

HPDI: Goals; - Examples of sod

- Shed a light on  $Q_n$ . Given a sm proj var.  $X \subseteq \mathbb{P}^n$

$X_H = X \cap H$  What can we say about  $D(X_H)$  in terms of  
sod of  $D(X)$ ?

Example (contd.): -  $X$  sm proj.  
-  $\mathcal{O}_X(1)$  be a bpf line bundle.

$V \cong H^0(X, \mathcal{O}_X(1))$  bpf  $\leadsto \exists X \xrightarrow{f} \mathbb{P}(V^*)$

Convention:  $\mathbb{P}(E)$  is covariant. (different from [Har 98] or Stacks)

$$\text{i.e. } \mathcal{E} \rightarrow \mathcal{F} \rightsquigarrow \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{F})$$

$$- \mathcal{O}_X(-1) \rightarrow \mathcal{E} \rightsquigarrow X \rightarrow \mathbb{P}(\mathcal{E})$$

$$- f: T \rightarrow X \quad T \rightarrow \mathbb{P}(\mathcal{E}) \rightsquigarrow \mathcal{L} \rightarrow f^* \mathcal{E}.$$



Define, universal hyperplane

$$\mathcal{H} = \{ (x, s) \in X \times \mathbb{P}(V) \mid s(x) = 0 \} \subseteq X \times \mathbb{P}(V)$$

Then, ①  $\boxed{\mathcal{H} = Z(\text{ev})}$

$$\text{ev} \in H^0(X \times \mathbb{P}(V), \mathcal{O}_X(1) \boxtimes \mathcal{O}_{\mathbb{P}(V)}(1))$$

$$\cong H^0(X, \mathcal{O}_X(1) \otimes V^*)$$

$$V \otimes \mathcal{O}_X \xrightarrow{\text{ev}} \mathcal{O}_X(1)$$

②  $0 \rightarrow \text{Ker} \rightarrow V \otimes \mathcal{O}_X \rightarrow \mathcal{O}_X(1) \rightarrow 0$

$$\boxed{\mathcal{H} = \mathbb{P}_X(\text{Ker})}$$

$\mathcal{H} \subset X \times \mathbb{P}(V)$  is a  $(1,1)$  hyperplane.  $\pi: \mathcal{H} \rightarrow X$

$$D(\mathcal{H}) = \left\langle \begin{array}{l} \pi^* D(X) \otimes \mathcal{L}^* \mathcal{O}_{\mathbb{P}(V)}(1), \dots, \\ \parallel \\ D(X)(0,1) \quad \pi^* D(X) \otimes \mathcal{L}^* \mathcal{O}_{\mathbb{P}(V)}(\dim V - 1) \end{array} \right\rangle$$

$$D(X \times \mathbb{P}(V)) = \langle \pi^* D(X), D(V) \rangle$$

This is a special case of Prop. 3-6.



Example 2:  $L \subseteq V$  linear subspace.

$$\mathcal{H}_L = \{ (x, s) \mid x_s(x) = 0 \} \subseteq X \times \mathbb{P}(L)$$

$$\underline{X_L = \mathcal{B}_s(L)} \quad (= : X_{L^\perp} \text{ notation in the notes})$$

Prop 3.6:  $\textcircled{1}$   $D(X_L) \xrightarrow{j_*^*} D(\mathcal{H}_L)$  full, faithful embedding  
Assume,  $\dim X_L = \dim X - l$

$$\textcircled{2} \underline{D(X)} \xrightarrow{\pi^*} D(\mathcal{H}_L)$$

$$\textcircled{3} D(\mathcal{H}_L) = \langle D(X_L), \pi^* D(X) \otimes \mathcal{L}^* \mathcal{O}_{\mathbb{P}(L)}(1), \dots, \pi^* D(X) \otimes \mathcal{L}^* \mathcal{O}_{\mathbb{P}(L)}(l-1) \rangle$$

