HPD I : Goals: - Examples of sod

- Shed a light on Qn. Given a sm proj var. $X \subseteq \mathbb{P}^n$

$X_H = X \cap H$ What can we say about $D(X_H)$ in terms of sod of $D(X)$?

Example (contd.): - X sm proj.

- $\mathcal{O}_X(1)$ be a bpf line bundle.

$V = H^0(X, \mathcal{O}_X(1))$ bpf $\rightsquigarrow$ $\exists\ X \xrightarrow{f} \mathbb{P}(V^*)$

Convention: $\mathbb{P}(\ )$ is covariant. (different from [Har 88] or Stacks

i.e. $\mathcal{E} \longrightarrow \mathcal{F} \rightsquigarrow \mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}(\mathcal{F})$

- $\mathcal{O}_X(-1) \to \mathcal{E} \rightsquigarrow X \to \mathbb{P}(\mathcal{E})$

- $f: T \to X \qquad T \to \mathbb{P}(\mathcal{E}) \rightsquigarrow \mathcal{L} \longrightarrow f^*\mathcal{E}.$

Define, universal hyperplane

$$\mathcal{H} = \{(x,s) \in X \times \mathbb{P}(V) \mid s(x) = 0\} \subseteq X \times \mathbb{P}(V)$$

Then, ① $\boxed{\mathcal{H} = Z(ev)}$ $\quad ev \in H^0(X \times \mathbb{P}(V), \mathcal{O}_X(1) \boxtimes \mathcal{O}_{\mathbb{P}(V)}(1))$

$$\simeq H^0(X, \mathcal{O}_X(1) \otimes V^*)$$

$$V \otimes \mathcal{O}_X \xrightarrow{ev} \mathcal{O}_X(1)$$

② $\quad 0 \to \mathrm{Ker} \to V \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X(1) \to 0$

$$\boxed{\mathcal{H} = \mathbb{P}_X(\mathrm{Ker})}$$

$\mathcal{H} \overset{\iota}{\subset} X \times \mathbb{P}(V)$ is a $(1,1)$ hyperplane. $\pi : \mathcal{H} \to X$

$$D(\mathcal{H}) = \Big\langle \underset{\substack{\| \\ D(X)(0,1)}}{\pi^* D(X) \otimes \iota^* \mathcal{O}_{\mathbb{P}(V)}(1)}, \dots, \pi^* D(X) \otimes \iota^* \mathcal{O}_{\mathbb{P}(V)}(\dim V - 1) \Big\rangle$$

$$D(X \times \mathbb{P}(V)) = \langle \pi^* D(X), D(\mathcal{H}) \rangle$$

This is a special case of Prop. 3.6.

Example 2: $L \subseteq V$ linear subspace.

$$\mathcal{H}_L = \{(x,s) \mid s(x)=0\} \subseteq X \times \mathbb{P}(L)$$

$X_L = Bs(L)$ ($=: X_{L^\perp}$ notation in the notes)

Assume, $\dim X_L = \dim X - \ell$

Prop 3.6: ① $D(X_L) \underset{j_*}{\xrightarrow{j p^*}} D(\mathcal{H}_L)$ full, faithful embedding

② $D(X) \xhookrightarrow{\pi^*} D(\mathcal{H}_L)$ " " "

③ $D(\mathcal{H}_L) = \langle D(X_L), \pi^* D(X) \otimes i^* \mathcal{O}_{\mathbb{P}(L)}(1), \ldots, \pi^* D(X) \otimes i^* \mathcal{O}_{\mathbb{P}(L)}(\ell-1)$

$\ell = \dim L$

§ Aside: $\operatorname{rk} L = 2$

$$L \otimes L \longrightarrow \Lambda^2 L \cong \mathbb{C}$$

$$\rightsquigarrow \quad L \xrightarrow{\sim} L^*$$

$$\begin{tikzcd}
\overset{n-1}{X_L \times \mathbb{P}(L)} \arrow[r, hook, "j"] \arrow[d, "p"] \arrow[rr, bend left, "id"] \arrow[dr, phantom, "\lrcorner"] & \overset{n+e-2}{\mathcal{H}_L} \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & \overset{n+e-1}{X \times \mathbb{P}(L)} \arrow[dl, "\rho"] \\
\underset{\substack{n-e \\ n-2}}{X_L} \arrow[r, hook, "i"] & \underset{n}{X} &
\end{tikzcd}$$

$$\begin{array}{ccc}
X \times \mathbb{P}(L) & \xrightarrow{\sim} & X \times \mathbb{P}(L^*) \\
\cup & & \cup \\
\mathcal{H}_L & \longrightarrow & \mathcal{H}_L'
\end{array}$$

