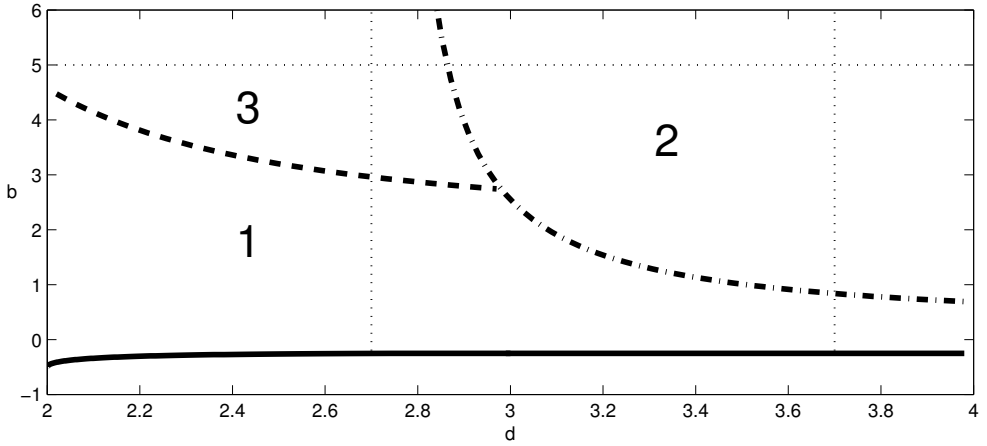


Figure 4.2.1: The existence region R as defined by (2.42), i.e. the region in which the m_2^{upper} -solution exists, divided into regions bounded by the thick lines indicating (for $\sigma = 1$) the number of zeros of $E(\lambda, a, \ell = 2)$ as defined by (4.33) with positive real part. The union of the Regions 1 – 3 is the existence region R as defined by (2.42). For the thin dotted lines, see Figure 4.2.2. See also Section 4.2.6 for a discussion.

For each point (d, b) in the union of Regions 1 to 3 in Figure 4.2.1, the four zeros of $E(\lambda, a, \ell = 2)$ as defined by (4.33) can be determined through combining Theorem 4.2.5 with the system of equations (2.26). Thus, as we change the parameters d and b such that we move over one of the thin dotted lines in Figure 4.2.1, we can construct figures that show the position of the zeros of $E(\lambda, a, \ell = 2)$ as defined by (4.33) in the complex plane. This is done in Figure 4.2.2.

In Figure 4.2.2a, the position of the zeros of $E(\lambda, a, \ell = 2)$ is displayed for $d = 2.7$ and b ranging from -0.2 (indicated with a $*$) to ∞ (indicated with a $\bullet$). Note that this corresponds to the thin dotted line $d = 2.7$ in Figure 4.2.1. As can be seen in Figure 4.2.2a, two additional eigenvalues destabilize (indicated with a $\circ$) which corresponds to the transition from Region 1 into Region 3 in Figure 4.2.1. The gray dashed line indicates the threshold at $\text{Re } \lambda = -\frac{d}{\sigma}|_{\sigma=1} = -a$, see Proposition 4.2.6. This proposition implies that everything left of the gray dashed line should be ignored: for these values of λ there exists a solution V of eigenvalue problem (4.11) that satisfies Property 3.1.1.a and 3.1.1.b, this solution does not satisfy Property 3.1.1.c and 3.1.1.d.

In Figure 4.2.2b, the position of the zeros of $E(\lambda, a, \ell = 2)$ is displayed for $d = 3.7$ and b ranging from -0.2 (indicated with a $*$) to ∞ (indicated with a $\bullet$). This corresponds to the thin dotted line $d = 3.7$ in Figure 4.2.1. The destabilization of one additional eigenvalue in Figure 4.2.2b (indicated with a $\circ$) corresponds to the transition from Region 1 into Region 2 in Figure 4.2.1.

