MODULI OF VECTOR BUNDLES

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Contents

1. Motivation

This course serves as an introduction to moduli of sheaves on smooth projective varieties over an algebraically closed field k. Sheaves on varieties keep track of various algebraic and topological data about the variety and moduli problems are the formal machinery to consider certain classes of objects (sheaves for instance) together. Another advantage of studying these moduli problems is that these give rise to interesting geometric objects of independent interests. For instance, moduli of sheaves of K3 surfaces are symplectic. When smooth or are smoothable via a resolution preserving the symplectic form, these moduli spaces serve as key examples of the so called *hyperkaehler manifolds*. Moduli of vector bundles on curves have been used in re-construction problems. For instance K3 surfaces which admit a smooth curve C of genus 11 can be reconstructed from certain locus in the moduli space $M_C(2,7)$ of rank 2 and degree 7 vector bundles on C. These are quite powerful techniques and has shaped much of modern geometry.

Take for example the variety given by $X = \operatorname{Spec} k$ for some field k. Vector bundles on it are all trivial. If we fix the rank of the vector bundle to be ℓ there is only isomorphism class of vector bundle on k, namely $k^{\oplus \ell}$. Thus the moduli in this case is a point. The same is true for the affine complex space since their geometry is very similar to that of a point.

If we upgrade to dimension 1, the picture is already quite interesting. For $X = \mathbb{P}^1$ we know line bundles can be classified by its degree. Thus if we fix rank to be 1 and degree to be d, there is only one vector bundle $\mathcal{O}(d)$ on $\mathbb{P}^1$. For higher rank we have Grothendieck's theorem.

Theorem 1.1. *A vector bundle of rank r over $\mathbb{P}^1$ can be written as a direct sum of line bundles unique up to isomorphism.*

Exercise 1. Let $\ell \simeq \mathbb{P}^1 \subset \mathbb{P}^n$ be a line in the projective n-space. Find the decomposition of the restriction of the cotangent bundle of $\mathbb{P}^n$ to this line, i.e. write $\Omega_{\mathbb{P}^n}\big|_\ell$ as direct sum of line bundles on ℓ.

Proof. Lets quickly recall the definition of complex vector bundles.

Definition 1.2. A vector bundle $\mathcal{E}$ on X is a scheme $\mathcal{E}$, together with a morphism $\pi\colon \mathcal{E} \to X$ such that π is locally trivial in the Zariski topology, i.e. there exists a Zariski open covering $\{U_i\}_{i\in I}$ of X and isomorphisms

$$\psi_i \colon \pi^{-1}(U_i) \to U \times \mathbb{A}^r$$

such that for any open affine $V = \operatorname{Spec} A \subseteq U_i \cap U_j$, there is a morphism

$$\psi_{ij} \coloneqq \psi_i \circ \psi_j^{-1} \in GL(r, A)$$

called the transition function. Furthermore ψ_{ij} is linear in the sense that we obtain a morphism[1]

$$\psi_{ij} \coloneqq \psi_i \circ \psi_j^{-1} \colon V \to \mathrm{GL}_r(\mathbb{C})$$

The tuples $(U_i, \psi_i, \psi_{ij})_{i\in I})$ is the *trivialization data* and the integer r is the *rank* of the vector bundle $\mathcal{E}$. We call vector bundles of rank 1, line bundles. In what follows we will identify vector bundles and locally free sheaves, i.e. the sheaf of continuous sections of the vector bundle (see [Har77, Ex. II.5.18]).

On $\mathbb{P}^1$ let $U_0 = \operatorname{Spec} k[s]$ and $U_1 = \operatorname{Spec} k[t]$, $U_{0,1} = \operatorname{Spec} k[s, s^{-1}] = U_0 \setminus \{0\}$ and $U_{1,0} = \operatorname{Spec} k[t, t^{-1}] = U_1 \setminus \{0\}$. Note that both U_0 and U_1 are $\mathbb{P}^1$ with two points taken out. $\mathbb{P}^1$ can be reconstructed by gluing together U_0 and U_1 along the isomorphism $U_{1,0} \xrightarrow{\sim} U_{0,1}$ induced by $k[s, s^{-1}] \to k[t, t^{-1}]$ sending $s \mapsto t^{-1}$.

Now a vector bundle $\pi\colon E \to \mathbb{P}^1$ of rank r is equivalent to the trivialisation data

$$(U_0, \psi_0 \colon E|_{U_0} \xrightarrow{\sim} U_0 \times \mathbb{A}^r, \psi_{0,1}) \text{ and } (U_1, \psi_1, \psi_{1,0})$$

[1] assume X is defined over $\mathbb{C}$.

Here $\psi_{0,1} \in GL(r, k[s, s^{-1}])$, i.e. a matrix with coefficients in the polynomial ring $k[s, s^{-1}]$ with nowhere zero determinant. Hence $\det \psi_{0,1} = cs^m$ for some integer m and $c \in \mathbb{C}^\times$ since otherwise it is a polynomial that admit zeros apart from the points $[0 : 1]$ and $[1 : 0]$ [2]. We may and do take $c = 1$ since any other choice of c would produce a vector bundle isomorphic to E. Now the result follows from the claim that any matrix $A(s, s^{-1})$ with determinant s^m can be diagonalised as follows: there exist $M_0(s) \in GL(r, k[s])$ and $M_1(s^{-1}) \in GL(r, k[s^{-1}])$ such that

$$M_1(s^{-1})A(s, s^{-1})M_0(s) = \begin{pmatrix} s^{m_1} & & & & \\ & s^{m_2} & & 0 & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & & \cdot \\ & & & & s^{m_r} \end{pmatrix}$$

with $m_1 \geq m_2 \geq \cdots \geq m_r$, $m_i \in \mathbb{Z}_{>0}$ and the m_i's are *uniquely* determined by the matrix $A(s, s^{-1})$. Indeed the vector bundle with $\psi_i(s)$ replaced by $M_i(s) \circ \psi_i(s)$ for $i = 0, 1$ is isomorphic to E since vector bundles on $\mathbb{A}^1$, the affine complex line are trivial.

To see the claim we follow [HM82] and proceed by induction. For $r = 1$ there is nothing to prove. For $r > 0$, multiply $A(s, s^{-1})$ by suitable power s^N to get rid of the denominators, i.e. turn it into a polynomial matrix $A'(s)$. By elementary column operations, i.e. by multiplying by an elementary matrix with coefficients in $k[s]$ on the right, we may assume that $A'(s)$ is lower triangular[3]. By induction we can find $M'_0(s)$ and $M'_1(s^{-1})$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & M'_1(s^{-1}) \end{pmatrix} A'(s) \begin{pmatrix} 1 & 0 \\ 0 & M'_0(s) \end{pmatrix} = \begin{pmatrix} s^{m'_1} & & & & \\ a_2 & s^{m'_2} & & & 0 \\ a_3 & & s^{m'_3} & & \\ \cdot & & & \cdot & \\ \cdot & & 0 & & \cdot \\ a_r & & & & s^{m'_r} \end{pmatrix} \tag{1}$$

with a_i assumed to be polynomials in $k[s]$ after clearing denominators once again. Lets assume for a moment that $m'_1 \geq m'_i$. Then by elementary column operation we may assume that $\deg c_i < m'_i \leq m'_1$ for all i. We now do row operation, i.e. multiplication on the left by elementary matrices with coefficients in $k[s^{-1}]$. We do this by multiplying the first row with suitable power of s^{-1} and subtract from from the i-th row in order to get rid of the a_i's. Putting all the steps together we got

$$M_1(s^{-1})s^{N'}A(s, s^{-1})M_0(s) = \begin{pmatrix} s_1^{m'_1} & & & & \\ a_2 & s^{m'_2} & & & 0 \\ a_3 & & s^{m'_3} & & \\ \cdot & & & \cdot & \\ \cdot & & 0 & & \cdot \\ a_r & & & & s^{m'_r} \end{pmatrix}$$

where N' came from clearing denominators several times in the process. Let $m_i = m'_i - N$ and we are done.

To see why m'_1 can be chosen to be maximal among m'_i's we argue as follows: Let the right hand side (call it $B(s)$) of (1) to have the largest m'_1 among all possible m'_1's which occur on such a matrix $B(s)$ equivalent to $A'(s)$. This exists since $\det(A'(s))$ is fixed and m'_1 must be smaller that the degree of the determinant polynomial. If there exists $m'_i > m'_1$, we replace the i-th row of $B(s)$

[2] here we use algebraically closed condition.

[3] triangularization over PID $k[s]$

by suitable multiple of the first row repeatedly so that all terms of degree smaller that m'_1 gets killed in a_i. So the i-th row of $B(s)$ now looks like

$$(s^{m'_1+1}\tilde{a}_i(s), 0, \cdots, 0, s^{m'_1+1}s^{(m'_i-m'_1)}, 0, \cdots 0).$$

Now the idea is to switch the first column and the i-th column and do the same operation on $B(s)$ so that we obtain $M''_0(s)$ and $M''_1(s^{-1})$ satisfying

$$M''_1(s^{-1})B(s)M''_0(s) = B'(s)$$

and $B'(s)$ is the form of the right hand side of (1). The new matrix is still equivalent to $A'(s)$ but its first element in the first row has degree at least $m'_1 + 1$. This contradicts the fact that we chose the first element of $B(s)$ maximal possible amongst all equivalent matrices of $A'(s)$.

The uniqueness is rather complicated; we will not get into it. For the details consult [HM82] which we have followed closely. □

Remark 1.3 (A remark about the definition of vector bundles). It should be noted that on a complex manifold we can define $\mathscr{C}^\infty$ (resp. holomorphic) vector bundle by demanding that the transition functions ψ_{ij}'s take values in the sheaf $\mathscr{C}^\infty_X$ (resp. holomorphic) and U_i's are analytic open covers. When X is additionally projective analytic space (or, equivalently a projective variety), the category of algebraic vector bundles we defined in the proof above coincides with the category of holomorphic vector bundles on X. If we take ψ_{ij} to be locally constant functions, i.e. on $V \subset U_i \cap U_j$ small enough $\psi_{ij} \colon V \to GL_r(\mathbb{C})$ then we obtain the so-called *locally constant sheaf*.