$l = \dim L$



§ Aside:  $\text{rk } L = 2$

$$L \otimes L \longrightarrow \wedge^2 L \cong \mathbb{C}$$

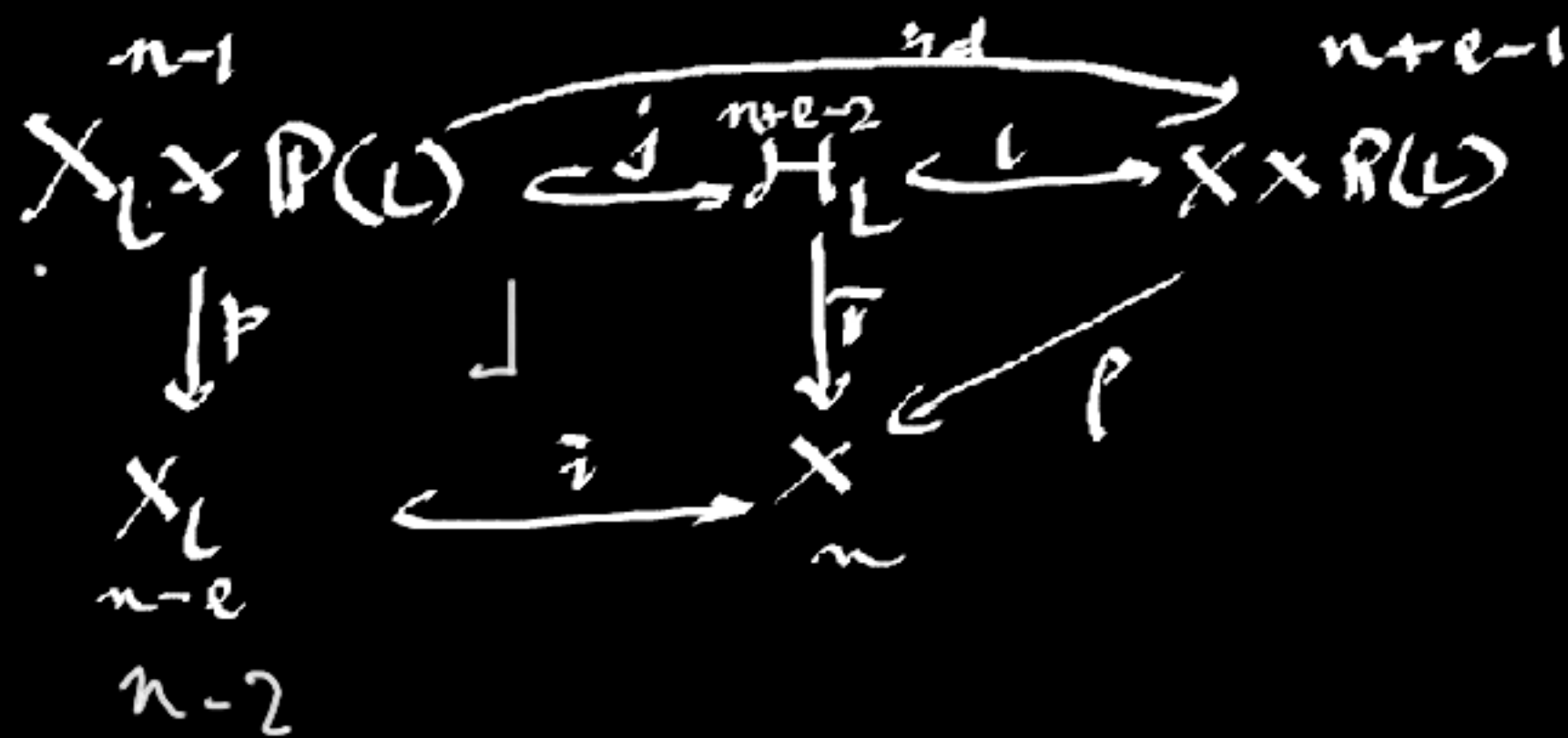
$$\cong L \xrightarrow{\sim} L^*$$

$$X \times \mathbb{P}(L) \xrightarrow{\sim} \mathbb{P}(L^*)$$

Q<sub>n</sub>: How to identify  $\mathcal{H}_L$  inside  $X \times \mathbb{P}(L^*)$

$$X \times \mathbb{P}(L) \hookrightarrow \mathcal{H}'_L \longrightarrow X \times \mathbb{P}(L^*)$$

Claim:  $\mathcal{H}'_L = \mathbb{B} \times X$



$H_{x_2} \quad 0 \rightarrow K \rightarrow L \otimes \mathcal{O}_X \xrightarrow{ev} \mathcal{O}_X(1) \rightarrow \mathcal{G}_{X_2} \rightarrow 0$

$\downarrow 0 \quad \downarrow \mathcal{O}_{X_2}(1)$

$B|_{X_2} X = Proj(\mathcal{G}_{X_2}^v)$   
 $X_2 \times IP(L) \hookrightarrow B|_{X_2} X$   
 is  
 $Proj(\mathcal{G}_X / \mathcal{G}_{X_2}^v)$

$\curvearrowright$   $0 \rightarrow K \rightarrow L \otimes \mathcal{O}_X \rightarrow \mathcal{G}_{X_2} \rightarrow 0$

$\downarrow \text{five lemma}$

Dual  $0 \rightarrow \mathcal{G}_{X_2}^* \rightarrow L^* \otimes \mathcal{O}_X \rightarrow Ker(K^* \rightarrow Ext^1(\mathcal{G}_X, \mathcal{O}_X)) \rightarrow 0$

$\downarrow \text{five lemma}$   
 0

$\left. \begin{array}{l} \mathcal{G}_{X_2} \text{ is torsion free} \\ + \mathcal{G}_{X_2} \simeq Ker|_{X_2} \end{array} \right\} \Rightarrow \mathcal{G}_{X_2} \hookrightarrow Ker \xrightarrow[\text{lemma}]{\text{five}} K \simeq \mathcal{G}_{X_2}^*$



$$\begin{array}{ccc} \mathbb{B} \times_L X & & \\ \downarrow & \searrow f & \\ X & \dashrightarrow & \mathbb{P}(L^*) \end{array}$$

$$\Gamma_f: \mathbb{B} \times_L X \hookrightarrow X \times \mathbb{P}(L^*)$$

Proof of Prop 3.6

$$\textcircled{1} \quad j_* p^*: \mathcal{D}(X_1) \longrightarrow \mathcal{D}(X)$$