In Figure 4.2.2c, the position of the zeros of $E(\lambda, a, \ell = 2)$ is displayed for $b = 5$ and d ranging from 2 (indicated with a $*$) to 4 (indicated with a $\bullet$). Again, this corresponds to the thin dotted line $b = 5$ in Figure 4.2.1. The destabilization of one additional eigenvalue in Figure 4.2.2c (indicated with a $\circ$) corresponds to the transition from Region 2 into Region 3 in Figure 4.2.1.

Finally, we note that the method presented in this section is not restricted to the case $\ell = 2$

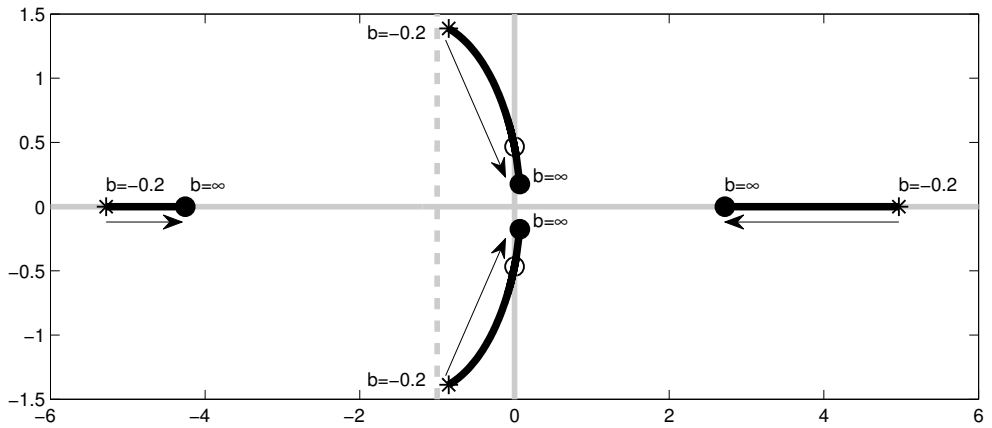
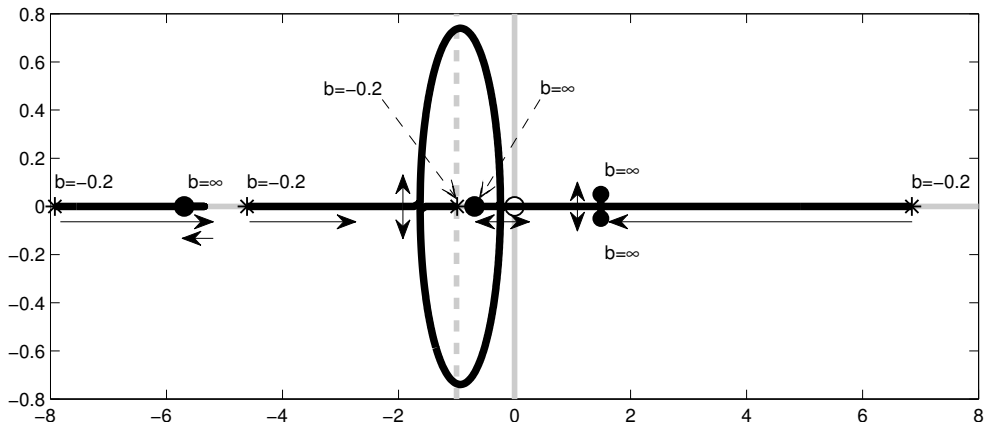
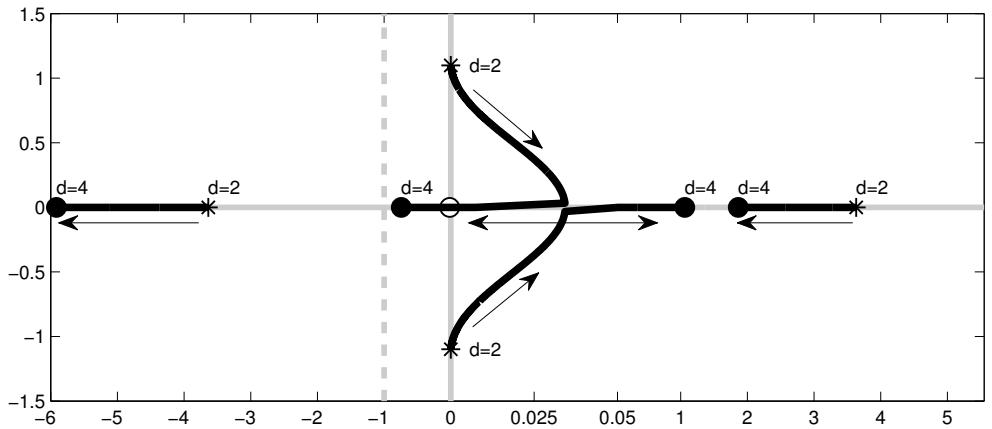
(a) $d = 2.7$ and b varies from -0.2 to infinity.(b) $d = 3.7$ and b varies from -0.2 to infinity.(c) $b = 5$ and d varies from 2 to 4.