Qn: How to identify $\mathcal{H}_L$ inside $X \times \mathbb{P}(L^*)$

$$X_L \times \mathbb{P}(L) \hookrightarrow \mathcal{H}_L' \longrightarrow X \times \mathbb{P}(L^*)$$

Claim: $\mathcal{H}_L' = Bl_{X_L} X$

$\mathcal{H}_L$

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r] & L \otimes \mathcal{O}_X \arrow[r, "ev"] \arrow[dr, "0"'] \arrow[rr, bend left, ""] & \mathcal{O}_X(1) \arrow[d] & \mathcal{I}_{X_L} \arrow[l] \\
& & & \mathcal{O}_{X_L}(1) &
\end{tikzcd}
$$

$Bl_{X_L} X = \mathrm{Proj}(\mathcal{I}_{X_L}^{\vee})$

$X_L \times \mathbb{P}(L) \hookrightarrow Bl_{X_L} X$

is

$\mathrm{Proj}(\mathcal{O}_X / \mathcal{I}_{X_L}^{2})^{\vee}$

$$
\begin{tikzcd}
& 0 \arrow[d, "\text{five lemma}"] & & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & L \otimes \mathcal{O}_X \arrow[r] \arrow[d, "\simeq"] & \mathcal{I}_{X_L} \arrow[r] \arrow[d] & 0 \\
\text{Dual} \quad 0 \arrow[r] & \mathcal{I}_{X_L}^{*} \arrow[r] & L^{*} \otimes \mathcal{O}_X \arrow[r] & \mathrm{Ker}\left(K^{*} \to \mathcal{E}xt^{1}(\mathcal{I}_X, \mathcal{O}_X)\right) \arrow[d, "\text{five lemma}"] & \\
& & & 0 &
\end{tikzcd}
$$

$$
\left.\begin{aligned}
&\mathcal{I}_{X_L} \text{ is torsion free} \\
&+\ \mathcal{I}_{X_L} \simeq \mathrm{Ker}|_{X \setminus X_L}
\end{aligned}\right\} \Rightarrow \mathcal{I}_{X_L} \hookrightarrow \mathrm{Ker} \xRightarrow[\text{lemma}]{\text{five}} K \simeq \mathcal{I}_{X_L}^{*}
$$

$$\begin{tikzcd} \mathrm{Bl}_{X_L} X \arrow[d] \arrow[dr, "f"] & \\ X \arrow[r, dashed] & \mathbb{P}(L^*) \end{tikzcd}$$

$$\Gamma_f : \mathrm{Bl}_{X_L} X \hookrightarrow X \times \mathbb{P}(L^*)$$

<u>Proof of Prop 3.6</u> ① $j_* p^* : D(X_L) \longrightarrow D(X)$

WTS $\mathrm{RHom}(j_* p^* E, j_* p^* F) \simeq \mathrm{RHom}(p^* E, p^* F)$

$$\mathrm{RHom}(j_* p^* E, j_* p^* F) \cong \mathrm{RHom}(j^* j_* p^* E, p^* F) \longrightarrow \mathrm{RHom}(p^* E, p^* F)$$

Counit $\quad j^* j_* \longrightarrow \mathrm{id}$

$$j^* \underline{j_*} \to \mathrm{id} \qquad j : A \hookrightarrow B$$

$$\begin{tikzcd}
\overset{n-1}{X_L \times \mathbb{P}(L)} \arrow[r, hook, "j"] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "p"] & \overset{n+e-2}{H_L} \arrow[d, "\pi"] & \overset{n+e-1}{X \times \mathbb{P}(L)} \arrow[l, hook', "\iota"'] \arrow[dl, "\rho"] \\
\underset{n-e}{X_L} \arrow[r, hook, "i"] & \underset{n}{X} &
\end{tikzcd}$$

$$-\mathrm{id} = \varphi_{\mathcal{O}_\Delta} \qquad \mathcal{O}_\Delta = \Delta_* A$$

$$\Delta : A \to A \times A .$$

$$\begin{tikzcd}
- D(A) \arrow[r, "j_*"] \arrow[r, bend left, "\varphi_{\Gamma_j}"] \arrow[rr, bend right, "\varphi_R"'] & D(B) \arrow[r, "j^*"] \arrow[r, bend left, "\varphi_{\Gamma_j}"] & D(A)
\end{tikzcd}$$

$$\begin{tikzcd}
& A \times B \times A \arrow[dl, "\pi_{AB}"'] \arrow[d, "\pi_{AA}"] \arrow[dr, "\pi_{BA}"] & \\
\underset{\cup}{A \times B} & \underline{A \times A} & \underset{\cup}{B \times A} \\
\Gamma_j & & \Gamma_j
\end{tikzcd}$$

$$R = \pi_{A \times A *}\left(\pi_{AB}^* \mathcal{O}_{\Gamma_j} \otimes \pi_{BA}^* \mathcal{O}_{\Gamma_j}\right)$$