1.1. **Classification of vector bundles on $\mathbb{P}^1$.** With Grothendieck's result at our disposal, the classification problem or the moduli problem for vector bundles on $\mathbb{P}^1$ somewhat simplifies. In other words, if we fix rank r and degree d, we know any vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ can be written as

$$\mathcal{E} \simeq \mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_r)$$

with $m_1 \geq \cdots \geq m_r$ and $\sum_i m_i = d$. However this moduli problem is in a way "infinite". Take for example the following $k \in \mathbb{N}$ indexed set of non-isomorphic vector bundles of rank r and degree d.

$$\mathcal{E}_k := \mathcal{O}(-k) \oplus \mathcal{O}^{\oplus r-2} \oplus \mathcal{O}(k+d). \tag{2}$$

I will convince you that there is no scheme S of *finite type* and a vector bundle $\mathscr{E}$ on $X \times S$ such that we can find a sequence of points s_k satisfying $\mathscr{E}|_{X_{s_k}} \simeq \mathcal{E}_k$. The reason is that if it did, the dimension of the vector space $H^0(X, \mathscr{E}|_{X_s})$ would be a upper semi-continuous function of s [Har77, Theorem III.12.8], i.e. only finitely many jumps in the dimension is allowed. Lets now calculate $H^0(\mathbb{P}^1, \mathcal{O}(-k) \oplus \mathcal{O}^{\oplus r-2} \oplus \mathcal{O}(k+d)) = k + d + r - 1$ which grows as $k \to \infty$.

How do we tackle this *unbounded* moduli problem? There are two major discourses in the literature. We will focus on one of them which is to restrict the class of vector bundle by imposing the so-called (Gieseker)-stability criterion. This gives rise to a fairly well-behaved moduli space.

It turns out that in order to obtain a complete moduli space one needs to allow the so called semi-stable sheaves as well. This is because stable bundles often in the limit splits of into the direct sum of sub-bundles. This prompts another serious issue.

Example 1.4. On a smooth projective curve of genus $g \geq 1$, the non-trivial extension E of a line bundle $\mathcal{L}$ of degree d by itself gives rise to a simple (i.e. non-splittable) *semi-stable* vector bundle of rank 2 and degree $2d$ [4]. These extensions are parametrised by $\mathrm{Ext}^1(\mathcal{L}, \mathcal{L}) \simeq H^1(C, \mathcal{O}_C)$ and hence the line $\mathbb{A}^1$ induces a vector bundle G on $C \times \mathbb{A}^1$ with $G_s \simeq E$ for $s \in \mathbb{A}^1 \setminus \{0\}$ and $G_0 \simeq \mathcal{L}_1 \oplus \mathcal{L}_2$. Take another family of semi-stable bundles given by $F := p_1^* E$ on $C \times \mathbb{A}^1$ so that $F_s \simeq E$ for all $s \in \mathbb{A}^1$. The moduli map (appropriately defined) will send both families to the point corresponding to $[E]$. But over 0, these two families disagree and thus should somehow be sent to different points. In other words the images of these two families under the moduli map cannot be separated in the

[4] find the definition of semi-stable bundle and check this!

moduli space. In order for a separated moduli space to exist we need to disallow certain sheaves. This is taken care of by the so called *S-equivalence* (See **??**). In this case the the two rank 2 bundles are indeed S-equivalent. We will modify the moduli problem so that we identify all such extensions to the S-equivalent class $\mathcal{L} \oplus \mathcal{L}$.

Over $\mathbb{P}^1$ this makes the moduli problem somewhat boring, since any semi-stable rank r degree dr S-equivalent class of vector bundles on $\mathbb{P}^1$ is given by just one vector bundle namely $\oplus_{i=1}^r \mathcal{O}(d)$. However for higher genus curve we obtain an irreducible projective variety as a coarse moduli space. It has dimension $r^2(g-1)+1$ and when r and d are co-prime, i.e. $(r,d)=1$ this moduli space is smooth and fine.

But before we come to these concepts it would be both interesting and convenient for later use to understand Grothendieck's Quot scheme construction. These are somewhat indispensable in the study of moduli problems in general. We will follow [HL10] closely for the most part of the course.

2. Coherent sheaves and their family

Let me start with two important exercises:

Exercise 2. Given a coherent sheaf $\mathcal{F}$ on a smooth projective scheme X, show that for $m \gg 0$, $H^i(X, \mathcal{F} \otimes \mathcal{O}_X(m)) =$ for $i > 0$ and the global sections of $\mathcal{F} \otimes \mathcal{O}_X(m)$ generates the sheaf.

Exercise 3. Given a globally generated vector bundle $\mathcal{E}$ on a smooth projective scheme X of rank at least $\dim X + 1$, show that there exists a short exact sequence of vector bundles

$$0 \to \mathcal{O}_X \to \mathcal{E} \to Q \to 0$$

where Q is the quotient bundle.

In the last section, we dismissed the possibility of finding a finite moduli for vector bundles of rank r and degree d on $\mathbb{P}^1$ by showing that there are no scheme S of finite type that would parametrise all such vector bundle. Parametrising in its finest form may be interpreted as the existence of a S-*flat* coherent sheaf $\mathcal{F}$ on $X \times S$ such that any object from the class of sheaves we are trying to parametrise is isomorphic to $\mathcal{F}|_{X_s}$ for some $s \in S$. In this direction we will see that for any projective morphism of Noetherian schemes $X \to S$ and a coherent sheaf $\mathcal{F}$ on X, S admits a cover by disjoint locally closed subschemes S_i such that $\mathcal{F}|_{X \times_S S_i}$ is flat over S_i. This is called a *flattening stratification*. Our next goal is to establish this phenomenon. Lets start by recalling an indispensible fairly good indicator of flatness, the Hilbert polynomial.

2.1. **Hilbert Polynomial.**

Recall that for a vector bundle $\mathcal{E}$ on a smooth projective curve C of genus g with fixed polarisation $\mathcal{O}_C(1)$, the only important numbers associated are the rank r and degree d of its determinant bundle $\bigwedge^r \mathcal{E}$. By this I mean that the Euler characteristic of any twist $\mathcal{E} \otimes \mathcal{O}_C(m)$, $m \in \mathbb{Z}_{>0}$ or the alternate sum of the dimension of the cohomologies of $\mathcal{E}(m)$ is completely determined in terms of r, d and g. Recall that

$$\chi(\mathcal{E} \otimes \mathcal{O}_C(m)) = \dim H^0(C, \mathcal{E} \otimes \mathcal{O}_C(m)) - \dim H^1(C, \mathcal{E} \otimes \mathcal{O}_C(m)) = r(1-g+\deg(C)m)+d. \quad (3)$$

So the Hilbert polynomial is given by $P(\mathcal{E},t) = tr\deg(C) + r(1-g)+d$. This is a linear polynomial and $P(\mathcal{E},m) = \chi(\mathcal{E}(m))$ for all m.

In higher dimensions, finding a suitable invariant is not so easy, in the sense that there may be many options. From the point of view of Gieseker stability the most important is the Hilbert polynomial $P(\mathcal{E},t)$. For m large enough $\chi(\mathcal{E}(m))$ [5] is still a polynomial in m in higher dimension. This is called the *Hilbert polynomial*, $P(\mathcal{E},t)$ in variable t. In particular, $P(\mathcal{E},m) = \chi(\mathcal{E}(m))$ for m large enough. More generally we will allow coherent sheaves F in place of $\mathcal{E}$. The polynomial $P(F,t)$ is of degree the same as $\dim \mathrm{Supp}(F)$.

[5] We use the notation $\mathcal{E}(m)$ for $\mathcal{E} \otimes \mathcal{O}_X(m)$.

The greatest advantage of the Hilbert polynomial is that if we let the coherent sheaf vary in flat family (the notion of which will be discussed in a moment) the Hilbert polynomial does not change. This is a big advantage since it let us control them simultaneously.

2.2. **Flat family.** Let $(X, \mathcal{O}(1))$ be a smooth projective variety with a fixed polarisation $\mathcal{O}(1)$. To solve the "moduli" problem, we would like to find a sheaf $\mathcal{F}$ on $X \times M$ such that for every $m \in M$, $X_m \simeq X$ is the fibre over m and $\mathcal{F}_m := \mathcal{F}|_{X_m}$ is a coherent sheaf on X. Furthermore we would like that for $m \neq m'$ the fibres $\mathcal{F}_m \not\simeq \mathcal{F}_{m'}$ and $P(\mathcal{F}_m, t)$ does not depend on the m. This is of course way too much to expect. We will come back to this point later on.

Let X be a Noetherian scheme over an algebraically closed field k. Let $f\colon X \to S$ be a morphism of finite type of Noetherian schemes. We denote $\mathcal{F}_s = \mathcal{F}|_{X_s}$. The sheaf $\mathcal{F}$ could be thought of as a collection of sheaves $\{\mathcal{F}_s\}$ parametrized by S. When we apply this of course we would like that all of the X_s are isomorphic to X. As mentioned before, instead of trying to classify arbitrary large set of coherent sheaves, it is wise to fix certain invariants, such as the Hilbert polynomial. It turns out (see Lemma 2.2) below that when S is reduced and irreducible, specifying that $P(\mathcal{F}_s, t)$ does not vary with s is equivalent to assume that $\mathcal{F}$ is a flat $\mathcal{O}_S$-module . We have the following

Definition 2.1. A flat family of coherent sheaves on the fibres of f is given by a coherent $\mathcal{O}_X$-module $\mathcal{F}$ which is flat over S.

The following proposition justifies why restricting to flat family is a good idea!

Lemma 2.2. *Let S be an integral Noetherian scheme. With notation from before the following are equivalent*

1. *$\mathcal{F}$ is S-flat*
2. *For all sufficiently large m the sheaves $f_*(\mathcal{F} \otimes \mathcal{O}_X(m))$ are locally free.*
3. *The Hilbert polynomial $P(\mathcal{F}_s)$ is independent of $s \in S$.*

Sketch of the proof. This is [Har77, Theorem 9.9]. Roughly the idea is to localise the problem to $S = \operatorname{Spec} A$ where A is a local Noetherian domain and $X = \mathbb{P}^n_A$. The statements above then reduce to almost commutative algebraic statements

1. $\mathcal{F}$ is A-flat.
2. For all sufficiently large m, $H^0(\mathbb{P}^n_A, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n_A}(m))$ is a free module of finite rank.
3. The Hilbert polynomial $P(\mathcal{F}_s, t)$ is independent of $s \in S$.

(1)⇒ (2): If $\mathcal{F}$ is flat, the terms in the Čech complexes computing the cohomology of $\mathcal{F}(m)$ are also flat A-modules. Then use Serre's vanishing to conclude that $H^i(\mathbb{P}^n_A, \mathcal{F}(m)) = 0$ for $m \geq m_0$ for some large enough m_0. Hence $H^0(\mathbb{P}^n_A, \mathcal{F}(m))$, the left-most kernel of the Čech complex is also flat and finitely generated and hence free of finite rank.