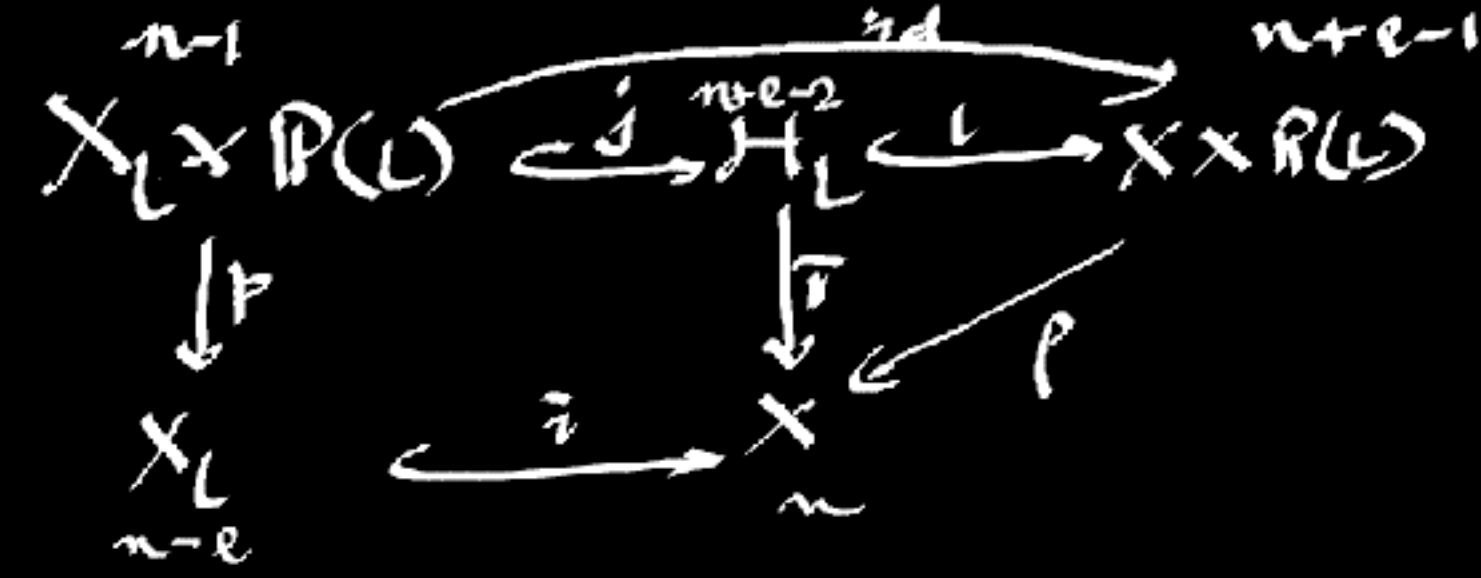
$$\text{WTS } \text{RHom}(j_* p^* E, j_* p^* F) \cong \text{RHom}(p^* E, p^* F)$$

$$\cong \text{RHom}(j^* j_* p^* E, p^* F)$$

$$\text{Counit } j^* j_* \longrightarrow \text{id}$$

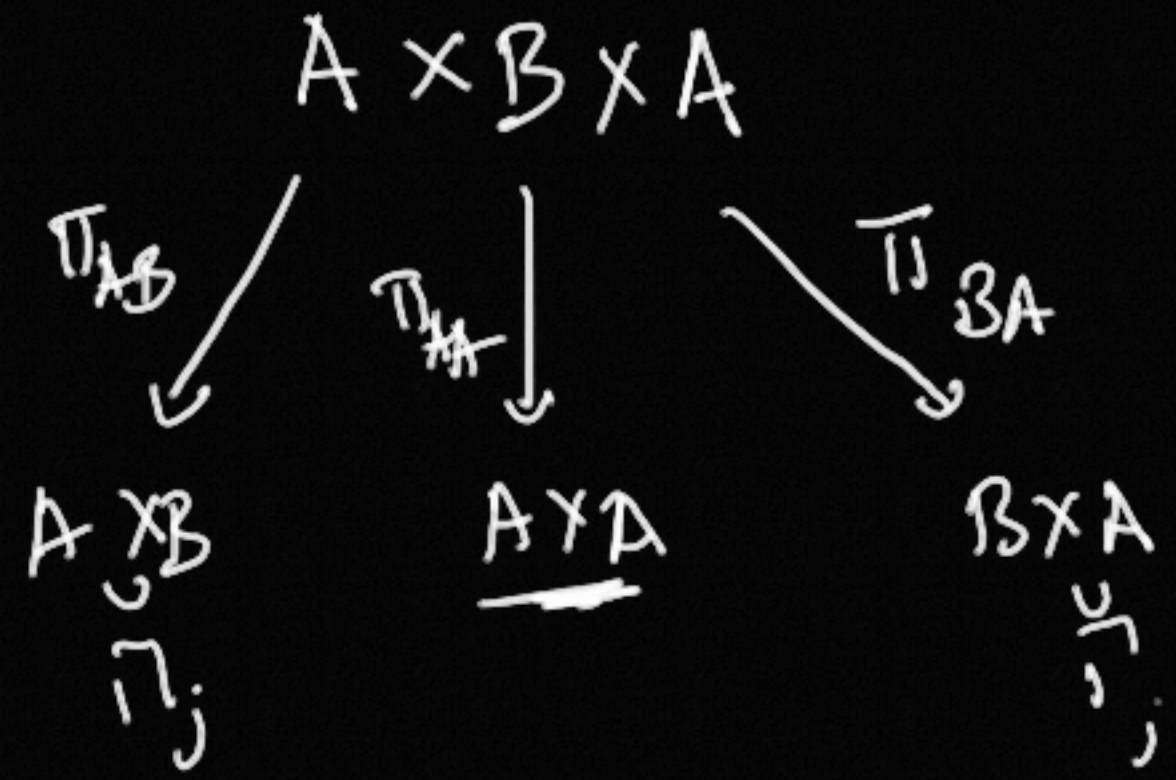
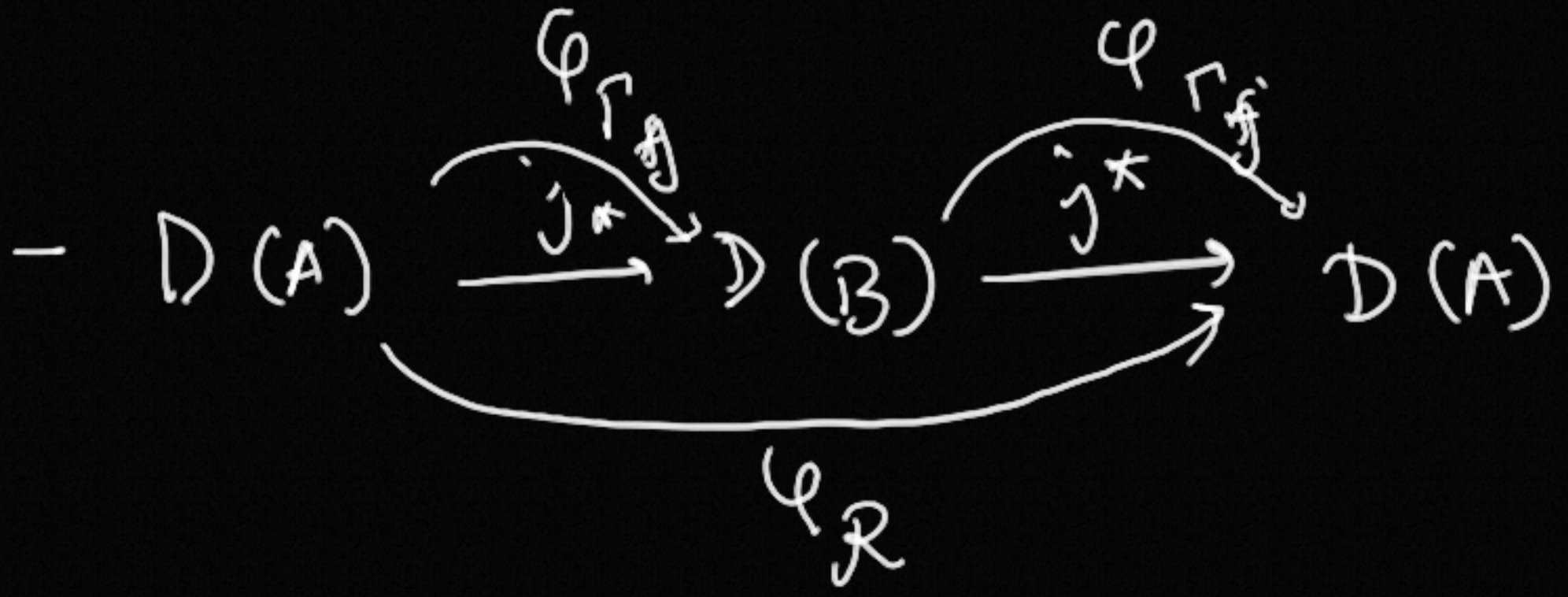