Figure 4.2.2: Position of the complex-valued zeros of $E(\lambda, a, \ell = 2)$ (for $\sigma = 1$), as defined by (4.33), divided by a in the complex plane on the thin dotted lines in Figure 4.2.1. The gray dashed lines indicate the threshold at $\text{Re } \lambda = -\frac{a}{\sigma}|_{\sigma=1} = -a$, see Proposition 4.36. The $\circ$'s indicate intersections with the imaginary axis. For more details, we refer to the discussion in Section 4.2.6.

and can be used for all $\ell = O(1)$.

4.3 Conclusion

In this chapter, we studied the spectral stability with respect to non-radially symmetric perturbations of the stationary m_2^{upper} -solution of equation (2.1) as defined by Definition 2.2.8. The m_2^{upper} -solution was constructed asymptotically in Section 2.2, see also [9, 40]. In Chapter 3, we showed that, when only the perturbed NLS spectrum is considered, the m_2^{upper} -solution is spectrally stable with respect to radially symmetric perturbations, see Theorem 3.4.1.

In order to do so, we showed that the set-up of Chapter 3 can be extended to the non-radially symmetric setting through using spherical harmonics Y_ℓ^m , see Section 4.1. This results in an eigenvalue problem and Evans function analysis for each mode, see (4.11).

In Section 4.2.1, we showed, by using an invariance under perturbations of the blowup point itself, that for $\ell = 1$ there is a positive eigenvalue $\lambda = a$. However, this eigenvalue is related to a perturbation in a direction tangent to a family of blowup solutions. Therefore, it is irrelevant in the setting in which we study spectral stability.

As in Chapter 3, the NLS point spectrum, consisting of $\lambda = 0$ with algebraic multiplicity 4 and geometric multiplicity 2, was taken as a starting point through considering solutions that are exponentially localized in the bump region, see Property 3.1.1. In Section 4.2.2, Evans function techniques are used to determine the perturbed spectrum if $\ell = O(1)$.

In Section 4.2.4, it is investigated whether exponentially localized solutions in the bump region decay in the far field. This leads to Proposition 4.2.6 which states that this is only possible if $\text{Re } \frac{\lambda}{a} > -\frac{1}{\sigma}$. Since for all points in the right half of the complex this inequality is satisfied, it has no impact for conclusions concerning spectral instability of the m_2^{upper} -solution.

Two particular cases, $\ell = 1$ and $\ell = 2$, are considered in more detail in Sections 4.2.5 and 4.2.6. We showed in the $\ell = 1$ case that there is, next to the irrelevant eigenvalue $\lambda = a$, always one additional eigenvalue with positive real part, see Lemma 4.2.9. And, if $\ell = 2$, we showed that there is always at least one eigenvalue with positive real part, see Figure 4.2.1. We thus conclude that the solution of which we studied the stability in this chapter is always unstable with respect to radially symmetric perturbations. We formulate this result in a theorem.