§11.1 Cor 11.4 + Prop 11.1

Let $N_{A/B}$ = normal bundle

$$\cdots \longrightarrow \Delta_* \overset{2}{\wedge} N_j^\vee \longrightarrow \Delta_* N_j^\vee \to R \longrightarrow \mathcal{O}_A \to 0$$

$\exists$ a ~~tor~~ iterated triangle

$$\bigoplus_{r}\Lambda^r N_j^\vee[r] \to j^* j_* \to id$$

$\bigoplus()$

$$R \simeq \bigoplus_{r=0}^{r} \Delta_* N_j^\vee[r] \; ; \qquad r=0 : \Delta_* \mathcal{O}_{X_L} \simeq \mathcal{O}_\Delta$$

Taking FM Kernel $\quad j^* j_* \simeq \varphi_R \simeq id \oplus (\Lambda^r N_j \otimes (\cdot))$

Back to our situation

$$
\begin{tikzcd}
\overset{n-1}{X_L \times \mathbb{P}(L)} \arrow[r, hook, "j"] \arrow[d, "p"] \arrow[rr, bend left, "\mathrm{id}"] & \overset{n+e-2}{H_L} \arrow[r, hook, "\iota"] \arrow[d, "\bar{\pi}"] & \overset{n+e-1}{X \times \mathbb{P}(L)} \arrow[dl, "\rho"] \\
\underset{n-e}{X_L} \arrow[r, hook, "i"] & \underset{n}{X} &
\end{tikzcd}
$$

$H_L = Z(\mathrm{ev})$ $\quad \mathrm{ev} \in H^0\big(X \times \mathbb{P}(L),\ \mathcal{O}_X(1) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(1)\big)$

$X_L = Z(\mathrm{ev}_L)$ $\quad \mathrm{ev}_L \in H^0\big(X, \mathcal{O}_X(1) \otimes L^*\big)$

Want to find $\quad N_{X_L \times \mathbb{P}(L)/H_L}$

Have, $X_L \times \mathbb{P}(L) \xhookrightarrow{j} \mathcal{H}_L \xhookrightarrow{i} X \times \mathbb{P}(L)$

$$
\begin{tikzcd}
0 \arrow[r] & N_{X_L \times \mathbb{P}(L)/\mathcal{H}_L} \arrow[r] \arrow[d, equal] & N_{X_L \times \mathbb{P}(L)/X \times \mathbb{P}(L)} \arrow[r] \arrow[d, phantom, "\cong"] & N_{\mathcal{H}_L/X \times \mathbb{P}(L)}\big|_{X_L \times \mathbb{P}(L)} \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\
& & \left(L^* \otimes \mathcal{O}_{X_L}(1)\right) \boxtimes \mathcal{O}_{\mathbb{P}(L)} \arrow[d, phantom, "\cong"] & \mathcal{O}_X(1) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(1)\big|_{X_L} \arrow[d, phantom, "\cong"] & \\
0 \arrow[r] & \boxed{\mathcal{O}_{X_L}(1) \boxtimes \Omega_{\mathbb{P}(L)}(1)} \arrow[r] & \mathcal{O}_X(1) \boxtimes L^* \otimes \mathcal{O}_{\mathbb{P}(L)} \arrow[r] & \mathcal{O}_{X_L}(1) \boxtimes \mathcal{O}_{\mathbb{P}(L)}(1) \arrow[r] & 0
\end{tikzcd}
$$

Use Relative Euler seq. for $X_L \times \mathbb{P}(L)$.

$$R\mathcal{H}om(\Lambda^r N_j^\vee \otimes p^* E, p^* F)$$

$$= R\mathcal{H}om(p^* E, p^*(F \otimes \mathcal{O}_{X_L}(r)) \otimes \Omega^r_{\mathbb{P}(L)}(r))$$

$$= R\mathcal{H}om(E, F \otimes \mathcal{O}_{X_L}(r) \otimes p_* \Omega^r_{\mathbb{P}(L)}(r))$$

$$\|$$

$$0$$

Recall

$$R\mathcal{H}om(i^* j_A^* p^* E, p^* F)$$

$$\downarrow \simeq$$

$$R\mathcal{H}om(p^* E, p^* F)$$

cone $= 0$

Using $H^0(\mathbb{P}^{l-1}, \Omega^r_{\mathbb{P}^{l-1}}(r)) = 0$ via Koszul resln

Eagon-Northcott for $p_* \Omega^r_{X \times \mathbb{P}(L)/X}(r)$.