(2)⇒ (1): This follows since $\mathcal{F}$ can be reconstructed from the graded $A[x_0, \cdots, x_n]$-module

$$\bigoplus_{m \geq m_0} H^0(\mathbb{P}^n_A, \mathcal{F}(m)).$$

(2) ⇒ (3) By Serre vanishing we can write $P(\mathcal{F}_s, m) = \dim H^0(X_s, \mathcal{F}_s(m))$ for all $m \geq m_0$. Now let $\mathfrak{m}$ denote the ideal corresponding to the closed point $s \in S$. Then (3) is a consequence of the isomorphism

$$H^0(X, \mathcal{F}(m)) \otimes \kappa(s) \simeq H^0(X, \mathcal{F}_s(m)).$$

To see this, tensor the exact sequence $\oplus A \to A \to \kappa(s) = A/\mathfrak{m}$ with $\mathcal{F}(m)$ and take global section and then compare this exact sequence with the one tensored with $H^0(X, \mathcal{F}(m))$.

(3)⇒ (2) On a Noetherian local domain $(A, \mathfrak{m})$ by the Nakayama lemma one can conclude [Har77, Theorem II.8.9] that if $\operatorname{rk}_{k(s)} H^0(X_s, \mathcal{F}_s(m))$ is the same as the generic rank of $H^0(X, \mathcal{F}(m))$, then it is a free A module. □

Example 2.3 (Why the integral domain condition is necessary). Let $A = \frac{k[x]}{(x^2)}$ be the ring of dual numbers and $X = \mathbb{P}^1 \times \operatorname{Spec} A$. Then the sheaf given by the skyscraper sheaf at point of $\mathbb{P}^1$ is not flat over A. Indeed, this is because $\frac{k[x]}{(x^2)} \to k$ is not flat. (Exercise: show this!).

When S is not reduced, the following criterion is often helpful

Lemma 2.4. *Let $S^{\mathrm{red}} \hookrightarrow S$ denote the reduced scheme given by the nilpotent ideal $\mathcal{I} \subset \mathcal{O}_S$. Then $\mathcal{F}$ is a flat $\mathcal{O}_S$-module if and only if it is flat over S^{red} and $\mathcal{I} \otimes_{\mathcal{O}_S} \mathcal{F} \to \mathcal{I}\mathcal{F}$ is an isomorphism.*

We have used the following facts from commutative algebra.

Exercise 4.
- Harshrone III.9.1
- Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is S-flat, then $\mathcal{F}''$ is S-flat if and only if for each $s \in S$ the homomorphism $\mathcal{F}'_s \to \mathcal{F}_s$ is injective.

Having established Lemma 2.2 it is natural to expect that given any morphism $f\colon X \to S$ and a coherent sheaf $\mathcal{F}$ on X, there should be a way to split S up into pieces where the Hilbert polynomial of $\mathcal{F}$ restricted to each individual pieces remain constant. When S is reduced and irreducible, such a decomposition should be flattening stratification of S. Mumford showed that this is indeed the case and moreover this stratification satisfies the universal property below [HL10, Theorem 2.1.5]

Theorem 2.5. *Let $f\colon X \to S$ be a projective morphism of Noetherian schemes and let F be a coherent sheaf on X. Then we have the following*

(1) The set $\mathcal{P} := \{P(F_s, t) | s \in S\}$ of Hilbert polynomials of $F_s := F|_{X_s}$ the restriction of F to the fibre X_s over s is finite.

(2) (Flattening Stratification) There are finitely many locally closed sets S_P indexed by $\mathcal{P}$ such that $j\colon \bigsqcup_{P \in \mathcal{P}} S_P \to S$ is a bijection and $F|_{S_P}$ is flat over S_P with Hilbert polynomial P.

(3) (Universal Property) If $g\colon S' \to S$ is a morphism of Noetherian schemes and $g_X^* F$ is flat over S' with Hilbert polynomial $P(F_{s'}, t) = P$ for all $s' \in S$, then g factors through $S_P \to S$ and in particular through j.

The first part of Mumford's theorem was shown by Grothendieck. He showed that there exists a *flattening stratification* in the sense that there exists pairwise disjoint finite cover $\{S_i\}$ by locally closed subsets of S such that $\mathcal{F}_i := \mathcal{F}|_{S_i}$ is flat over S_i, which implies the first part of Theorem 2.5.

Proof. We first show Grothendieck's statement above. The idea is to use results from commutative algebra to reduce the problem to generic flatness of morphism of schemes of finite type. We may and do assume that S is irreducible. Moreover, since the problem is local we assume that $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ where B is a finitely generated A algebra. Finally, since we want to find an open set $U \subset S$ such that $\mathcal{F}$ is flat over U^{red}, it is enough to assume that A is integral.

Let $\mathcal{F}$ be given by the sheafification of the finitely generated B-module M. Since M is a finitely generated B-module, a basic fact from commutative algebra [Mat87, Theorem 6.4] implies that M admits a filtration with quotients isomorphic to $B/\mathfrak{p}_i$ for prime ideals $\mathfrak{p}_i$[6]. Since $B/\mathfrak{p}_i$ is an integral domain, we reduce to the case when $B = M$ is an itegral domain and $A \to B$ injective. Recall that the quotient field $K(A)$ of A is given by inverting all elements of A but 0. Then by Noether's normalisation theorem there exists finitely many elements $b_1, \cdots, b_n$ such that $K \otimes_A B$ is a finite module over $K(A)[b_1, \cdots, b_n]$. Clearing denominators we may find $f \in A$ such B_f is a finite module over the polynomial ring $A_f[b_1, \cdots, b_n]$. Since finite extensions are flat and the polynomial ring $A_f[b_1, \cdots, b_n]$ is flat over A_f we obtain that B_f is flat over A_f. We continue by restricting $\mathcal{F}$ over $X \setminus \operatorname{Spec}(B_f)$ to find the remaining locally closed sets S_i.

[6] Here $\mathfrak{p}_i$'s are associated primes of M. Exercise: Show this fact! Hint: if $M' \subset M$ is the maximal sub-module that admit such filtration then an associpated prime of M/M' would contradict maximality of M'.

Since the fibres of f is quasi-compact and S is of finite dimension, this process will stop. Pulling back to the disjoint union under the natural map $j\colon \sqcup_i S_i \to S$ we note that $j^*\mathcal{F}$ is flat over $\sqcup_i S_i$ and hence the $P(\mathcal{F}_s, t)$ is locally constant with repect to $s \in \sqcup_i S_i$. Thus set $\mathcal{P}$ in the statement of the theorem is indeed finite and S_i's can be reindenxed by S_P for each $P \in \mathcal{P}$. □

To see the universal property in (3) we need the following

2.2.1. *Key Observation.* Let F be a coherent sheaf on a Noetherian scheme S. Then the flattening stratification for F (i.e. $f = \mathrm{id}\colon S \to S$) is simply saying that the locus where rank of F (or the dimension of the fibre of F) remains constant. In particular, the flattening stratification coincides with locally closed stratification $\{S_r\}$ of S such that $F|_{S_r}$ is locally free of rank r. In this case the universal property (3) is rather easy to observe. Let $s \in S_r$, i.e. the dimension of the fibre at s, i.e. $\dim F \otimes \kappa(s) = r$ for some positive integer r. Then there exists an open set U around s, such that the basis of $F \otimes \kappa(s)$ lifts to sections $\mathcal{O}_U^{\oplus r}$. Note that since S and hence U is Noetherian, the kernel K of this lifting $\mathcal{O}_U^{\otimes r} \to F|_U$ is finitely generated. Say r_1 is its dimension at s. By possibly shrinking U we consider the following exact sequence

$$\mathcal{O}_U^{\oplus r_1} \xrightarrow{\psi} \mathcal{O}_U^{\oplus r} \to F|_U.$$

Note that, restricted to the point s, the map on the right is an isomorphism by definition. Hence the matrix of $\psi(s) = (\psi_{ij}(s)) = 0$. Let $Z \subseteq U$ be the scheme defined by simultaneous vanishing locus $Z(\psi_{ij})$. Thus $S_r \cap U = Z$.

Moreover, pulling this back under any morphism $g\colon S' \to S$ we obtain

$$\mathcal{O}_{g^{-1}(U)}^{\oplus r_1} \xrightarrow{g^*\psi} \mathcal{O}_{g^{-1}(U)}^{\otimes r} \to g^*F|_{g^{-1}(U)}.$$

Now g^*F is locally free of rank r if and only $g^*\psi = 0$ if and only if $\psi = 0$. Thus g must factor through $Z \subset U$.

To see this in action, consider the ideal sheaf $\mathfrak{m} = (x, y)$ of the origin in $\mathbb{A}^2 = \operatorname{Spec} k[x, y]$. Then on $S_1 \coloneqq \mathbb{A}^2 \setminus \{(0, 0)\}$ we have $\mathfrak{m}|_{S_1} \simeq \mathcal{O}_{S_1}$ and at $(0, 0)$, $\mathfrak{m}/\mathfrak{m}^2$ is of rank 2.

The strategy for proving the universal property in Theorem 2.5 (3) for any morphism f is to interpret the flattening stratification $\{S_i\}$ in (1) in terms of the flattening stratifications of $f_*F(m)$ for m large enough.

Proof of (3). Let $F' \coloneqq j^*F$ on $\bigsqcup_{P \in \mathcal{P}} S_P$. By Lemma 2.2 we know there exists m_0 such that for all $m \geq m_0$, $f_*F'(m)$ is locally free on S_P for all $P \in \mathcal{P}$. Moreover by Serre's vanishing m_0 can be chosen large enough such that $R^i f_*F'(m) = 0$ for all $i > 0$. Thus $f_*F'(m)$ is locally free of rank $P(m)$. Thus we have

$$S_P \coloneqq \bigcap_{m \geq m_0} S_{m,P(m)}$$

where $S_{m,P(m)} \coloneqq \{s \in S | f_*F(m) \text{ is locally free at } s \text{ of rank } P(m)\}$. Indeed, the inclusion $S_{m,P(m)} \supseteq S_P$ follows from the fact that $f_*F(m)$ is locally free on S_P and the key observation. The equality follows since the Hilbert polynomial is a numerical polynomial determined by its values $P(m)$ for $m \gg 0$.

As noted in the Key observation, the sets on the right hand side are locally closed subschemes. Since S and hence any open subscheme U are assumed to be Noetherian the intersection on the right hand side is finite.

Now let $g\colon S' \to S$ be such that g_X^*F is flat over S' with Hilbert polynomial P. Consider the following diagram

$$
\begin{tikzcd}
X_{S'} \arrow[r, "g_X"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}
\tag{4}
$$

By Lemma 2.2 this is equivalent to saying that for m large enough $f'_* g_X^* F(m)$ is locally free. We will see in Lemma 2.6 that there exists m_0 depending on g such that $f'_* g_X^* F(m) \simeq g^* f_* F(m)$. If m_0 is selected while taking into account the above-mentioned properties of $F(m)$, from Key observation it is evident that g factors through the intersection of $S_{m,P(m)}$ for all $m \geq m_0$ and hence through S_P.