Theorem 4.3.1. *Let $0 < a \ll 1$, let $\sigma = 1$, consider the region R in the (d, b) -plane as defined by (2.42) and let $(d, b) \in R$. Then, the radially symmetric m_2^{upper} -solution of (2.3), i.e. the solution on the upper branch of the saddle-node bifurcation as denoted in Figure 3.4.1a, is spectrally unstable with respect to non-radially symmetric perturbations.*

Corollary 4.3.2. *Under the conditions stated in Theorem 4.3.1, the m_2^{upper} -solution is linearly unstable with respect to non-radially symmetric perturbations.*

Proof. Spectral instability caused by an eigenvalue implies linear instability, see Remark 1.4.3. □

Note that Theorem 4.3.1 and Corollary 4.3.2 do not contradict the numerical simulations in [9]. The numerical simulations in [9], for the case $d = 3$ and $b = 0$, only concern the stability with respect to radially symmetric perturbations.

Appendix A

Proof of Theorem 2.1.1

In this appendix we present the proof of Theorem 2.1.1.

Proof. For the main part we follow the proof in [40] and we therefore adopt the notation of [40] in this proof. Due to a slight error in [40], we adapt the proof of Lemma 7.1 and 7.2 in [40]. We reformulate Lemma 7.1 and 7.2 as follows.

Lemma A.0.3. *Let $0 < a \ll 1$, $2 < d < 4$, $b > 0$ be given. And let $0 < \varepsilon \ll 1$ be such that $\varepsilon = Ka$ for some positive constant K . Then, for every $\xi_b \leq \xi \leq \xi_{\max}$, the phase $|\phi| = o(\sqrt{a})$ iff the following two statements hold:*

(A.0.3.a) *there exists a K_0 such that $K = K_0 + O(a)$;*

(A.0.3.b) $\left| \left(\frac{d-2}{2} - K_0 - \frac{2}{3}(2b-1)K_0 \left(1 - \frac{\eta^2}{4} \right) \right) \right| = 0.$

There is a slight error in [40] Formula (4.3), it should read

$$\begin{aligned} \phi(\xi) = & \frac{a\xi^{1-d}K^2}{A(\xi)^2(1+\varepsilon^2)} \int_0^\xi A^2\tilde{\xi}^{d-1} \left(\frac{d-2}{2} + K - K(b+1)A^2 + aK\tilde{\xi}\psi + O(a^4) \right) d\tilde{\xi} \\ & + \frac{\xi a^3 K^2}{2(1+a^2 K^2)}, \end{aligned}$$

where K is such that $\varepsilon = Ka$, i.e. $K = K_0 + O(a)$. As in Appendix D of [40], we have

$$\begin{aligned} |\phi(\xi)| \leq & \frac{a\xi^{1-d}}{A(\xi)^2(1+\varepsilon^2)} \left(\left| \int_{\xi_b}^\xi A(\tilde{\xi})^2 \tilde{\xi}^{d-1} \left(\frac{d-2}{2} + K - K(b+1)A(\tilde{\xi})^2 - \frac{Ka^2\tilde{\xi}^2}{2} \right) d\tilde{\xi} \right| \right. \\ & \left. + CA(\xi_b)^2 \xi_b^d \right) + c_1 a^2. \end{aligned}$$

Furthermore, as derived in Appendix D of [40], we have $A(\xi) \geq A(\xi_b)a^{\frac{1}{3}}$. It follows

$$|\phi(\xi)| \leq \frac{a\xi^{1-d}}{A(\xi)^2(1+\varepsilon^2)} \left| \int_{\xi_b}^\xi A^2\tilde{\xi}^{d-1} \left(\frac{d-2}{2} + K - K(b+1)A^2 - \frac{Ka^2\tilde{\xi}^2}{2} \right) d\tilde{\xi} \right| + O(a^{\frac{3}{4}}),$$

where we used $\xi_b = O(\log \frac{1}{a})$. From [40] Section 5, we conclude that for the 2-bump solution $A = A_0 + O(a)$, where