$$j^* j_* \rightarrow \text{id} \quad j: A \hookrightarrow B$$



$$- \text{id} = \varphi_{\mathcal{O}_\Delta} \quad \mathcal{O}_\Delta = \Delta_* \mathbb{A}$$

$$\Delta: A \rightarrow A \times A$$



$$\mathcal{R} = \pi_{A \times R}^* \left( \pi_{AB}^* \varphi_{\Gamma_j} \otimes \pi_{BA}^* \varphi_{\Gamma_j} \right)$$

§ 11.1 Cor 11.4 + Prop 11.1

Let  $\mathcal{N}_{A/B}$  = normal bundle

$$\dots \rightarrow \Delta_* \mathcal{N}_j^{\vee} \rightarrow \Delta_* \mathcal{N}_j^{\vee} \rightarrow \mathcal{R} \rightarrow \mathcal{O}_A \rightarrow 0$$



$\exists$  a ~~iterated~~ iterated triangle

$$\underbrace{\begin{array}{c} \Lambda^r N_j^v[\Gamma] \rightarrow j^* j_* \rightarrow \text{id} \\ \otimes(\gamma) \end{array}}$$

$$\mathcal{R} \simeq \bigoplus_{r=0}^{\infty} \Delta_* N_j^v[\Gamma]; \quad r=0: \Delta_* \mathcal{O}_{X_0} = \mathcal{O}_\Delta$$

$$\underline{j^* j_* \simeq \varphi_{\mathcal{R}} \simeq \text{id} \oplus (\Lambda^r N_j^v \otimes \mathcal{O}(\gamma))}$$