□

2.3. **A base change property.** We prove the result employed at the end of the proof. Remember that when a map $g\colon S' \to S$ is flat, using notations in (4) we have the isomorphism $f'_* g_X^* F \simeq g^* f_* F$. However when g is not necessarily flat but f is a projective morphism of Noetherian schemes, at the expense of twisting F by a relatively ample line bundle enough number of times we can ensure that base change still holds. This is the content of the next result.

Lemma 2.6. *Let $f\colon X \to S$ be a projective morphism of Noetherian schemes, let $\mathcal{O}_X(1)$ be a line bundle on X which is very ample relative to S , and let F be a coherent $\mathcal{O}_X$-module. Then for $m \gg 0$*

$$f'_* g_X^* F(m) \simeq g^* f_* F(m).$$

Proof. Let $S = \operatorname{Spec} A$ and $X = \mathbb{P}^n_S = \operatorname{Proj} A[x_0, \cdots, x_n]$. The $\mathcal{O}_S$ algebra

$$\Gamma_X \coloneqq \oplus_{m \geq 0} H^0(\mathbb{P}^n_k, \mathcal{O}(m)) \otimes \mathcal{O}_S$$

recovers $\mathcal{O}_X$ as a $\mathcal{O}_S$ algebra. Similarly, the graded Γ_X-module $\Gamma_{X,F} \coloneqq \oplus_{m\geq 0} f_* F(m)$ recovers the $\mathcal{O}_X$-module F. This reconstruction functor is inverse to the functor $\Gamma_{X,\text{-}}$.

This construction is also functorial. This means for any map $g\colon S' \to S$, the graded $\Gamma_{S'}$-module $g^*\Gamma_{X,F} = \oplus_{m \geq 0} g^* f_* F(m)$ recovers $g_X^* F$. Thus $g_X^* F$ can be reconstructed from $\Gamma_{S'}$-modules

$$\Gamma_{X_{S'}, g_X^* F} \text{ and } g^* \Gamma_{X,F}.$$

By [Har77, Ex. II.5.14(c)] we conclude that there exists m_0 such that for all $m \geq m_0$,

$$f'_* g_X^* F(m) \simeq g^* f_* F(m)$$

□

3. A glimpse toward semistability: Curves

Definition 3.1 (Slope of a vector bundle)**.** A vector bundle $\mathcal{E}$ on a smooth projective curve C over an algebraically closed field k of rank r and degree $d \coloneqq \deg \bigwedge^r \mathcal{E}$ is said to have slope

$$\mu \coloneqq \frac{d}{r}.$$

Then we have

Definition 3.2 ((semi)-stable vector bundles)**.** A vector bundle $\mathcal{E}$ on a smooth projective curve C of rank r and degree d is called semistable (resp. stable) if for any subbundle $0 \neq \mathcal{F} \hookrightarrow \mathcal{E}$, we have $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ (resp. $\mu(\mathcal{F}) < \mu(\mathcal{E})$).

Remark 3.3 (stability via reduced Hilbert polynomial). Define the reduced Hilbert polynomial to be the Hilbert polynomial divided by its leading coefficient. On curves we have

$$p(\mathcal{E},t)=\frac{P(\mathcal{E},t)}{r}=t+\frac{d}{r}+(1-g).$$

Thus (semi)-stability may also be defined as $p(\mathcal{F})\leq p(\mathcal{E})$ (resp. $p(\mathcal{F})<p(\mathcal{E})$). In higher dimensions this definition generalises to the so-called *Gieseker stability*. We will come back to this point.

3.1. **Some basic properties of bundles.** Lets analyse some immediate and not-so-immediate properties of stability. We will revisit this again in higher dimension. For now, consider them as exercises.

(1) Note that $\mathcal{F}\subset\mathcal{E}\Rightarrow \mathrm{rk}(\mathcal{F})\leq \mathrm{rk}(\mathcal{E})$. Thus if $\mu(\mathcal{F})<\mu(\mathcal{E})$ and $\mathrm{rk}(\mathcal{F})<\mathrm{rk}(\mathcal{E})$, the degree of $\mathcal{F}$ must seriously compensate by being much smaller that $\deg(\mathcal{E})$.
(2) similarly if $\mathcal{F}\subset\mathcal{E}$ and $\mu(\mathcal{F})>\mu(\mathcal{E})$, it must be the case that $\mathrm{rk}(\mathcal{F})<\mathrm{rk}(\mathcal{E})$. Indeed, otherwise $\mathcal{E}/\mathcal{F}$ would be supported along points and $\deg\mathcal{E}=\deg\mathcal{F}+\mathrm{length}(\mathcal{E}/\mathcal{F})$ where $\mathrm{length}(\mathcal{E}/\mathcal{F})$ is the sum of the dimensions of all the sky-scraper sheaves $\mathcal{E}/\mathcal{F}$.
(3) *Saturated subsheaf:* A subsheaf $\mathcal{F}\subset\mathcal{E}$ can be chosen so that $\mathcal{E}/\mathcal{F}$ is torsion free, hence locally free on C. The remark above suggests that on curves saturation of a subsheaf of same rank is isomorphic.
(4) A bundle $\mathcal{E}$ is (semi-)stable if and only if fall quotient bundles, i.e. $\mathcal{E}\twoheadrightarrow\mathcal{G}$ satisfy $\mu(\mathcal{G})(\geq$ -$)>\mu(\mathcal{F})$.
(5) If $\mathcal{E}$ and $\mathcal{F}$ are stable vector bundles and $\mu(\mathcal{E})=\mu(\mathcal{F})$, then any non-trivial homomorphism $\varphi\colon\mathcal{E}\to\mathcal{F}$ is an isomorphism. In particular $Hom(\mathcal{E},\mathcal{E})\simeq k$ (i.e. $\mathcal{E}$ is simple). In particular, all line bundles are stable.
(6) The category $SS(\mu)$ of semistable bundles of a fixed slope μ is abelian.
(7) Every semi-stable bundle $\mathcal{E}$ admits a *Jordan–Hölder filtration* (JH), namely a filtration by subbundles

$$0=\mathcal{E}_0\subset\mathcal{E}_1\subset\cdots\subset\mathcal{E}_\ell=\mathcal{E}$$

such that the graded pieces $gr^i(\mathcal{E})\coloneqq\mathcal{E}_i/\mathcal{E}_{i-1}$ are stable with slope $\mu(\mathcal{E})$.
(8) S-equivalence: The filtration is not unique but the graded bundle $gr^\bullet(\mathcal{E})\coloneqq\oplus_i gr^i(\mathcal{E})$ does not depend on the filtration. Moreover, for two semi-stable sheaves $\mathcal{E}$ and $\mathcal{E}'$ if $gr^\bullet(\mathcal{E})\simeq gr^\bullet(\mathcal{E}')$ then we say they are S-equivalent and do not distinguish them in the Moduli space. For instance, any non-trivial extensions $\mathcal{E}$ of two line bundles $\mathcal{L}_1$ by $\mathcal{L}_2$ of degree d is semistable and the Jordan–Hölder filtration is given by $\mathcal{L}_2\subset\mathcal{E}$. Thus in the separated moduli we see only one point $\mathcal{L}_2\oplus\mathcal{L}_1$.
(9) When $\mathcal{E}$ is not even semi-stable, a unique filtration by semistable sheaves exists. This is called the *Harder-Narasimhan* (HN) filtration and is an increasing filtration by subbundles

$$0=\mathcal{B}_0\subset\mathcal{B}_1\subset\mathcal{B}_2\subset\cdots\subset\mathcal{B}_\ell=\mathcal{E}$$

such that $gr_i\mathcal{E}\coloneqq\mathcal{B}_i/\mathcal{B}_{i-1}$ is a semistable sheaf with slope μ_i satisfying

$$\mu_{\max}(\mathcal{E})\coloneqq\mu_1>\cdots>\mu_\ell=:\mu_{\min}(\mathcal{E})$$

Exercise 5. Show Property (4) above, i.e. show that given a short exact sequence of vector bundles $0\to\mathcal{F}\to\mathcal{E}\to\mathcal{G}\to 0$ $\mu(\mathcal{F})(\leq)<\mu(\mathcal{E})$ if and only if $\mu(\mathcal{E})(\leq)<\mu(\mathcal{G})$.

3.2. **Properness: Langton's theorem.** The result of Langton gives additional justifications for why (semi)-stable sheaves are just the right choice. The result says even without constructing the moduli space we can tell that the space (if exists) will be compact[7]. In other words, he gives a recipe to fill-up any non-compact family of (semi-) stable sheaves over a punctured disc to a family

[7]Nowadays this follows from GIT construction of the moduli.

over the entire disc whose central fibre is semi-stable. In algebraic language this is called the *the valuative criterion* for properness. Recall [Har77, Chapter II.4]

Proposition 3.4 (Valuative criterion for properness)**.** *Let $f\colon M \to \operatorname{Spec} k$ be a noetherian scheme of finite type over a field k. Then M is proper if and only if for any discrete valuation ring R with maximal ideal $\mathfrak{m} = (m)$, $k \simeq R/\mathfrak{m}$ and quotient field K, any morphism $\operatorname{Spec}(K) \to X$ extends uniquely to $\operatorname{Spec}(R)$. In other words, there exists a unique morphism $\operatorname{Spec}(R) \to X$ such that the following diagram commutes*

$$\begin{tikzcd} \operatorname{Spec}(K) \arrow[r] \arrow[d] & M \arrow[d, "f"] \\ \operatorname{Spec}(R) \arrow[r] \arrow[ur, dashed, "\exists!"] & \operatorname{Spec} k \end{tikzcd}$$

Theorem 3.5 (Langton's theorem)**.** *Let C be a smooth projective curve over an algebraically closed field k. Let $(R, \mathfrak{m})$ be a discrete valuation ring with $R/\mathfrak{m} \simeq k$ and $K =$ the field of fractions of R. Let F be a vector bundle on $C \times \operatorname{Spec} R$ flat over R such that F_K is semi-stable. Then there exists a subsheaf $E \subset F$ such that $E_K = F_K$ and E_k is semistable.*

Remark 3.6. Technically we should start with a flat family F_K of sheaves over $C_K \coloneqq C \times \operatorname{Spec} K$ and proceed by first completing it to F and then showing that F can be chosen so that F_k is semi-stable. Note that, one can always extend the coherent sheaf to F' on X_R such that $F'_K \simeq F_K$. Let $F'^{\vee\vee}$ denote the double dual of F', i.e. $\mathcal{H}om(\mathcal{H}om(F', \mathcal{O}_{C_R}), \mathcal{O}_{C_R})$. Then it is torsion free. Torsion free sheaves are flat over DVR. So we let $F \coloneqq F'^{\vee\vee}$.

