Stabilization of pulses by diffusion

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Introduction

Stationary and traveling pulses appear generically in the dynamics generated by nonlinear partial differential equations (PDEs) of evolution type. In the context of applications they for instance represent pulses through nerves, signals through transmission wires or optical fibers, solitary water waves, etc.. A typical example of a stationary pulse 'pattern' in the Gierer-Meinhardt system, a model that was originally introduced in the context of biological pattern formation ('morphogenesis') [6], is shown in Figure 1. The *existence* and *stability* of solutions of pulse type to a given system is a fundamental question in the study of nonlinear PDEs.

One-dimensional scalar parabolic equations of reaction-diffusion type are among the most simple nonlinear PDEs,

$$U_t = U_{xx} + F(U), \tag{1}$$

with $U(x,t) : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}$ and $F(U)$ a nonlinear function of U. A stationary pulse solution to this PDE corresponds to a so-called *'localized'* solution $u(x)$ of the ordinary differential equation (ODE) associated to the stationary problem: $u_{xx} + F(u) = 0$. A 'localized' solution is a solution that is close to a critical point of the ODE except on bounded x-intervals. Here, a critical point of the ODE corresponds to a solution $U(x,t) \equiv U_0$ of the PDE (1) with $F(U_0) = 0$; such a solution is called a (trivial) 'background state'. In the terminology of dynamical systems, a pulse solution corresponds to a *homoclinic* solution $u_h(x)$ of the ODE; it satisfies $\lim_{x\to\pm\infty} u_h(x) = U_0$.

The stability of the pulse can be established by linearizing around the pulse, i.e. by introducing the eigenvalue $\lambda \in \mathbf{C}$ and the perturbation $v : \mathbf{R} \to \mathbf{C}$ through $U(x,t) = u_h(x) + e^{\lambda t} v(x)$, substituting this 'Ansatz' into (1), and neglecting the nonlinear terms in v:

$$[\mathcal{L}(x) - \lambda]v = v_{xx} + [F'(u_h(x)) - \lambda]v = 0. \tag{2}$$

This is a singular Sturm-Liouville eigenvalue problem, or a (linear) Schrödinger equation. Note that eigenvalue problems of this type also occur naturally in quantum-mechanics [10]. The spectrum of the differential operator $\mathcal{L}(x)$ consists of essential spectrum, in this case given by the set $\{\lambda \in \mathbf{R} : \lambda \le F'(U_0)\}$ and a finite number (N) of discrete eigenvalues $\lambda_N \le ... \le \lambda_1$ [11]. Thus, the essential spectrum is determined by the background state U_0 [8]. A discrete eigenvalue λ_i

corresponds to a localized and integrable eigenfunction/perturbation $v_i(x)$. It can be expected that the homoclinic pulse solution is an asymptotically stable solution to the PDE if $F'(U_0) < 0$ and $\lambda_i < \lambda_1 = 0$ for all $i = 2, ..., N$ (note that $\lambda_i = 0$ is always an eigenvalue with eigenfunction $\frac{d}{dx}u_h(x)$) – see also [8].

However, the largest eigenvalue of the linearized stability problem for a pulse solution of (1) is always positive. This implies that the pulse solution is unstable [8]. As a consequence, the pulse cannot be observed in applications, or in numerical simulations of (1). The proof of this assertion for instance follows immediately from singular Sturm-Liouville theory [11], or can be obtained by various other classical approaches, such as by the use of maximum principles. Moreover, the characteristics of the problem can sometimes be computed explicitly (with the aid of classical special function theory). For instance, for the case $F(U) = U^2 - U$, which will be considered in the next section, $U_0 = 0$, $u_h(x) = \frac{3}{2}\mathrm{sech}^2(\frac{1}{2}x)$, $N = 3$ and $\lambda_1 = \frac{5}{4}$, $\lambda_2 = 0$, $\lambda_3 = -\frac{3}{4}$ [10].

Stabilization by diffusion

Although pulse solutions of scalar equations can thus never be stable, there are many systems of PDEs, such as the Gray-Scott model for autocatalytic reactions [7] and the Gierer-Meinhardt model, that exhibit pulse solutions that are very similar to scalar pulses, see Figure 1. These systems can be written in the following form

$$\left\{ \begin{array}{rclcr} U_t & = & U_{xx} & + & G(U,V) \\ \varepsilon^2 V_t & = & V_{xx} & + & \varepsilon^2 H(U,V) \end{array} \right. \tag{3}$$

where $0 < \varepsilon \ll 1$ is an asymptotically small parameter. Since both $U(x,t)$ and $V(x,t)$ are assumed to be bounded for $x \in \mathbf{R}$ – a natural condition for solutions to PDEs on unbounded domains – it follows that $V(x,t) \to V_0$, a constant, as $\varepsilon \to 0$. Hence, (3) reduces to (1), with $F(U) = G(U, V_0)$ in the limit $\varepsilon \to 0$. Note that a coupled system of reaction-diffusion equations in which the ratio of the diffusion coefficients can be assumed to be small, as is the case for the Gray-Scott and the Gierer-Meinhardt model, can always be written in the form (3).