$$A_0 = \sqrt{2\left(1 - \frac{a\xi^2}{4}\right)} \operatorname{sech}\left(\sqrt{\left(1 - \frac{a\xi^2}{4}\right)y}\right) + O(a)$$

and $y = \xi - \frac{\eta}{a}$. Thus, for the 2-bump, we obtain

$$\begin{aligned} |\phi(\xi)| &\leq \frac{a\xi^{1-d}}{A(\xi)^2} \left| \underbrace{\int_{\xi_b}^{\frac{\eta}{a}-a^{-\gamma}} A_0^2 \tilde{\xi}^{d-1} \left(\frac{d-2}{2} + K_0 - K_0(b+1)A_0^2 - \frac{K_0 a^2 \tilde{\xi}^2}{2} \right) + O(a^2) d\tilde{\xi}}_{=O(a)} \right. \\ &\quad + \int_{\frac{\eta}{a}-a^{-\gamma}}^{\frac{\eta}{a}+a^{-\gamma}} A_0^2 \left(\frac{\eta}{a} \right)^{d-1} \left(\frac{d-2}{2} + K_0 - K_0(b+1)A_0^2 - \frac{K_0 \eta^2}{2} \right) + O(a^2) d\tilde{\xi} \\ &\quad \left. + \int_{\frac{\eta}{a}+a^{-\gamma}}^{\xi} A_0^2 \tilde{\xi}^{d-1} \left(\frac{d-2}{2} + K_0 - K_0(b+1)A_0^2 - \frac{K_0 a^2 \tilde{\xi}^2}{2} \right) + O(a^2) d\tilde{\xi} \right| + O\left(a^{\frac{3}{4}}\right) \\ &\leq \frac{a\xi^{1-d}}{A(\xi)^2} \left| \int_{-a^{-\gamma}}^{a^{-\gamma}} A_0(y)^2 \left(\frac{\eta}{a} \right)^{d-1} \left(\frac{d-2}{2} + K_0 - K_0(b+1)A_0(y)^2 - \frac{K_0 \eta^2}{2} \right) dy \right| + O\left(a^{\frac{3}{4}}\right). \end{aligned} \quad (\text{A.1})$$

Here $\frac{3}{4} < \gamma < 1$ and we assumed without loss of generality that $\frac{\eta}{a} + a^{-\gamma} < \xi < \xi_{\max}$. For $|y| < a^{-\gamma}$, we have

$$A_0 = \sqrt{2\left(1 - \frac{\eta^2}{4}\right)} \operatorname{sech}\left(\sqrt{\left(1 - \frac{\eta^2}{4}\right)y}\right) + O(a).$$

Thus, upon explicitly evaluating (A.1), we obtain

$$|\phi(\xi)| \leq \frac{a\xi^{1-d}}{A(\xi)^2} 4 \left(1 - \frac{\eta^2}{4}\right) \left(\frac{\eta}{a}\right)^{d-1} \left| \frac{d-2}{2} - K_0 - \frac{2}{3}(2b-1)K_0 \left(1 - \frac{\eta^2}{4}\right) \right| + O\left(a^{\frac{3}{4}}\right).$$

The last formula differs from the one derived in Appendix E of [40] due to a small calculation error. We conclude that in order for $|\phi|$ to be of $o(\sqrt{a})$ we require that at leading order

$$\left| \left(\frac{d-2}{2} - K_0 - \frac{2}{3}(2b-1)K_0 \left(1 - \frac{\eta^2}{4}\right) \right) \right|$$

is exponentially small in a . This condition corresponds to the condition $I_2 = 0$, see equations (2.102), (2.26a) and (2.26b) and also the discussion in Section 2.5.1.