③ SOD

Recall $D(\mathcal{H}_L) = \langle D(X_L), \pi^* D(X)(0,1), \ldots, \pi^* D(X)(0,\ell-1) \rangle$

$$\begin{tikzcd}
\overset{n-1}{X_L \times \mathbb{P}(L)} \arrow[r, hook, "j"] \arrow[d, "p"] \arrow[rr, bend left, "i\times \mathrm{id}"] & \overset{n+e-2}{\mathcal{H}_L} \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & \overset{n+e-1}{X \times \mathbb{P}(L)} \arrow[dl, "p"] \\
\underset{\underset{E}{n-e_1}}{X_L} \arrow[r, hook, "i"] & \underset{\underset{F}{n}}{X} &
\end{tikzcd}$$

WTS $\quad R\mathrm{Hom}\left(\pi^* F \otimes \iota^* \mathcal{O}_{\mathbb{P}(L)}(k),\ j_* p^* E\right) = 0$

$$\overset{\shortparallel}{\pi^* F(0,k)}$$

$$\simeq R\mathrm{Hom}\left(j^* \pi^* F \otimes \mathcal{O}_{\mathbb{P}(L)}(k),\ p^* E\right)$$

$$\simeq R\mathrm{Hom}\left(p^* i^* F \otimes \mathcal{O}_{\mathbb{P}(L)}(k),\ p^* E\right)$$

$$\simeq R\mathrm{Hom}\left(i^* F,\ E \otimes \underbrace{p_* \mathcal{O}_{\mathbb{P}(L)}(-k)}_{=0}\right) \qquad k > 0$$

3 b) $\mathrm{RHom}\left(\pi^*E(0,k+1),\ \pi^*F(0,k)\right)$

$$\begin{tikzcd}
\overset{n-1}{X_L \times \mathbb{P}(L)} \arrow[r, hook, "j"] \arrow[d, "p"] \arrow[rr, bend left, "id"] & \overset{n+e-2}{H_L} \arrow[r, hook, "\iota"] \arrow[d, "\bar{p}"] & \overset{n+e-1}{X \times \mathbb{P}(L)} \arrow[dl, "\rho"] \\
X_L \arrow[r, hook, "i"] & \underset{n}{X} &
\end{tikzcd}$$

$$= \mathrm{RHom}\left(\iota^*\left(\rho^*E \otimes \mathcal{O}_{\mathbb{P}(L)}(k+1)\right),\ \iota^*\left(\rho^*F \otimes \mathcal{O}(k)\right)\right)^{n-e}$$

$$= \mathrm{RHom}\left(\rho^*E \otimes \mathcal{O}(k+1),\ \underbrace{\iota_*\iota^*}_{id}\left(\rho^*F \otimes \mathcal{O}(k)\right)\right)$$

$$= 0$$

$$id(-1,-1) \to id \longrightarrow \iota_*\iota^*$$

$$\mathrm{RHom}\left(\rho^*E \otimes \mathcal{O}\quad,\ \rho^*F(-1) \otimes \mathcal{O}(-2)\right)$$

$$= \mathrm{RHom}\left(E,\ F(-1) \otimes \overset{=0}{\rho_*\mathcal{O}_{\mathbb{P}(L)}(-2)}\right) = 0$$

② $\pi^*: D(X) \longrightarrow D(\mathcal{H}_L)$

Similar.

$$\begin{tikzcd} X_L \times \mathbb{P}(L) \arrow[r, hook, "j"] \arrow[d, "p"] \arrow[rr, bend left, "id"] & \mathcal{H}_L \arrow[d, "\pi"] & X \times \mathbb{P}(L) \arrow[l, hook', "\iota"] \arrow[dl, "p"] \\ X_L \arrow[r, hook, "i"] & X & \end{tikzcd}$$

$n-1$, $n+e-2$, $n+e-1$, $n-e$, n

Generation: $E \in {}^\perp\langle D(X_L), \pi^*D(X)(0,1), \dots \rangle$

Then $E \simeq 0$

ETS $E \otimes k(x) = 0 \qquad x \in \mathcal{H}_L$

Claim $E|_{\pi^{-1}(x)} = 0 \qquad x \in X \quad F = \pi^{-1}(x)$

$$\begin{tikzcd} \mathbb{P}(L) \arrow[r, hook, "j"] \arrow[d, "p"] & \mathcal{H}_L \arrow[d, "\pi \text{ flat}"] \\ x \arrow[r, hook] & X \setminus X_L \end{tikzcd}$$