The limit $\varepsilon \to 0$ intuitively yields two conclusions for the situation $0 \neq \varepsilon \ll 1$: *(i)* it can be expected that (3) indeed has solutions $(U_h(x), V_h(x))$ of pulse type of which the U-component $U_h(x)$ is close to a pulse solution $u_h(x)$ of a scalar PDE (1); *(ii)* the pulse solution $(U_h(x), V_h(x))$ can also not be stable: the spectrum of the linearization around $(U_h(x), V_h(x))$ should be close to that of the scalar case (which has an $\mathcal{O}(1)$ unstable eigenvalue). In general, neither of these assertions is (completely) true. Moreover, both the existence *(i)* and the stability *(ii)* question require the development of novel ideas and techniques. For instance, the scalar problem (1) with $F(U) = G(U, V_0)$ will have a pulse solution for open V_0-sets, the limit procedure is expected to select only a discrete number of the V_0's and the corresponding pulses $u_h(x; V_0)$. However, the ODE associated to the stationary problem for the PDE (3) is a dynamical system in $\mathbf{R}^4$. A priori,

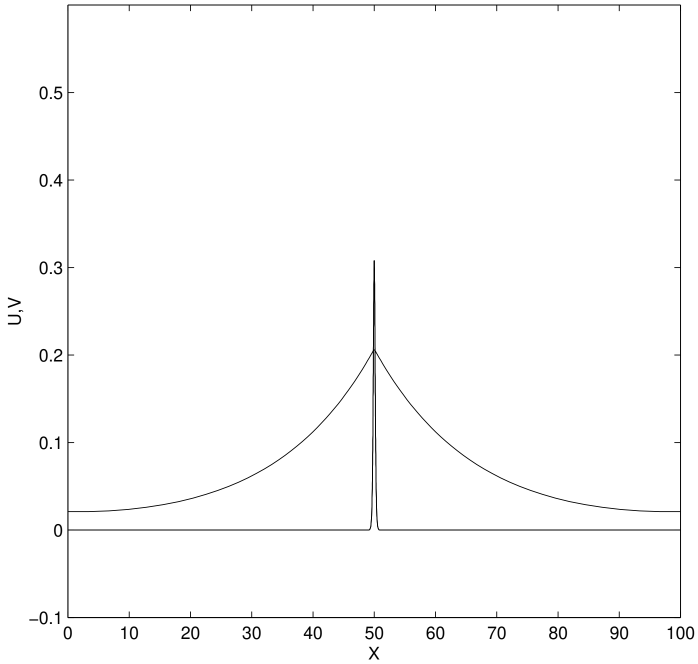

Figure 1: The homoclinic pulse solution of (4) for $\varepsilon^2 = 0.1$ and $\mu = 0.38$. Here, the scaled pulses $\varepsilon U_h(x)$ and $\varepsilon V_h(x)$ are plotted on the long spatial scale $X = \varepsilon x$. The singular 'sharp' pulse represents the U-component, it is $\mathcal{O}(\varepsilon)$ close to a homoclinic solution of the scalar equation (1) with $F(U) = \frac{2}{\sqrt{\mu}} U^2 - U$ (Theorem 1). This figure has been obtained by direct numerical simulation of (4) using the moving-grid code described in [2].

there are no general techniques by which the existence of homoclinic orbits in such problems can be established. Furthermore, the stability problem yields a linear system that is neither of Sturm-Liouville type, nor self-adjoint, which implies that one should expect complex eigenvalues λ. Moreover, the Sturm-Liouville equation (2) associated to the scalar limit indeed is a singular limit, the 'perturbations' due to the coupling to the additional slow diffusion equation for V have a leading order effect on the eigenvalues. In other words: the coupling of U to a slow diffusion equation for V might be able to bring the unstable eigenvalue of the scalar limit problem to the stable $\{\mathrm{Re}(\lambda) \leq 0\}$ half-plane.

As an example we present here two theorems that settle both the the existence problem and the stability problem in the special case of the classical Gierer-Mienhardt problem,

$$\left\{ \begin{array}{rclcrcr} U_t & = & U_{xx} & + & [\frac{U^2}{V} & - & U] \\ \varepsilon^2 V_t & = & V_{xx} & + & \varepsilon^2[U^2 & - & \mu V] \end{array} \right. \tag{4}$$

It should be remarked, however, that this case is less special than suggested by comparing (3) to (4). In fact, (4) can be seen as a prototypical system: under certain (generic) conditions equation (3) can be scaled into a *normal form* of the type (4) [3]. Essential in the derivation of the normal form is that both the U-component $U_h(x)$ as well as the V-component $V_h(x)$ of the pulse solution to (3) scale with a certain negative power of ε as $\varepsilon \to 0$, i.e. the amplitude of the pulse $(U_h(x), V_h(x))$ is in general asymptotically large [3].