For the n -bump, where n is even, the estimate is similar. Note that an n -bump solution has $\frac{n}{2}$ maxima on the halfline $\xi \geq 0$. Let one of these bumps have a maximum at $\xi = \frac{\eta_l}{a}$ in the interval $(\frac{\eta}{a} - a^{-\gamma}, \frac{\eta}{a} + a^{-\gamma})$. From [40], it follows that the maxima are $O(\log a)$ apart. As for

the 2-bump we integrate and obtain to leading order

$$\begin{aligned} \int_{\frac{\eta_i}{a} + \log a}^{\frac{\eta_i}{a} - \log a} A_0^2 \left(\frac{\eta_i}{a} \right)^{d-1} \left(\frac{d-2}{2} + K_0 - K_0(b+1)A_0^2 - \frac{K_0\eta_i^2}{2} \right) + O(a^2) d\tilde{\xi} \\ = 4 \left(1 - \frac{\eta_i^2}{4} \right) \left(\frac{\eta_i}{a} \right)^{d-1} \left| \frac{d-2}{2} - K_0 - \frac{2}{3}(2b-1)K_0 \left(1 - \frac{\eta_i^2}{4} \right) \right|, \end{aligned}$$

where

$$A_0 = \sqrt{2 \left(1 - \frac{\eta_i^2}{4} \right)} \operatorname{sech} \left(\sqrt{2 \left(1 - \frac{\eta_i^2}{4} \right)} y \right) + O(a)$$

and $y = \xi - \frac{\eta_i}{a}$. Now, we use the fact that $\eta_i = \eta + O(a \log a)$. It follows that

$$\begin{aligned} 4 \left(1 - \frac{\eta_i^2}{4} \right) \left(\frac{\eta_i}{a} \right)^{d-1} \left| \frac{d-2}{2} - K_0 - \frac{2}{3}(2b-1)K_0 \left(1 - \frac{\eta_i^2}{4} \right) \right| \\ = 4 \left(1 - \frac{\eta^2}{4} \right) \left(\frac{\eta}{a} \right)^{d-1} \left| \frac{d-2}{2} - K_0 - \frac{2}{3}(2b-1)K_0 \left(1 - \frac{\eta^2}{4} \right) \right| + o \left(\frac{1}{a^{d-1}} \right). \end{aligned}$$

Thus, integrating over $\frac{n}{2}$ of such bumps again yields that

$$\frac{n}{2} \left| \left(\frac{d-2}{2} - K_0 - \frac{2}{3}(2b-1)K_0 \left(1 - \frac{\eta^2}{4} \right) \right) \right|$$

should be exponentially small. □

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Samenvatting

Dit proefschrift gaat over oplossingen van de complexe Ginzburg-Landau vergelijking waarvan de amplitude oneindig groot wordt in eindige tijd. In deze samenvatting zal ik in een voor iedereen begrijpbare taal proberen uit te leggen wat dit precies inhoudt.

De Ginzburg-Landau vergelijking is een voorbeeld van een partiële differentiaalvergelijking. Partiële differentiaalvergelijkingen kunnen (onder andere) gebruikt worden om processen te modelleren die afhangen van zowel de plaats als de tijd, denk bijvoorbeeld aan: de temperatuur in een ruimte, de concentratie van reagerende chemische stoffen in een glas, maar ook aan zeer complexe systemen zoals het weer.

In tegenstelling tot vakgebieden als natuurkunde, scheikunde en biologie, houdt men zich in de wiskunde in eerste instantie niet bezig met de processen die deze partiële differentiaalvergelijkingen kunnen beschrijven, maar richt men zich met name op het begrijpen van de vergelijking an sich. Om ervoor te zorgen dat het wiskundig onderzoek aansluit bij toepassingen is het cruciaal dat er intensief wordt samengewerkt met aangrenzende vakgebieden. Een intensieve samenwerking kan ervoor zorgen dat de wiskunde vooruit geholpen wordt door de toepassingen en dat de toepassingen vooruit geholpen worden door het wiskundig onderzoek. Tegelijkertijd ontstaat er door het bestuderen van vergelijkingen ook “een groot boek” waarin alle vergelijkingen staan en waarin men kan opzoeken wat er allemaal over bekend is. Dit is een manier waarop veel mensen met wiskunde in aanraking komen: als je bijvoorbeeld de inhoud van een voetbal wilt uitrekenen, dan zoek je de bijbehorende formule op in een wiskundeboek.