On curves, torsion-free coherent sheaves are locally free. As a result, we do not need to consider non-locally free sheaves on curves. However, in higher dimensions, allowing pure sheaves becomes critical. All of the preceding notions generalise to pure coherent sheaves. These are similar to torsion free sheaves; torsion-free along their support.

Proof of Theorem 3.5. Suppose we cannot find such E. This means we can construct a chain of subsheaves $F \supset F^1 \supset F^2 \supset \cdots$ such that $F^i_K \simeq F_K$ but none of F^i_k's on the curve C_k are semistable. Thus F^n_k will always admit a destabilizing sub-sheaf (or, equivalently quotient sheaf). If we construct this chain strategically and inductively, we will show that eventually we can find a destabilzing quotient sheaf of $F^n_{R/\mathfrak{m}^n}$ for n large enough. Hence of $F^n_{\hat{R}}$. Lets first argue this.

Say F^n has already been constructed. Since F^n_k is not semi-stable, one can fine a HN filtration of F^n_k, such that B^n is the smallest piece or equivalently the maximal destabilising subbundle. Said differently, we have $\mu(B^n) > \mu(F^n)$ as well as for any other $\widetilde{B}^n \subset F^n_k$, $\mu(\widetilde{B}^n) < \mu(B^n)$.

Let $G^n \coloneqq F^n_k / B^n$ and note that $\mu(G^n) > \mu(F^n_k)$, i.e. the quotient G^n destabilises F^n_k. Define the next step

$$F^{n+1} \coloneqq \ker(F^n \to F^n_k \to G^n).$$

First note that $F^{n+1} \subset F$ is a sub-module of an R-flat module and hence is itself R-flat[8]. Furthermore, we have a short exact sequence

$$0 \to F^{n+1} \to F^n \to G^n \to 0$$

restricting to the special point, i.e. tensoring $\_ \otimes_R \operatorname{Spec}(k)$ we get

$$0 \to Tor_1^R(G^n, k) \to F^{n+1}_k \to F^n_k \to G^n \to 0.$$

The exactness on the left follows from the fact that F is flat over R. The right-most map factors through F^n_k, truncating there we obtain

$$0 \to Tor_1^R(G^n, k) \to F^{n+1}_k \to B^n \to 0. \tag{5}$$

[8] R is a discrete valuation ring and hence a PID, so flat = torsion free.

Compute $Tor_1^R(G^n, k)$ using the free resolution $R \xrightarrow{\cdot m} R \to k$ where $\mathfrak{m} = (m)$. We obtain $Tor_1^R(G^n, k) \simeq G^n$. Thus we can see G^n as a subsheaf of F_k^{n+1}.

Assume that $B^{n+1} \subset F^{n+1}$ is like before the maximal saturated destabilising subsheaf of F^{n+1}. Let $C^n = B^{n+1} \cap G^n$. We have $\mu(C^n) < \mu(B^n)$. An inequality follows from maximality of B^n but the strict inequality is due to the fact that $\mu(C^n) \leq \mu_{\max}(G^n) < \mu(B^n)$. Thus $\mathrm{rk}(C^n) < \mathrm{rk}(B^n)$ and eventually $C^n = 0$.

On the other hand, since we have the inclusion

$$B^{n+1}/C^n \subseteq F_k^{n+1}/G^n \simeq B^n,$$

Eventually we have $B^{n+1} = B^n$ for n large enough; we denote this $\mathcal{O}_{C_k}$-module by B. Thus inclusion $B^n \hookrightarrow F_k^{n+1}$ and the short exact sequence (5) is split. we have for all n large enough $F_k^n = B \oplus G$, as G^n's also stabilize to G.

We now ignore all F^n's for n small and assume that $F_k^n \simeq B \oplus G$ for all $n \geq 0$. The goal now is to extend G over to the thickened point $\operatorname{Spec} R/\mathfrak{m}^n$. To this end, let $Q^n = F/F^n$ be the $R/\mathfrak{m}^n$-module. Note that it is a quotient of $F/\mathfrak{m}^n F$. Furthermore, comparing the sequence $F_k^n = B \oplus G \to F_k^{n-1} = B \oplus G \to G \to 0$ to $0 \to B \to F^{n-1} \to G \to 0$, we obtain that the maps $F^i \to F^j$ is given by $\mathrm{id}_B \oplus 0$. Hence $F_k/F_k^n \simeq Q_k^n \simeq G$. One can check that Q^n is a flat $R/\mathfrak{m}^n$-module and hence, $F_{\hat{R}} \to Q_{\hat{R}}$ is a quotient such that the Hilbert polynomial $P(Q_{\hat{R}}) = \mathrm{rk}(G) \cdot \mu(G) + (1-g) = P(G)$.

At this point we use the fact that the Quot functor $\mathcal{Q}uot_{C_R/R}(F, P(G))$ of quotients of F on C_R with slope Hilbert polynomial $P(G)$ satisfies faithfully flat descent and $R \to \hat{R}$ is faithfully flat. Thus there exists a quotient $F \twoheadrightarrow Q$ on C_R such that $P(Q) = P(G)$. Pulling back to K, we found $F_K \twoheadrightarrow Q_K$ such that $\mu(Q_K) = \mu(G)$. That Q_K destabilise F_K follows from the fact that G destabilises F_k^n. Indeed, note that $\mu(Q_K) = \mu(G) > \mu(F_k^n) = \mu(F_K)$. The last equality follows since F and F^n's are flat over R, $\mu(F_k) = \mu(F_k^n) = \mu(F_K)$.

□

Back when Langton proved this theorem, the moduli space of semistable sheaves in dimension > 1 was not constructed yet. Thus his result established the fact that semistability was indeed the optimal choice for obtaining proper moduli.

4. The Grassmannian and Grothendieck's Quot scheme

In this part, we will look at an intriguing classification problem. A moduli space that exists and is used as a building block to solve a variety of moduli problems, including our own semistable sheaves moduli. This entails classifying all subsheaves with a fixed Hilbert polynomial P of pullbacks of a fixed coherent sheaf F on a fixed k-scheme S of finite type. The easiest case is when $S = \operatorname{Spec} k$ and we consider the r-dimensional quotient spaces of a fixed k-vector space V of dimension n. In this case the moduli space is the Grassmannian variety $\mathrm{Gr}(r, V)$. Define the moduli functor

$$\mathcal{G}r(r, V) \colon (\mathbf{Sch}/k)^{\mathrm{op}} \to \mathbf{Sets}$$

by

$$S \mapsto \{\mathcal{O}_S^{\oplus n} \twoheadrightarrow \mathcal{E} | \mathcal{E} \text{ is locally free of rank } r\}/\sim.$$

Here we identify $\mathcal{O}_S \otimes V \simeq \mathcal{O}_S^{\oplus n}$ and the relation $\sim$ is given by an isomorphism $\mathcal{E} \simeq \mathcal{E}'$ that commutes with the surjection $\mathcal{O}_S \otimes V \twoheadrightarrow \mathcal{E}$. We will learn that this functor is *representable* (see Theorem 4.5) by the well-known scheme $\mathrm{Gr}(r, V)$, the Grassmannian of r-dimensional linear quotient-spaces of V. To see how it can be endowed with the structure of a variety consult [Ses07, Chapter 1].

4.1. **Moduli and representability.** Let $\mathscr{C}$ be any category. Given an object $X \in \mathscr{C}$, consider the functor of points associated to X, denoted by

$$h_X : \mathscr{C}^{\mathrm{op}} \to \mathbf{Sets}$$

defined by sending $T \to \mathrm{Hom}_{\mathscr{C}}(T, X)$.

The functor category $\mathbf{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Sets})$ or the *category of presheaves* is a category whose objects are the functors $F\colon \mathscr{C}^{\mathrm{op}} \to \mathbf{Sets}$ and whose morphisms are the *natural transformations* $\mathrm{Nat}(F_1, F_2)$ between functors. This category can be thought of as an enlargement of $\mathscr{C}$ via the Yoneda lemma

Exercise 6 (Yoneda). For any object $X \in \mathscr{C}$ and any functor $F\colon \mathscr{C} \to \mathbf{Sets}$, there is a bijection (in $\mathbf{Sets}$) between

$$\mathrm{Nat}(h_X, F) = F(X)$$

given by sending $\xi \mapsto \xi(\mathrm{id}_X)$.[9]

In particular, the functor of points $h\colon \mathscr{C} \to \mathbf{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Sets})$ given by $X \mapsto h_X$ is fully faithful, i.e. it embeds $\mathscr{C}$ as a full subcategory of $\mathbf{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Sets})$.

A priori, our moduli functor would be an element of $\mathbf{Func}(\mathbf{Sch}^{\mathrm{op}}, \mathbf{Sets})$. The problem of representability measure how far it is from being naturally isomorphic to a functor of points.

Definition 4.1. A functor $F\colon \mathscr{C}^{\mathrm{op}} \to \mathbf{Sets}$ is said to be *corepresentable* by X if there exists an object $X \in \mathscr{C}^{\mathrm{op}}$ and a natural transformation (or a morphism in $\mathbf{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Sets})$) $\alpha\colon F \to h_X$ such that for any other object $X' \in \mathscr{C}^{\mathrm{op}}$ and a morphism $\alpha'\colon F \to h_{X'}$ there is a transformation $\beta\colon h_X \to h_{X'}$ so that $\alpha = \beta \circ \alpha'$.

The functor F is said to be *universally corepresentable* by X if for any $T \in \mathscr{C}^{\mathrm{op}}$ transformation $\phi\colon h_T \to h_X$ the fibre product $F_T \coloneqq h_T \times_{h_X} F$ is corepresented by T.

The functor is *representable* if it is universally corepresentable by X and the transformation $\alpha\colon F \to h_X$ is an isomorphism in $\mathbf{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Sets})$.

For us $\mathscr{C} = \mathbf{Sch}/k$ or $\mathbf{Sch}/S$ for some noetherian scheme S of finite type. A *moduli problem* (e.g. $\mathcal{G}r(r, V)$ above) will be a functor

$$\mathcal{P}\colon (\mathbf{Sch}/k)^{\mathrm{op}} \to \mathbf{Sets}$$

When $\mathcal{P}$ is representable by a scheme P, we say P is the *fine moduli space* associated to $\mathcal{P}$.

We say a scheme P is a *coarse moduli* of $\mathcal{P}$, if $\mathcal{P}$ is universally corepresentable by P and $\alpha(k)\colon \mathcal{P}(\operatorname{Spec} k) \to h_X(\operatorname{Spec} k)$ is a bijection.

Exercise 7. If X corepresents (resp. univ. corepresents) a functor, show that it is the unique object of $\mathscr{C}$ that does so.