Taking FM  
Kornd

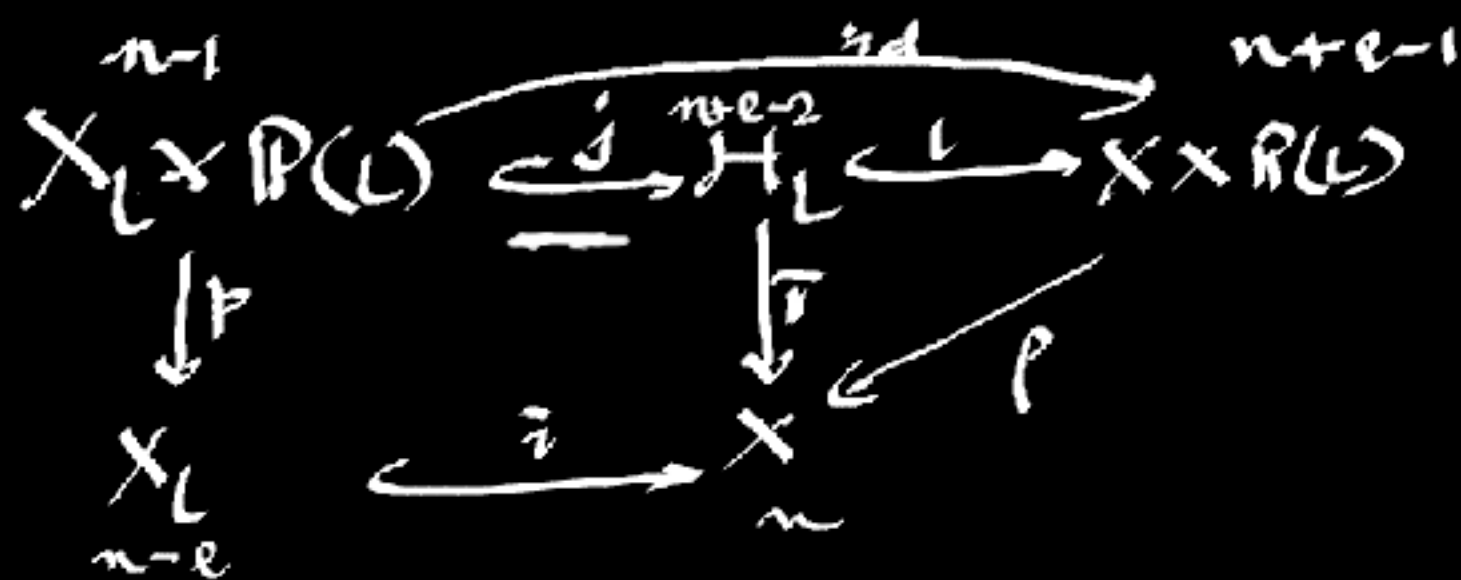
Back to our situation

$$H_L = Z(\text{ev}) \quad \text{ev} \in H^0(X \times P(L),$$

$$(\mathcal{O}_X(1) \boxtimes \mathcal{O}_{P(L)}(1)))$$

$$X_L = Z(\text{ev}_L) \quad \text{ev}_L \in H^0(X, \mathcal{O}_X(1) \otimes L^*)$$

Want to find  $\mathcal{N}_{X_L \times P(L) / H_L}$





Have,  $X \times \mathbb{P}(L) \xrightarrow{j} \mathcal{H}_L \xrightarrow{\nu} X \times \mathbb{P}(L)$

$$\begin{array}{c}
 0 \rightarrow \mathcal{N}_{X \times \mathbb{P}(L)} \xrightarrow{\quad} \mathcal{N}_{X \times \mathbb{P}(L)} / \mathcal{H}_L \xrightarrow{\quad} \mathcal{N}_{\mathcal{H}_L / X \times \mathbb{P}(L)} \rightarrow 0 \\
 \quad \quad \quad \downarrow \cong \quad \quad \quad \downarrow \cong \quad \quad \quad \downarrow \cong \\
 0 \rightarrow \left[ \mathcal{O}_{X \times \mathbb{P}(L)} \boxtimes \mathcal{O}_{\mathbb{P}(L)} \right] \rightarrow \left[ \mathcal{O}_{X \times \mathbb{P}(L)} \boxtimes \mathcal{O}_{\mathbb{P}(L)} \right] \rightarrow \left[ \mathcal{O}_{X \times \mathbb{P}(L)} \boxtimes \mathcal{O}_{\mathbb{P}(L)} \right] \rightarrow 0 \\
 \quad \quad \quad \downarrow \cong \quad \quad \quad \downarrow \cong \quad \quad \quad \downarrow \cong \\
 0 \rightarrow \left[ \mathcal{O}_{X \times \mathbb{P}(L)} \boxtimes \mathcal{O}_{\mathbb{P}(L)} \right] \rightarrow \left[ \mathcal{O}_{X \times \mathbb{P}(L)} \boxtimes \mathcal{O}_{\mathbb{P}(L)} \right] \rightarrow \left[ \mathcal{O}_{X \times \mathbb{P}(L)} \boxtimes \mathcal{O}_{\mathbb{P}(L)} \right] \rightarrow 0
 \end{array}$$

Use Relative Euler seq. for  $X \times \mathbb{P}(L)$ .

$$\begin{aligned}
& R\mathrm{Hom}(\wedge^r N_j^\vee \otimes p^*E, p^*F) \\
&= R\mathrm{Hom}(p^*E, p^*(F \otimes \mathcal{O}_{X_L}(r)) \otimes \Omega_{\mathbb{P}^1}^r(r)) \\
&= R\mathrm{Hom}(E, F \otimes \mathcal{O}_{X_L}(r) \otimes p_* \Omega_{\mathbb{P}^1}^r(r)) \\
&\quad \parallel \\
&\quad 0
\end{aligned}$$

Using  $H^0(\mathbb{P}^{e-1}, \Omega_{\mathbb{P}^{e-1}}^r(r)) = 0$  via Koszul res'n

Eagon-Northcott for  $p_* \Omega_{X \times \mathbb{P}^1}^r(r)$ .