Bij het bestuderen van (partiële) differentiaalvergelijkingen stellen wiskundigen vaak de volgende vragen: bestaat er een (unieke) oplossing, kunnen we begrijpen wat voor soort oplossingen er allemaal bestaan en zijn er stabiele toestanden. Deze vragen zal ik toelichten aan de hand van een proces dat gemodelleerd kan worden met een gewone differentiaalvergelijking. Dat is een differentiaalvergelijking waarin alleen de tijd een rol speelt en het proces niet afhangt van de plaats. Beschouw een plank die aan het uiteinde met een spijker aan de muur bevestigd is. De plank kan nu rond deze spijker in het vlak van de muur roteren. In de natuurkunde heet dit een mathematische slinger. Het meest voor de hand liggend is misschien de situatie waarin de plank stil hangt en de plank aan de spijker hangt, d.w.z. de spijker zit bovenaan de plank. Wanneer we de plank uit evenwicht brengen begint deze heen en weer te slingeren.

Het is mogelijk om dit proces met een differentiaalvergelijking te beschrijven. Gegeven de initiële positie en snelheid kan de positie van de plank op toekomstige tijdstippen berekend worden door deze differentiaalvergelijking op te lossen. De vraag of de opgeschreven differentiaalvergelijking een oplossing heeft kan, in dit geval, gezien worden als antwoord geven op de vraag of het opgeschreven model, dat is in dit geval de differentiaalvergelijking, überhaupt zinnig is. Het model beoogt immers een situatie in de praktijk beschrijven en in de praktijk

bevindt de plank zich op ieder tijdstip op precies één positie, dus zou dat in het model ook zo moeten zijn. Het bestaan van een unieke oplossing in het model kan, in dit geval, ook aangetoond worden.

In dit voorbeeld zullen veronderstellen dat er ook wrijving is en spreken dan van een gedempte mathematische slinger. Wat we verwachten is dat, wanneer de plank aan het oscilleren is, hij uiteindelijk stil zal komen te hangen aan de spijker (de spijker zit dan bovenaan de plank). Deze zogenaamde evenwichtstoestand noemen we stabiel, omdat wanneer we de plank een kleine uitwijking geven hij uiteindelijk weer naar dit evenwicht toegaat. Er is echter ook nog een andere evenwichtstoestand: namelijk, de plank kan in principe ook naar boven wijzen met de spijker aan de onderkant en daarop rusten. Deze evenwichtstoestand noemen instabiel: met een kleine duw zal dit evenwicht immers verstoord worden en komt de plank niet terug in het betreffende evenwicht.

De laatste vraag is het begrijpen van de structuur van de oplossingen. In dit voorbeeld is dat betrekkelijk eenvoudig. Kort samengevat, verwachten we dat de plank, wanneer deze voldoende snelheid heeft, een aantal keren om de spijker heen roteert, vervolgens wat heen en weer slingert en uiteindelijk in het stabiele evenwichtstoestand terecht komt. Het andere evenwicht waarbij de plank op z'n kop op de spijker rust bestaat wel en we kunnen er in terecht komen wanneer we de plank een duw geven die precies hard genoeg is. Echter, we verwachten we dit in de praktijk niet vaak te zien. Verder kunnen we, wanneer de initiële positie, initiële snelheid, wrijvingsconstante, massa etc. allemaal gegeven zijn, ook precies aangeven hoe vaak de plank heen en weer slingert.