Example 4.2. The functor

$$\Gamma\colon (\mathbf{Sch}/k)^{\mathrm{op}} \to \mathbf{Sets}$$

given by $T \mapsto \Gamma(T, \mathcal{O}_T)$ is represented by $\mathbb{A}^1_k$. Indeed, $h_{\mathbb{A}^1}(Y) \coloneqq \{f\colon Y \to \mathbb{A}^1\}$, which is the set of holomorphic functions on Y[10].

Exercise 8. Find a scheme that represents $T \mapsto \Gamma(T, \mathcal{O}^*_T)$.

4.2. **A representability criterion.** First and foremost, in order for $\mathcal{P}$ to be representable it should "glue" in the Zariski topology. More precisely, for any k-scheme X and for any Zariski open set $U \subset X$, the presheaf $U \mapsto \mathcal{P}(U)$ must form a sheaf, i.e. for any Zariski cover of $\{U_i\}$ of S, if two elements $a, b \in \mathcal{P}(S)$ satisfy $a|_{U_i} = b|_{U_i}$ for all i then $a = b$, furthermore if for the collection of elements $a_i \in U_i$, we have $a_i|_{U_i \cap U_j} = a_j|_{U_i \cap U_j}$ then there exists $a \in \mathcal{P}(X)$ such that $a|_{U_i} = a_i$. This condition is usually summed up by saying the following diagram is an equaliser

$$\mathcal{P}(X) \to \prod \mathcal{P}(U_i) \rightrightarrows \prod_{i,j} \mathcal{P}(U_i \cap U_j).$$

[9] Hint: If $\xi_1(\mathrm{id}) = \xi_2(\mathrm{id})$, for any $A \in \mathscr{C}$ and $\varphi \in h_X(A)$, draw diagrams applying $\xi_i\colon h_X \to F$ on the arrow $\varphi\colon A \to X$.

[10] One can replace k by S

In addition to being sheaf in the Zariski site, if $\mathcal{P}$ admits a *cover* by representable *open subfunctors* $\{\mathcal{P}_\alpha\}_{\alpha\in I}$, the original functor is representable. We will see this shortly in Lemma 4.4 and use this to show that the Grassmannian functor $\mathcal{G}r(r,V)$ is representable. But first let us define what these terminologies mean.

Definition 4.3. A subfunctor $\mathcal{P}' \subset \mathcal{P}$ is a functor such that for any $S \in \mathscr{C}^{\mathrm{op}}$, $\mathcal{P}'(S) \subset \mathcal{P}(S)$. Furthermore, $\mathcal{P}'$ is said to be *open subfunctor* (resp. *closed*) if for any scheme T admitting a natural transformation $h_T \to \mathcal{P}$ the pull-back functor $h_T \times_{\mathcal{P}} \mathcal{P}'$ is representable by an open (resp. closed) subscheme $T' \subset T$.

In moduli language this means that the natural transformation given by the inclusion $\mathcal{P}' \subset \mathcal{P}$ is representable. Note that this is different from representability of $\mathcal{P}'$ itself.

Exercise 9. Let $\mathcal{P}' \hookrightarrow \mathcal{P}$ be a subfunctor of $\mathcal{P}$ such that the inclusion is representable. If $\mathcal{P}$ is representable, show that $\mathcal{P}'$ is also representable.

Lemma 4.4. *Let $\mathcal{P}\colon (\mathbf{Sch})^{\mathrm{op}} \to \mathbf{Sets}$ be a functor that is a sheaf in the the Zariski topology. The functor $\mathcal{P}$ is representable if additionally there exists a collection of representable open subfunctors $\{\mathcal{P}_\alpha\}_{\alpha\in I}$ such that they cover $\mathcal{P}_\alpha$(i.e. for every scheme T together with a nartual functor $\xi\colon h_T \to \mathcal{P}$, the collection of open subschemes $\{U_\alpha\}$ representing $\{\mathcal{P}_\alpha \times_{\mathcal{P}} h_T\}$ covers T).*

Proof. Let X_α be a scheme representing $\mathcal{P}_\alpha$. The goal is to glue X_α's to construct a scheme X that would represent $\mathcal{P}$. By Yoneda we have $\mathrm{Hom}(h_{X_\alpha}, \mathcal{P}_\alpha) = \mathcal{P}_\alpha(X_\alpha)$. Let $\xi_\alpha \in \mathcal{P}_\alpha(X_\alpha) \hookrightarrow \mathcal{P}(X_\alpha)$ be the image of id_{X_α}. Since the inclusion $\mathcal{P}_\alpha \hookrightarrow \mathcal{P}$ is also representable, the pull back $\mathcal{P}_{\alpha'} \times_{\mathcal{P}} X_\alpha$ is representable by an open subscheme $X_{\alpha,\alpha'} \subset X_\alpha$. Similarly $X_{\alpha',\alpha}$ represents the pullback of the inclusion $\mathcal{P}_{\alpha'} \hookrightarrow \mathcal{P}$ to the scheme $X_{\alpha'}$. Hence $X_{\alpha,\alpha'}$ and $X_{\alpha',\alpha}$ both represent the functor $h_{X_\alpha} \times_{\mathcal{P}} h_{X_{\alpha'}}$ and thus are isomorphic. Let us denote this isomorphism by $\varphi_{\alpha,\alpha'}\colon X_{\alpha,\alpha'} \to X_{\alpha',\alpha}$ and note that $\mathcal{P}(\varphi_{\alpha,\alpha'})^*\xi_\alpha = \xi_{\alpha'}$. Note also that the last condition uniquely determines $\varphi_{\alpha,\alpha'}$.

In order to glue X_α's along these isomorphisms, we need the cocyle condition. To make sense of it, first note that the open sets

$$X_{\alpha,\alpha'} \cap X_{\alpha,\alpha''} \overset{\varphi_{\alpha,\alpha'}}{\simeq} X_{\alpha',\alpha} \cap X_{\alpha',\alpha''}$$

are isomorphic. By Yoneda's lemma $\varphi_{\alpha,\alpha'}$ satisfies the cocycle condition

$$\varphi_{\alpha',\alpha''} \circ \varphi_{\alpha,\alpha'} = \varphi_{\alpha',\alpha''}.$$

Indeed, by construction, $\mathcal{P}(\varphi_{\alpha,\alpha'})\mathcal{P}(\varphi_{\alpha',\alpha''})\xi_\alpha = \mathcal{P}(\varphi_{\alpha',\alpha''})\xi_\alpha = \xi_{\alpha''}$. Hence we can glue $\{X_\alpha\}$ to construct X. Note also that since $\mathcal{P}$ is a sheaf in the Zariski topology, ξ_α's glue to give $\xi \in \mathcal{P}(X)$ the unique representative of id_X.

To see that X represents $\mathcal{P}$, we need to construct a map $\mathcal{P} \to h_X$ that is an isomorphism. Pick $\xi' \in \mathcal{P}(T)$ for some scheme T. By representability of the inclusion $\mathcal{P}_\alpha \hookrightarrow \mathcal{P}$, we obtain $\xi'_\alpha \in \mathcal{P}_\alpha(T_\alpha)$ and by representability of $\mathcal{P}_\alpha$, we view this as $g_\alpha\colon T_\alpha \to X_\alpha$. By a similar compatibility argument as above, also glue together to $g\colon T \to X$. Since $\mathcal{P}(g)\xi_\alpha = \xi'_\alpha$ for each α, $\mathcal{P}(g)^*\xi = \xi'$ hence g is unique. To see surjectivity, given any element $g\colon T \to X$, define $\xi' \coloneqq \mathcal{P}(g)(\xi)$ □

4.3. **The Grassmannian.**

Theorem 4.5. *The Grassmannian is representable by a projective k-scheme* $\mathrm{Gr}(r,V)$.

Proof. We use the criterion for representability in Lemma 4.4. Define the collection of subfunctor $\mathcal{G}_W$ indexed by all subspace $W \subset V$ of dimension r [11] by

$$\mathcal{G}_W(S) \coloneqq \{[\varphi\colon \mathcal{O}_S \otimes V \twoheadrightarrow \mathcal{E}] \in \mathcal{G}r(r,V)(S) | \exists \text{ an isomorphism } \mathcal{O}_S \otimes W \hookrightarrow \mathcal{O}_S \otimes V \twoheadrightarrow \mathcal{E}\}/\sim.$$

[11] if you want to be economical, it is enough to fix a basis for V and consider all subspace spanned by basis vectors with indices from size r subset of $\{1,\cdots,n\}$

We want to show that $\mathcal{G}_W$'s are representable and form an open cover of $\mathcal{G}r(r,V)$. The representability comes for free, since the isomorphism given by the composition $\mathcal{O}_S \otimes W \hookrightarrow \mathcal{O}_S \otimes V \twoheadrightarrow \mathcal{E}$ splits the first injection and is thus determined via the splitting $\mathcal{O}_S \otimes V \twoheadrightarrow \mathcal{O}_S \otimes W$. The splitting is determined by first fixing $n-r$ dimensional complement W'[12] of W and then determining the ways to project these to W. Thus it is given by the affine space $\mathbb{A}_W := \mathrm{End}_k(W', W)$ of dimension $\mathbb{A}^{r(n-r)}$.

To see openness, given $W \subset V$ and an element $[\varphi\colon \mathcal{O}_S \otimes V \twoheadrightarrow \mathcal{E}] \in \mathcal{G}r(r,V)(S)$, we consider the maximal set of points $S_W \subset S$ such that for all $x \in S_W$ the inclusion $\mathcal{O}_S \otimes W \hookrightarrow \mathcal{O}_S \otimes V$ composed with φ pulled back under $\{x\} \hookrightarrow X$ is an isomorphism. We claim that this set is open. Indeed, consider the exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_S \otimes W \to \mathcal{E} \to \mathcal{Q} \to 0$$

where $\mathcal{K}$ and $\mathcal{Q}$ are kernel and cokernel of the composition. Since the support of the coherent sheaves $\mathcal{K}$ and $\mathcal{Q}$ are closed, on the complement we have an isomorphism. This is S_W. Thus $[\varphi|_{S_W}] \in \mathcal{G}_W(S_W)$[13]. Finally, we want to show that S_W represents $\mathcal{G}_{W,S} := h_S \times_{\mathcal{G}r(r,V)} \mathcal{G}_W$, i.e. given a morphism of schemes $f\colon U \to S$, $[f^*\varphi] \in \mathcal{G}_{W,S}(U)$ if and only if f factors through S_W. One direction is easy since pullback of an isomorphism of vector bundle remains an isomorphism. To see the converse, let $[f^*\varphi] \in \mathcal{G}_{W,S}(U)$, i.e. $f^*\varphi\colon \mathcal{O}_U \otimes W \to f^*\mathcal{E}$ is an isomorphism, or equivalently $f^*\mathcal{Q} = 0$. Since the stalk of $f^*\mathcal{Q}$ at any point $x \in U$ is spanned by the same number of elements as the dimension of the fibre $f^*\mathcal{Q} \otimes \kappa(x)$ and this dimension is the same that the dimension $\mathcal{Q} \otimes \kappa(f(x))$, we conclude that $f^*\varphi$ is surjective at x if and only if φ is surjective at $f(x)$ or equivalently $f(x) \in S_W$. Hence the image of f lies in S_W.