Recall

$$R\mathrm{Hom}(i^* j_* E, p^* F)$$

$\downarrow \cong$

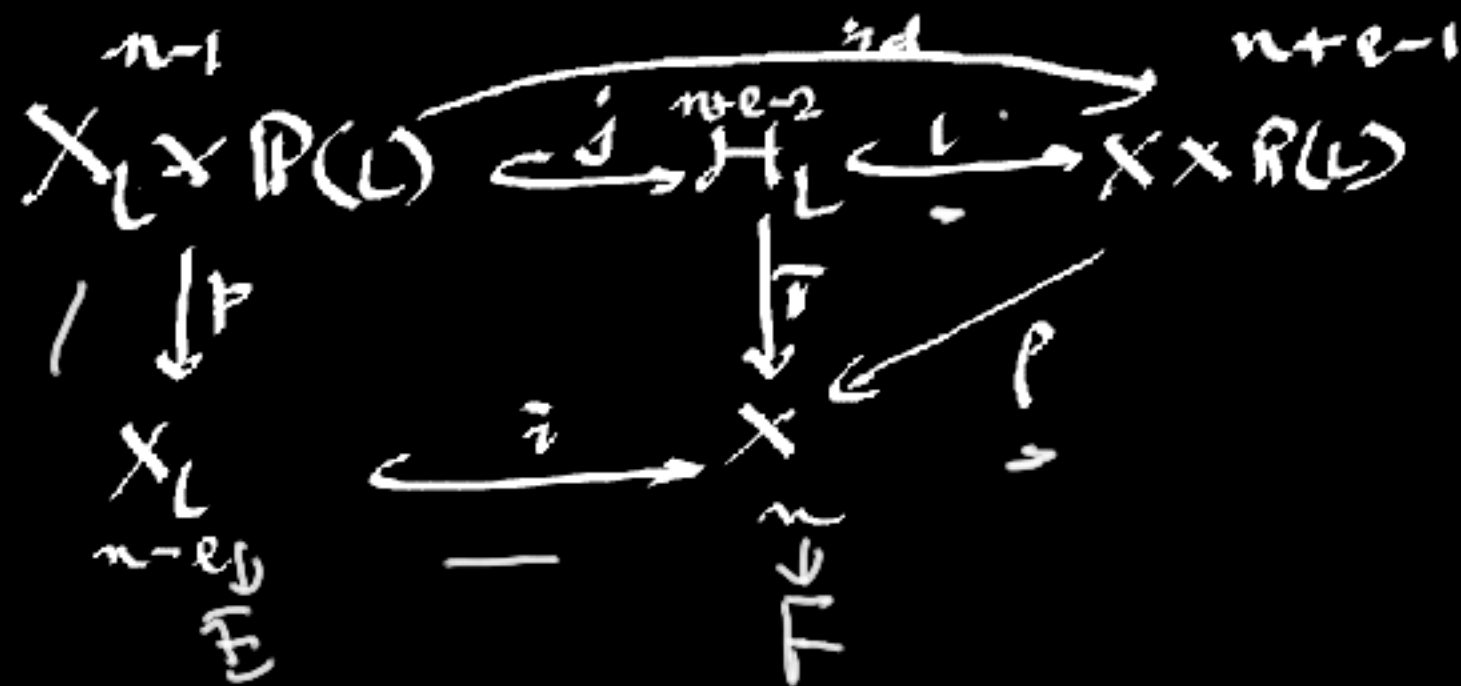
$$R\mathrm{Hom}(p^* E, p^* F)$$

conclude  $\rightarrow 0$



③ SOD

Recall  $D(\mathcal{H}_L) = \langle D(x_0), \pi^* D(x)(0, 1), \dots, \pi^* D(x)(0, l-1) \rangle$



WTS  $R\text{Hom}(\pi^* F \otimes \mathcal{O}_{P(L)}(k), j_* p^* E) = 0$

$\pi^* F(0, k)$

$\simeq R\text{Hom}(j^* \pi^* F \otimes \mathcal{O}_{P(L)}(k), p^* E)$

$\simeq R\text{Hom}(p^* j^* F \otimes \mathcal{O}_{P(L)}(k), p^* E)$

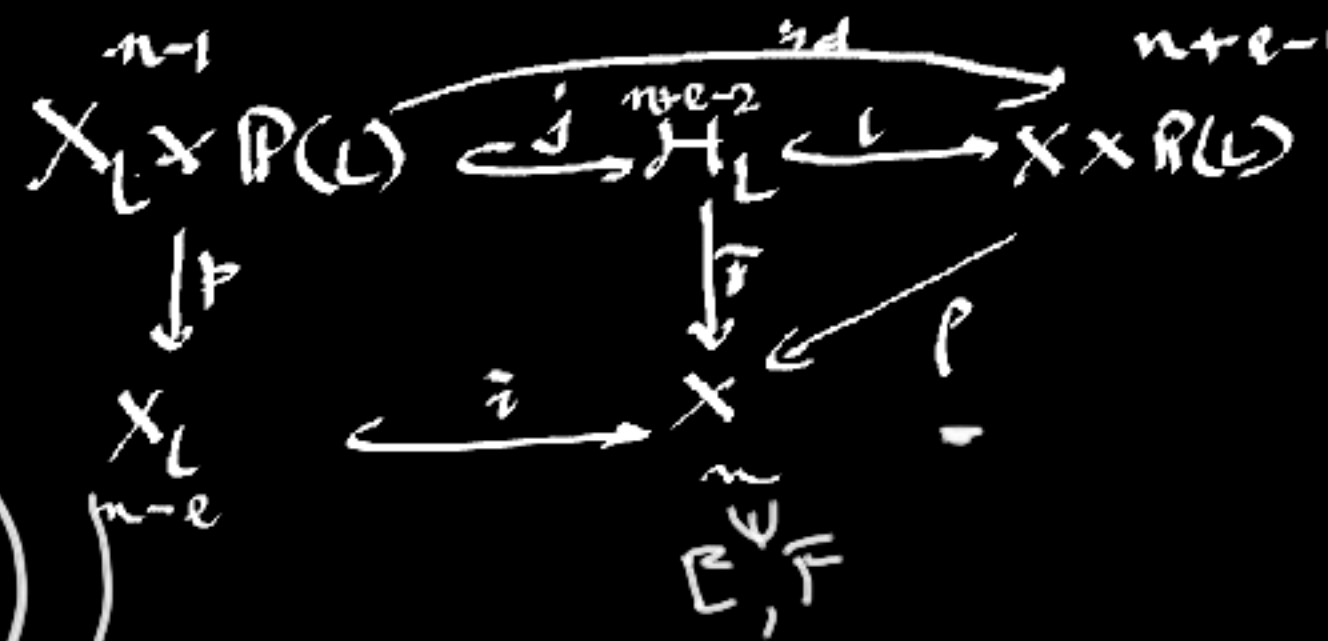
$\simeq R\text{Hom}(v^* F, E \otimes p_* \mathcal{O}_{P(L)}(-k))$