Bij ingewikkeldere (partiële differentiaal)vergelijkingen stellen we in principe dezelfde vragen. Echter, in dat geval is de structuur van de oplossingen in zijn algemeenheid vaak lastig te beschrijven. Dit kan vergeleken worden met een situatie waarin we proberen het weer te beschrijven met een groot aantal ingewikkelde partiële differentiaalvergelijkingen. Het aantal mogelijkheden van wat er kan gebeuren is nu zo groot dat we in plaats daarvan ons afvragen hoe patronen kunnen ontstaan. Met patronen bedoelen we dan bijvoorbeeld een hogedrukgebied, lagedrukgebied etc. Deze vraag sluit aan bij observaties van het systeem in de praktijk. Een van doelen is uiteraard het begrijpen van deze observaties aan de hand het model. Wat we ons nu wederom afvragen is of deze patronen stabiel zijn. Wanneer ze instabiel zijn, verwachten we dat we deze patronen in de praktijk vrijwel nooit zullen waarnemen.

Of bepaalde patronen kunnen ontstaan hangt af van (exogene) parameters van het model. In dit voorbeeld zou wanneer het model alleen de luchtdruk beschrijft de temperatuur een parameter kunnen zijn. Er kan nu een grenswaarde zijn waarbij als we daar doorheen gaan bepaalde patronen wel en anders bepaalde patronen niet kunnen ontstaan. Dit heet een bifurcatie. Een duidelijk uitlegbaar voorbeeld van een bifurcatie is een faseovergang: boven de 100° Celsius verdampt al het water, net eronder wordt het vloeibaar.

In dit proefschrift heb ik niets gedaan met de toepassingen die ik zojuist beschreven heb en ook niet met vergelijkingen die deze processen beschrijven. De gegeven voorbeelden zijn uitsluitend bedoeld uit te leggen hoe wiskundigen in dit vakgebied partiële differentiaalvergelijkingen bestuderen. De complexe Ginzburg-Landau vergelijking, die ik in dit proefschrift bestudeerd heb, is ook een partiële differentiaalvergelijking. Deze partiële differentiaalvergelijking heeft te maken met patroonvorming (zie het voorbeeld over het weer) in een algemene klasse van partiële differentiaalvergelijkingen. De amplitude van een patroon dat kan ontstaan in deze algemene klasse van partiële differentiaalvergelijkingen kan in de buurt van een bifurcatie (onder bepaalde voorwaarden) bij benadering worden beschreven met de complexe Ginzburg-Landau vergelijking. De benadering is alleen geldig zolang de amplitude niet al te

groot is.

Wanneer we de Ginzburg-Landau vergelijking bestuderen kunnen we wederom dezelfde drie vragen stellen als zojuist besproken. Gegeven de achtergrond dat de Ginzburg-Landau vergelijking de amplitude van het patroon slechts benadert als deze niet al groot is, zijn we geïnteresseerd in de vraag of er in de Ginzburg-Landau vergelijking oplossingen zijn waarvan de amplitude wel groot wordt. Dergelijke oplossingen worden bestudeerd in dit proefschrift. Dit zegt dan iets over de geldigheid van de Ginzburg-Landau vergelijking om dergelijke patronen te beschrijven. In dit proefschrift wordt één specifieke oplossing bestudeerd waarvan de amplitude oneindig groot wordt in eindige tijd. In het laatste hoofdstuk van dit proefschrift wordt bewezen dat deze oplossing instabiel is, hetgeen betekent dat we hem in de praktijk “niet” zullen zien.

Curriculum vitae

Martin van der Schans was born on the 30th of May, 1982 in Rotterdam, the Netherlands. After completing his pre-university education in 2000 at Farel College in Ridderkerk, he started studying Mathematics and Physics at Leiden University. In 2001, he received his Propaedeutic diploma in both Mathematics and Physics. In 2006, he obtained his Master's degree in Mathematics. His Master's thesis was titled "Harmonic map heat flow" and presents a study on blowup behavior in the heat equation from the disk to the sphere.

After his graduation in 2006, he started his PhD research at the Mathematical Institute of Leiden University under supervision of Prof. Arjen Doelman and Dr. Vivi Rottschäfer; the results are presented in this thesis. As of 2011, he works as a financial researcher at Ortec-Finance bv located in both Rotterdam and Amsterdam, the Netherlands.