Finally, we argue that $\mathcal{G}_W$'s cover $\mathcal{G}r(r,V)$, or equivalently S_W's cover S. This is a statement solely about locally free sheaves. Indeed, note that for any $s \in S$, there exists a Zariski open set $s \in U \subset S$ such that $\mathcal{E}|_U \simeq \mathcal{O}_U^k$. Hence we obtain a surjection given by the composition $\mathcal{O}_U \otimes V \twoheadrightarrow \mathcal{E}|_U \simeq \mathcal{O}_U^k$. We let W be determined by the kernel of the restriction of this to the residue field $\kappa(s)$, i.e. $W := \mathrm{Ker}(\kappa(s)^{\oplus n} \twoheadrightarrow \mathcal{E} \otimes \kappa(s) \simeq \kappa(s)^{\oplus r})$ and then consider it as a vector space over k via the extension $k \hookrightarrow \kappa(s)$. Then $S_W \supset U$. Finitely many such U and thus S_W's cover S.

We have shown that the Grassmannian is representable. Lets call the scheme representing the Grassmannian $\mathrm{Gr}(r,V)$. We are now left to show the projectivity of $\mathrm{Gr}(r,V)$. First we use the valuative criterion to show that it is proper over $\operatorname{Spec} k$. First note that Gr is of finite type over k. Indeed it is covered by affine varieties of the form $\mathrm{End}_k(V,W)$. Secondly, consider a dvr R over $\mathbb{Z}$ with fraction field K. We have

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \mathrm{Gr}(r,V) \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

Since $\mathrm{Gr}_k(r,V)$ represents the Grassmannian, the top horizontal arrow gives a surjection $K^{\oplus n} \twoheadrightarrow W$, where W is a K-vector quotientspace of rank r. To see properness, we want to argue that there is a unique way to lift this surjection $R^{\oplus n} \twoheadrightarrow \widetilde{W}$ for a unique lift of W to a free R-module $\widetilde{W}$. The choice is somewhat obvious, namely we let $\widetilde{W}$ to be W with the induced module structure coming from $R \twoheadrightarrow K$ and $\widetilde{W} \otimes_R K \simeq W$ by Exercise 10.

To see projectivity note that the map $\mathcal{G}r(r,V) \to h_{\mathbb{P}_k}(\wedge^r V)$ defined by

$$[\mathcal{O}_S \otimes V \twoheadrightarrow \mathcal{E}] \mapsto [\mathcal{O}_S \otimes \bigwedge^r V \to \det(\mathcal{E})]$$

[12] The economical approach would have come in handy here

[13] In fact we do not need to worry about $\mathcal{K}$ since surjectivity of locally free sheaves of same rank implies isomorphism. Show this!

is a monomorphism; the so-called Plücker embedding. Indeed, on the open set S_W we have an isomorphism given by the composition $\mathcal{O}_S \otimes W \hookrightarrow \mathcal{O}_S \otimes V \twoheadrightarrow \mathcal{E}$ and as seen before it is determined by the splitting $\mathcal{O}_S \otimes V \twoheadrightarrow \mathcal{O}_S \otimes W$ that sends $\mathcal{O}_S \otimes W$ to itself. This determines a splitting $\mathcal{O}_S \otimes \wedge^r V \twoheadrightarrow \det \mathcal{E} \simeq \mathcal{O}_S \otimes \det W \simeq \mathcal{O}_S$, which in turn gives $\binom{n}{r}$ global sections of $\mathcal{O}_S$ and thus is represented by $\mathbb{A}^{\binom{n}{r}}$ (see Exercise 9). The affine spaces Gr_W subschemes of $\mathrm{Gr}(r, V)$ representing $\mathcal{G}r_W$ embeds in $\mathbb{A}^{\binom{n}{r}}$. This together with the fact that $\mathrm{Gr}(r, V)$ is proper implies that it is a projective scheme. See [Ses07] for more details on Plücker embedding.

□

Exercise 10. Let $i\colon A \hookrightarrow B$ be a morphism of rings. When is $B \otimes_A B \simeq B$? Find an example when it is not the case. Show that when $B = S^{-1}A$ for some multiplicatively closed set $S \subset A$ then it is the case.

The construction is functorial, i.e. in place of V we could start with a free $\mathbb{Z}$-module and everything said above would work over $\mathbb{Z}$. Furthermore of Grassmannian can be extended to include locally free quotients of a fixed rank of any coherent sheaf $\mathcal{V}$ on a k-scheme S of finite type. The updated functor would look as follows:

$$\mathcal{G}r(r, \mathcal{V})\colon (\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Sets}$$

defined by

$$(f\colon T \to S) \mapsto \{[\varphi\colon f^*\mathcal{V} \to \mathcal{E}] | \mathcal{E} \text{ is locally free on } T \text{ of rank } r\}/\sim$$

This functor is representable by a projective S-scheme $\mathrm{Gr}_S(r, \mathcal{V})$. When $\mathcal{V} \simeq \mathcal{O}_S^{\oplus n}$, $\mathrm{Gr}_S(r, \mathcal{V}) \simeq S \times \mathrm{Gr}(r, k^{\oplus n})$. The scheme $\mathrm{Gr}_S(r, \mathcal{V})$ is constructed by reducing the problem to this case. For more detail see [HL10, p. 42].

Remark 4.6 ($\mathbb{P}^n_{\mathbb{Z}}$ as Grassmannian). It is well-known that $\mathbb{P}^n_{\mathbb{Z}} \simeq \mathrm{Gr}_{\mathbb{Z}}(1, V)$ for $V = \mathbb{Z}^{\oplus n}$. In this new light, we can view $\mathbb{P}^n_{\mathbb{Z}}$ as the scheme representing the function $T \mapsto \{\mathcal{O}_T^{\oplus n+1} \twoheadrightarrow \mathcal{L}\}/\sim$ for some line bundle $\mathcal{L}$.

Remark 4.7 (The universal bundle). Since $\mathrm{Gr}(r, V)$ represents $\mathcal{G}r(r, V)$, i.e. there exists an isomorphism $\mathcal{G}r(r, V) \xrightarrow{\sim} h_{\mathrm{Gr}(r,V)}$, the universal object $\xi \in \mathcal{G}r(r, V)(\mathrm{Gr}(r, V))$ corresponding to $\mathrm{id}_{\mathrm{Gr}(r,V)}$ is given by $\xi\colon \mathcal{O}_{\mathrm{Gr}(r,V)} \otimes V \to \mathcal{U}$. The bundle $\mathcal{U}$ is called the universal bundle or *the tautological bundle* of the Grassmannian. The fibres of $\mathcal{U}$ at a point $[V \twoheadrightarrow W] \in \mathrm{Gr}(r, V)$ is given by W. Similarly, for any k-scheme S, the element $[\varphi\colon \mathcal{O}_S \otimes V \twoheadrightarrow \mathcal{E}] \in \mathcal{G}r(r, V)$ are in one-to-one correpondence with the datum of a map $u_\varphi\colon S \to \mathrm{Gr}(r, V)$. Furthermore we have $u_\varphi^*\xi = \varphi$. This is all Yoneda.

4.4. **The Quot.** The goal of the Quot functor is to classify quotients that are compatible with a fixed family $f\colon X \to S$.

Definition 4.8. Given a projective S-scheme $f\colon X \to S$ and a coherent sheaf $\mathcal{V}$ on X, the Quot functor

$$\mathcal{Q}uot_{X/S}(\mathcal{V}, P)\colon (\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Sets}$$

is defined by

$$(p\colon T \to S) \mapsto \{[\mathcal{V}_T \twoheadrightarrow F] | F \text{ is a } T\text{-flat coherent sheaf on } X_T \coloneqq X \times_S T \text{ with } P(F_t) = P\}/\sim .$$

Here $\sim$ is given by an isomorphism $F \simeq F'$ that is comptible with the projections $p^*\mathcal{V} \twoheadrightarrow F$ and $p^*\mathcal{V} \twoheadrightarrow F'$ and $\mathcal{V}_T \coloneqq p^*\mathcal{V}$.

Example 4.9. Let $X = \mathbb{P}^n_k$ and $S = \operatorname{Spec} k$, the set $\mathcal{Q}uot_{\mathbb{P}^n_k}(\mathcal{O}^{\oplus r}_{\mathbb{P}^n_k}, P)(T)$ consists of T-flat coherent sheaves F on $\mathbb{P}^n_T$ together with a surjection of $\mathcal{O}_{\mathbb{P}^n_T}$-modules $\mathcal{O}^{\oplus r}_{\mathbb{P}^n_T} \twoheadrightarrow F$ such that restricted to any closed fibre $\mathbb{P}^n_{\kappa(t)}$, we have $P(F_t) = P$.

Exercise 11. Show that $\mathcal{Q}uot_{X/S}(\mathcal{V}, P)$ satisfies fppf descent, i.e. $u_i \colon T_i \to T$ be a collection of finitely presented, faithfully flat maps such that $f_i(T_i)$ covers T then show that the following diagram is an equaliser

$$\mathcal{Q}uot_{X/S}(\mathcal{V}, P)(T) \to \prod_i \mathcal{Q}uot_{X/S}(\mathcal{V}, P)(T_i) \rightrightarrows \prod_{i,j} \mathcal{Q}uot_{X/S}(\mathcal{V}, P)(T_i \times_T T_j)$$

4.4.1. *Castelnuovo–Mumford's m-regularity.* Various properties of flat sheaves on a projective family $f \colon X \to S$ over a Noetherian scheme S that we have studied in §2 will come in handy at this point. Recall that given a coherent sheaf F on X we know that F is S-flat if and only if $f_*F(m)$ is locally free for $m \gg 0$. In this section we will get a better hold on m based on the Hilbert polynomial of $F|_{X_s}$. To this end we recall the following

Definition 4.10. Given a projective variety $(X, \mathcal{O}_X(1))$ with a fixed polarisation $\mathcal{O}_X(1)$, a coherent sheaf F is said to be (m-regular with respect to $\mathcal{O}(1)$) if $H^i(X, F(m-i)) = 0$ for all $i > 0$. The smallest such m is the CM-regularity (or m-regularity) of F and denoted by $\operatorname{reg}_{\mathcal{O}_X(1)}(F)$, when there is a no room for confusion, we simply write $\operatorname{reg}(F)$.