$j_* \mathcal{O}_F(k) = j_* p^* k(x) = \pi^* k(x)$

Case ⓐ: $x \in X \setminus X_L$, $R\mathrm{Hom}(E|_F, \mathcal{O}_F(k)) = R\mathrm{Hom}(E, \pi^* k(x))$

$= 0 \quad [\because \pi^* k(x) \in \pi^* D(X)]$

Case (b) $x \in X_L$

$$\begin{tikzcd} \overset{n-1}{X_L \times \mathbb{P}(L)} \arrow[r, hook, "j"] \arrow[rr, bend left, "\hat{j}"] \arrow[d, "p"] & \overset{n+e-2}{H_L} \arrow[r, hook, "\iota"] \arrow[d, "\bar{\pi}"] & \overset{n+e-1}{X \times \mathbb{P}(L)} \arrow[dl, "\rho"] \\ \underset{n-e}{X_L} \arrow[r, hook, "i"] & \underset{n}{X} & \end{tikzcd}$$

$$\underline{\pi^* k(x)} = i^* \rho^* k(x) = \boxed{i^* \iota_*}\, j_* \mathcal{O}_{\{x\} \times \mathbb{P}(L)}$$

$$\mathrm{id}(1,1)[1] \to i^* \iota_* \to \mathrm{id}$$

Get $j_* \mathcal{O}_{\{x\} \times \mathbb{P}(L)}(-1,-1)[1] \to \pi^* \mathcal{O}_x \to \hat{j}_* \mathcal{O}_{\{x\} \times \mathbb{P}(L)}$

Note, WTS $R\mathrm{Hom}(E|_F, \mathcal{O}_{\mathbb{P}(L)}) = 0$ ✓

Rmk: $\odot$ $R\mathrm{Hom}(E, j_* \underbrace{\mathcal{O}_{\{x\} \times \mathbb{P}(L)}}_{= p^* k(x)}) = R\mathrm{Hom}(E, j_* \underbrace{p^* k(x)}_{D(X_L)})$

② $R\mathcal{H}om(E, \pi^* k(x)) = 0 \qquad \pi^* k(x) \in D(X)$

$$R\mathcal{H}om(E_x, \mathcal{O}_{\mathbb{P}(L)}(k-1))[1] \longrightarrow \overset{=0}{R\mathcal{H}om(E_x, \pi^* k(x)|_F)} \longrightarrow R\mathcal{H}om(E_x, \mathcal{O}_{\mathbb{P}(L)}(k))$$

$$R\mathcal{H}om(E_x, \mathcal{O}_{\mathbb{P}(L)}) = 0 \implies R\mathcal{H}om(E_x, \mathcal{O}_{\mathbb{P}(L)}(k)) = 0 \quad \forall k$$

$$E_x = 0 \quad \forall\, E_x \in D(\mathbb{P}(L))$$

§ Cohomology ring of $\mathcal{H}_L$:

$$[\mathcal{H}_L] = [X \setminus X_L][\mathbb{P}^{l-2}] + [X_L][\mathbb{P}^{l-1}]$$

$$H^*(\mathcal{H}_L, \mathbb{Q}) \qquad R\pi_* \mathbb{Q}|_{X \setminus X_L} = \bigoplus R^i \pi_* \mathbb{Q}[-i]\Big|_{X \setminus X_L}$$

$$= \bigoplus \mathbb{Q}[-i]$$

$$\mathbb{R}\pi_*\mathbb{Q} \simeq \bigoplus_{i=0}^{l-2} \mathbb{Q}[-2i] \oplus \mathcal{B} \qquad \operatorname{Supp}\mathcal{B} = X_L.$$

$$\mathcal{H}^i(\mathbb{R}\pi_*\mathbb{Q})^{\otimes\kappa(x)} \simeq \mathbb{Q} = \mathbb{Q} \oplus \mathcal{H}^i(\mathcal{B})^{\otimes\kappa(x)} \quad 0 \le i \le 2l-4$$

$$R^i\pi_*\mathbb{Q}\otimes\kappa(x)$$

$$\mathcal{H}^{2l-2}(\mathbb{R}\pi_*\mathbb{Q}) = \mathbb{Q} = \mathcal{H}^{2l-2}(\mathcal{B}) \Rightarrow \mathcal{B} = \mathbb{Q}[-2l+2]$$

$$\boxed{H^*(\mathcal{H}_L, \mathbb{Q}) = \sum_{i=0}^{l-2} h^i \cup H^*(X) + \underline{H^*(X_L) \cup h^{l-1}}}$$

$\square$