Theorem 1. [3] *For any $\mu > 0$ there exists an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ there exists a homoclinic pulse solution $(U_h(x), V_h(x))$ to (4). Both components are of $\mathcal{O}(\frac{1}{\varepsilon})$, i.e. both $\lim_{\varepsilon\to 0} \varepsilon U_h(x)$ and $\lim_{\varepsilon\to 0} \varepsilon V_h(x)$ exist (and are not 0). Moreover, $\lim_{\varepsilon\to 0} \varepsilon U_h(x) = u_h(x)$, the homoclinic solution of the associated scalar limit problem (1) with $F(U) = \frac{U^2}{V_0} - U$ and $V_0 = \frac{1}{2}\sqrt{\mu}$.*

See Figure 1. The proof of this theorem is based on the ideas of *geometric singular perturbation theory*, see [9].

Theorem 2. [3] *The homoclinic pulse solution $(U_h(x), V_h(x))$ of (4) is unstable for $0 < \mu < \mu_{\mathrm{Hopf}} = 0.36... + \mathcal{O}(\varepsilon)$, and spectrally stable for $\mu > \mu_{\mathrm{Hopf}}$.*

The proof of this result relies heavily on the *Evans function approach* as was developed in [1]. This method was extended in [4] to systems of the type (3), in the context of an explicit model problem – the Gray-Scott equation. Later, this method has been generalized, so that it is possible to consider the general equation (3). Note that one can go from a situation where the above intuitive argument *(ii)* is valid to a bifurcation that contradicts the intuition, by varying the parameter μ: if $\mu > 0$ is small enough there is a unstable eigenvalue $\lambda_1(\mu) \in \mathbf{R}$ that merges with the unstable eigenvalue $\lambda_1 = \frac{5}{4}$ of the scalar limit problem (2) in the limit $\mu \to 0$ [3]. However, at a certain critical value $\mu_{\mathrm{complex}} < \mu_{\mathrm{Hopf}}$ $\lambda_1(\mu)$ merges with another real (positive) eigenvalue $\lambda_2(\mu)$ and a pair of complex conjugated eigenvalues is formed. This pair crosses the imaginary axis as μ passes through μ_{Hopf}. Furthermore, it follows from the methods developed in [4, 3] that the number of eigenvalues of the linearized stability problem associated to (4) is 4, two of which can be complex, one more than that of the limit problem (2), which only has real eigenvalues. Hence, the relation between (4) and its scalar limit (1) with $F(U) = \frac{U^2}{V_0} - U$ is more singular than expected at first sight.

Conclusion

The main conclusion is that slow diffusion can act as a *stabilization mechanism* in nonlinear reaction-diffusion equations. Patterns that are unstable solutions of a scalar equation (1) can be stabilized by the a priori negligible effects of coupling a slow diffusion equation to this scalar problem (as in (3)). The general theory developed in [4, 3] can be used to determine whether this 'control mechanism' is effective; it can be applied to large classes of singularly perturbed reaction-diffusion equations, including the well-studied Gray-Scott and Gierer-Meinhardt models. Moreover, the theory of [4, 3] is formulated in the setting of the Evans function approach. As a consequence, it is possible to extend the method both to more general classes of 'patterns' (including multi-pulse and periodic patterns) and to more general systems than two-component reaction-diffusion equations in one spatial dimension [5].

References

[1] J. Alexander, R. A. Gardner, and C.K.R.T. Jones [1990], A topological invariant arising in the stability of traveling waves, *J. Reine Angew. Math.* **410** 167–212.

[2] J.G. Blom and P.A. Zegeling [1994], Algorithm 731: a moving-grid interface for systems of one-dimensional time-dependent partial differential equations, *ACM Trans. Math. Software* **20** 194–214.

[3] A. Doelman, R. A. Gardner and T.J. Kaper [2001], Large stable pulse solutions in reaction-diffusion equations, *Indiana U. Math. J.* **50**(1) 443–507.

[4] A. Doelman, R. A. Gardner and T.J. Kaper [2002], A stability index analysis of 1-D patterns of the Gray–Scott model, *Memoirs AMS* **155** number **737**.

[5] A. Doelman, H. van der Ploeg [2002], Homoclinic stripe patterns, to appear in *SIAM Appl. Dyn. Syst.*

[6] A. Gierer and H. Meinhardt [1972], A theory of biological pattern formation, *Kybernetik* **12** 30–39.

[7] P. Gray and S.K. Scott [1983], Autocatalytic reactions in the isothermal, continuous stirred tank reactor: isolas and other forms of multistability, *Chem. Eng. Sci.* **38** 29–43.

[8] D. Henry [1981], *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics **840**, Springer.

[9] C.K.R.T. Jones [1995], *Geometric Singular Perturbation Theory*, in *Dynamical systems, Montecatibi Terme, 1994*, Lecture Notes in Mathematics **1609**, R. Johnson (ed.), Springer.

[10] L.D. Landau and E.M. Lifschitz [1977], *Quantum Mechanics: Non-Relativistic Theory*, *Course on Theoretical Physics* **3** (3rd edn.), Pergamon.

[11] E.C. Titchmarsh [1962], *Eigenfunction Expansions Associated with Second-Order Differential Equations* (2nd edn.), Oxford U. Press.