$\simeq 0$

$k > 0$

3 b)  $R\text{Hom}(\pi^* E(0, R+1), \pi^* F(0, K))$

$= R\text{Hom}(L^*(p^* E \otimes \mathcal{O}_{P(L)}(K+1)), L^*(p^* F \otimes \mathcal{O}(K)))$



$= R\text{Hom}(p^* E \otimes \mathcal{O}(K+1), \underline{L_X L^*}(p^* F \otimes \mathcal{O}(K)))$

$\xrightarrow{id} \cong$

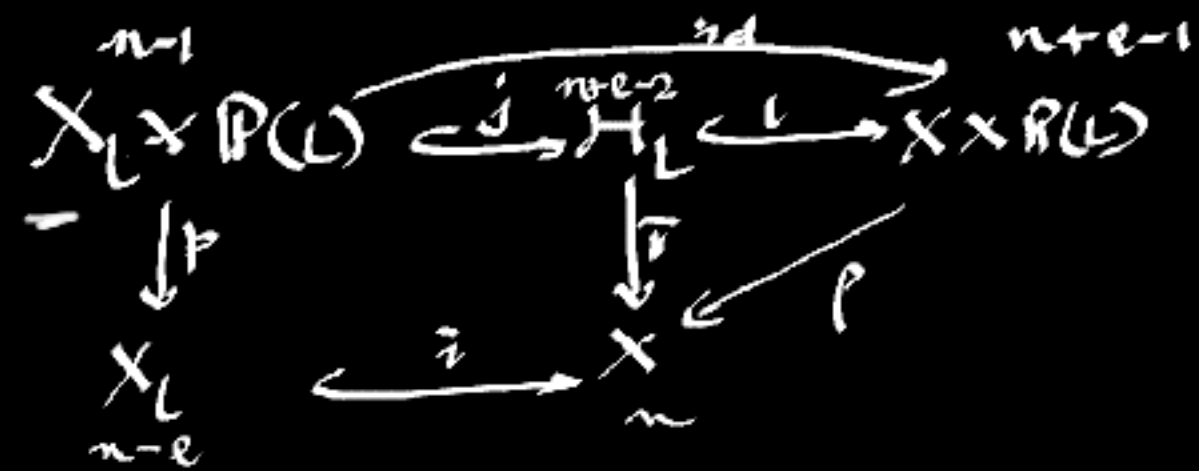
$id(-1, -1) \rightarrow id \rightarrow v_* L^*$

$R\text{Hom}(p^* E \otimes \mathcal{O}, p^* F(-1) \otimes \mathcal{O}(-2))$

$= R\text{Hom}(E, F(-1) \otimes p_* \mathcal{O}_{P(L)}(-2)) = 0$



②  $\pi^*: D(X) \rightarrow D(\mathcal{H}_L)$   
 Similar.

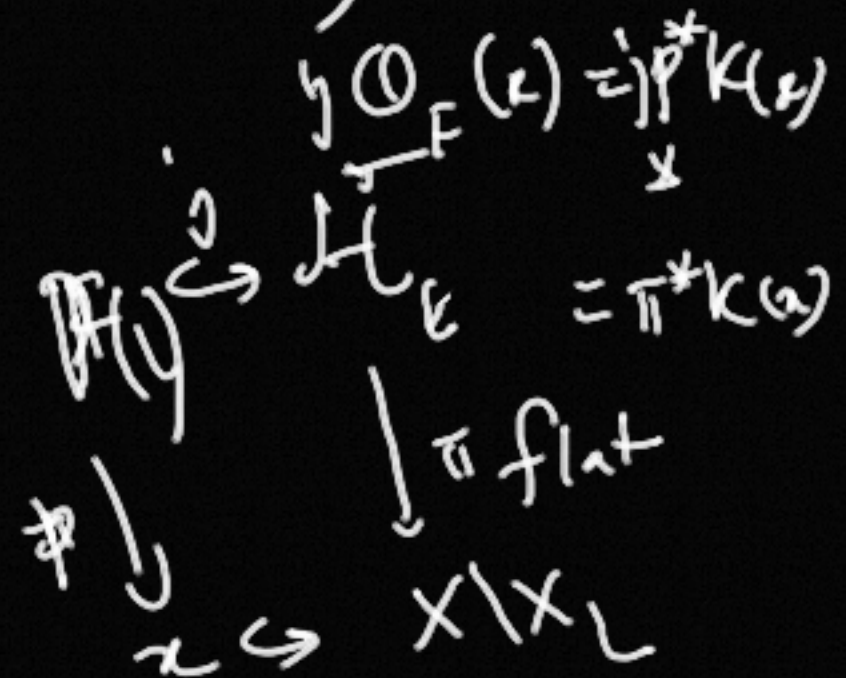


Generation:  $E \in \langle D(X), \pi^* D(X)(0,1), \dots \rangle$

Then  $E \cong 0$

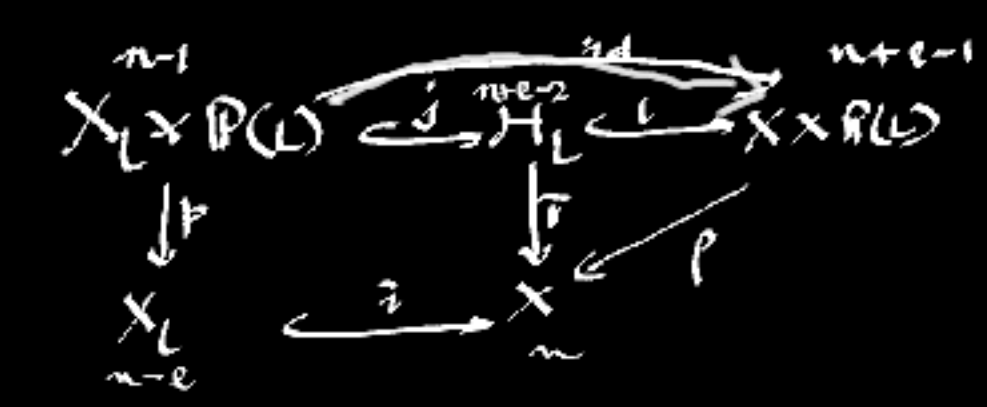
ETS  $E \otimes K(x) = 0 \quad x \in \mathcal{H}_L$

Claim  $E|_{\pi^{-1}(x)} = 0 \quad x \in X \quad F = \pi^{-1}(x)$



Case @:  $x \in X \setminus X_L, \mathcal{R}Hom(E|_F, \mathcal{O}_F(x)) = \mathcal{R}Hom(E, \pi^* K(x)) = 0$  [  $\because \pi^* K(x) \in D(X)$  ]

Case ①  $x \in X_1$



$$\frac{\pi^* \kappa(\alpha)}{j_*} = i^* p^* \kappa(\alpha) = \boxed{i^* \kappa(\alpha)} \Big|_{j_*} \mathcal{O}_{\{x\} \times P(L)}$$

$$j_* \mathcal{O}_{\{x\} \times P(L)}(-1, -1) \rightarrow \pi^* \mathcal{O}_x \rightarrow \hat{j}_* \mathcal{O}_{\{x\} \times P(L)}$$

Get  $j_* \mathcal{O}_{\{x\} \times P(L)}(-1, -1) \rightarrow \pi^* \mathcal{O}_x \rightarrow \hat{j}_* \mathcal{O}_{\{x\} \times P(L)}$

Note, WTS  $R\text{Hom}(E|_E, \mathcal{O}_{P(L)}) = 0 \quad \checkmark$

Remark:  $0 \rightarrow R\text{Hom}(E, \underbrace{j_* \mathcal{O}_{\{x\} \times P(L)}}_{\pi^* \kappa(\alpha)}) = R\text{Hom}(E, \underbrace{j_* p^* \kappa(\alpha)}_{D(x)})$



$$\textcircled{2} R\text{Hom}(E, \pi^* K(X)) = 0 \quad \pi^* K(X) \in \mathcal{D}(X)$$

$$R\text{Hom}(E_n, \mathcal{O}_{\mathbb{P}(L)}(k-1)[1]) \rightarrow R\text{Hom}(E_n, \pi^* K(X)|_E) \xrightarrow{\cong} R\text{Hom}(E_n, \mathcal{O}_{\mathbb{P}(L)}(k))$$

$$R\text{Hom}(E_n, \mathcal{O}_{\mathbb{P}(L)}) = 0 \Rightarrow R\text{Hom}(E_n, \mathcal{O}_{\mathbb{P}(L)}(k)) = 0 \quad \forall k$$

$$E_n = 0 \quad \pi^* K(X) \in \mathcal{D}(\mathbb{P}(L))$$

### § Cohomology ring of $\mathcal{H}_L$ :

$$[\mathcal{H}_L] = [X \setminus X_L][\mathbb{P}^{l-2}] + [X_L][\mathbb{P}^{l-1}]$$

$$H^*(\mathcal{H}_L, \mathbb{Q}) \quad R\pi_* \mathbb{Q}|_{X \setminus X_L} = \bigoplus R^i \pi_* \mathbb{Q}[-i]|_{X \setminus X_L} \\ = \bigoplus \mathbb{Q}[-i]$$

$$\mathbb{R}\pi_* \mathbb{Q} \simeq \bigoplus_{i=0}^{l-2} \mathbb{Q}[-2i] \oplus \mathcal{B} \quad \text{Supp } \mathcal{B} = X_L$$

$$H^i(\mathbb{R}\pi_* \mathbb{Q})^{\otimes K(L)} = \mathbb{Q} = \mathbb{Q} \oplus H^i(\mathcal{B})^{\otimes K(L)} \quad 0 \leq i \leq 2l-4$$

$$R^{2l-2} \pi_* (\mathbb{Q} \otimes K(L))$$

$$H^{2l-2}(\mathbb{R}\pi_* \mathbb{Q}) = \mathbb{Q} = H^{2l-2}(\mathcal{B}) \Rightarrow \mathcal{B} = \mathbb{Q}[-2l+2]$$

$$H^*(H_L, \mathbb{Q}) = \sum_{i=0}^{l-2} h^i \cup H^*(X) + \underline{H^*(X_L) \cup L^{l-1}}$$