Example 4.11. A line bundle $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^n}(-\ell)$ on $\mathbb{P}^n$ is ℓ-regular. Indeed, $H^i(\mathbb{P}^n, \mathcal{O}(\ell)) = 0$ for $i > 0$ unless $i = n$ and $\ell \leq -n-1$ (Exercise!). Thus, in order for $H^n(\mathbb{P}^n, \mathcal{O}(m+\ell-n))$ to be zero, we need that $m + \ell - n > -n-1$.

Exercise 12. Given a hypersurface $i \colon X \hookrightarrow \mathbb{P}^n$ of degree d, show that the coherent sheaf $F \coloneqq i_*\mathcal{O}_X$ is $d-1$-regular with respect to $\mathcal{O}(1)$.

The m-reularity indicates at which point cohomological complexities of F vanish. This is captured by the following result of Mumford.

Theorem 4.12. *If F is m-regular then $F(m)$ is globally generated. Furthermore, the multiplication map*

$$H^0(X, F(m)) \otimes H^0(X, \mathcal{O}_X(k)) \to H^0(X, F(m+k))$$

is surjective for all $k \geq 0$.

Proof sketch. For details see [Laz04, Thm. 1.8.3]. For $k \gg 0$, the sheaf $F(m+k)$ is globally generated, i.e. the evaluation map

$$H^0(X, F(m+k)) \otimes \mathcal{O}_X \xrightarrow{\text{ev}} H^0(X, F(m+k))$$

is surjective. Assume the surjectivity of in the second part of the theorem, we obtain

$$H^0(X, F(m)) \otimes H^0(X, \mathcal{O}_X(k)) \otimes \mathcal{O}_X(-k) \twoheadrightarrow H^0(X, F(m+k)) \otimes \mathcal{O}_X(-k) \twoheadrightarrow F(m).$$

The composition factors through the surjection

$$H^0(X, F(m)) \otimes H^0(X, \mathcal{O}_X(k)) \otimes \mathcal{O}_X(-k) \twoheadrightarrow H^0(X, F(m)) \otimes \mathcal{O}_X.$$

Hence we obtain $H^0(X, F(m)) \otimes \mathcal{O}_X \twoheadrightarrow F(m)$, i.e. $F(m)$ is globally generated.

Thus it is enough to show the second part. Consider the evaluation

$$V \otimes \mathcal{O}(-1) \twoheadrightarrow \mathcal{O}$$

and the Koszul complex associated to it

$$0 \to \bigwedge^{n+1} V(-n-1) \to \cdots \bigwedge^{2} V(-2) \to V \otimes \mathcal{O}(-1) \twoheadrightarrow \mathcal{O}$$

Since F is m-regular, $H^i(\mathbb{P}^n, \bigwedge^i V(m-i) \otimes F) = H^i(\mathbb{P}^n, F(m-i))^{\oplus \binom{n+1}{i}} = 0$. This shows the desired surjection from the sequence obtained by tensoring the Koszul complex with $F(m+1)$ and taking cohomologies. $\square$

The following result shows that m-regularity of certain sheaves behave well in flat family, in the sense that it only depends on the Hilbert polynomial.

Lemma 4.13 (m-regularity in family)**.** *For any $p, n \in \mathbb{Z}_{\geq 0}$ there exists a polynomial $m_{p,n}(t_1, \cdots, t_n)$ with integral coefficients such that for any coherent sheaf F on $\mathbb{P}^n_k$ together with a surjection $q\colon \mathcal{O}_{\mathbb{P}^n}^{\oplus p} \twoheadrightarrow F$, if the Hilbert polynomial of F is written in terms of binomial coefficients as*

$$P(F, r) = \sum_{i=0}^{n} a_i \binom{r}{i}$$

where $a_i \in \mathbb{Z}$, then F is m-regular for $m = m_{p,n}(a_0, ..., a_n)$.

Proof. We show this by induction on n. The case $n = 0$ being the case of a point, the theorem is true for any polynomial m. In general, let H be a very general hyperplane in $X = \mathbb{P}^n$ such that H does not contain any associated points of F. Then $\mathcal{T}or_1^{\mathcal{O}_X}(F, \mathcal{O}_H) = 0$. Thus we obtain a short exact sequence

$$0 \to \mathrm{Ker}(q)|_H \to \mathcal{O}_H^{\oplus p} \to F|_H \to 0.$$

This is the main framework for the induction. Lets estimate $\chi(F|_H)$ using $0 \to F(-H) \to F \to F|_H \to 0$. We have for $m \gg 0$

$$P(F|_H, m) = P(F, m) - P(F, m-1) = \sum_{i=0}^{n-1} b_i \binom{r}{i}$$

This is again a numerical polynomial written with coefficients $b_i(a_0, \cdots, a_n)$, polynomial in a_i. By induction there exists $m_0 \coloneqq m_{p,n-1}(b_0, \cdots, b_n)$ such that $F|_H$ is m_0-regular. From the long exact sequence of cohomology we get

$$H^0(X, F(m)) \to H^0(X, F|_H(m)) \to H^1(X, F(m-1)) \to H^1(X, F(m)) \to 0.$$

Indeed, by m_0-regularity $H^1(X, F|_H(m)) = 0$ for $m \geq m_0 - 1$. Similarly, $H^i(X, F(m-1)) \simeq H^i(X, F(m))$ for $i \geq 2$ and $m \geq m_0 - 1$. Increasing m this way, by Serre vanishing we get $H^i(X, F(m-1)) \simeq H^i(X, F(m-1+k)) = 0$ for $k \gg 0$. Thus

$$H^i(X, F(m)) = 0 \text{ for } \text{ for } i \geq 2 \text{ and } m \geq m_0 - 2.$$

For $i = 1$, first note that $h^1(X, F(m-1)) \geq h^1(X, F(m))$. If equality holds for some $m \geq m_0$, we have a surjection $H^0(X, F(m)) \twoheadrightarrow H^0(H, F|_H(m))$. We claim that it implies $H^0(X, F(m+i)) \twoheadrightarrow H^0(H, F|_H(m+i))$ are surjective for all $i \geq 0$. Hence from this step onwards $h^1(X, F(m-1+i)) = h^1(X, F(m+i))$. Since $h^1(X, F(m+i)) = 0$ for $i \gg 0$, this implies already $h^1(X, F(m)) = 0$. So until that happens, it must be the case that $h^1(X, F(m-1)) > h^1(X, F(m))$, i.e. the dimension drops at every step. This means for $m = m_0 + h^1(X, F(m_0))$ we must have $h^1(X, F(m)) = 0$.

To see the claim in the previous paragraph, consider surjective multiplication map coming from m_0-regularity of $F|_H$

$$H^0(H, F|_H(m)) \otimes H^0(H, \mathcal{O}(i)) \twoheadrightarrow H^0(H, F|_H(m+i)),$$

for $i \geq 0$. The surjective map $H^0(H, F(m)) \otimes H^0(\mathbb{P}^n, \mathcal{O}(i)) \twoheadrightarrow H^0(H, F|_H(m)) \otimes H^0(H, \mathcal{O}(i))$ composed with the muliplication above factors through

$$H^0(X, F(m+i)) \to H^0(H, F|_H(m+i)).$$

Hence this must be a surjection as well for all $i \geq 0$.

So we are left to polynomially estimate $h^1(X, F(m_0))$. Since we already established that $h^i(X, F(m_0)) = 0$ for $i \geq 2$, we have

$$h^1(X, F(m_0)) = h^0(X, F(m_0)) - P(F, m_0) \leq r\binom{n+m_0}{m_0} + h^1(X, \mathrm{Ker}(q)(m_0)) - \sum_{i=0}^{n} a_i \binom{m_0}{i}.$$

The latter can be written as a linear polynomial $m_1(t_0, \cdots, t_n)$. Since b_i's are polynomial in $a_i's$ we think of $m_{p,n-1}(b_0, \cdots, b_{n-1})$ also as a polynomial $m_0(t_0, \cdots, t_n)$ in $n+1$ variable. We let

$$m_{p,n} \coloneqq m_1(t_0, \cdots, t_n) + m_0(t_0, \cdots, t_n)$$

□

More generally, by [HL10, Lem. 1.7.6] given a S-flat family of coherent sheaves F on the S-scheme $X \times S$ with $P(F_s, t) = P(t)$ for some fixed polynomial P, together with a fixed coherent sheaf $\mathcal{V}$ on X and surjections $\mathcal{V} \twoheadrightarrow F_s$ for all $s \in S$, there exists an integer m_0 such that F_s is m_0-regular for all $s \in S$.

A consequence of this result is that given a S-flat coherent sheaf F on $\mathbb{P}^n_S$, after fixing a surjection $\mathcal{O}_{\mathbb{P}^n}^{\oplus p} \twoheadrightarrow F$, we can find an integer m_0 such that F_s is m-regular on $\mathbb{P}^n_{\kappa(s)}$ for all $s \in S$. By Theorem 4.12 this means $f_*F(m_0)$ is locally free (use Grauert's theorem [Har77, III.12]) of rank $P(m_0)$ and $R^if_*F(m_0) = 0$.

4.4.2. *Construction.* We now come to representability of the Quot functor $\mathcal{Q}uot_{X/S}(\mathcal{V}, P)$. The idea is to see it as subfunctor of some Grassmannian.

Theorem 4.14. *The quot functor $\mathcal{Q}uot_{X/S}(\mathcal{V}, P)$ is representable by a projective S-scheme $\mathrm{Quot}_{X/S}(\mathcal{V}, P)$.*

Proof.

Step 1. *Assume $X = \mathbb{P}^n_k$ and $S = \operatorname{Spec} k$*

For any k-scheme T, let $f_T \colon \mathbb{P}^n_T \to T$ denote the pull-back of $f\colon X \to \operatorname{Spec} k$. Given, any element $[\varphi\colon \mathcal{V}_T \twoheadrightarrow F] \in \mathcal{Q}uot(\mathcal{V}, P)(T)$, note that $K = \ker(\varphi)$ is also T-flat (Exercise). We know by Lemma 4.13 that for all $m > \max\{\mathrm{reg}(F_t), \mathrm{reg}(\mathcal{V}_t), \mathrm{reg}(K_t)\}$ we have the following short exact sequence of locally free sheaves

$$0 \to f_{T_*}K(m) \to f_{T_*}\mathcal{V}_T(m) \to f_{T_*}F(m) \to 0$$

Indeed, by m-regularity of K we also have $R^1f_{T_*}K(m) = 0$. Furthermore, note that $f_{T_*}\mathcal{V}_T(m) \simeq \mathcal{O}_T \otimes_k H^0(\mathbb{P}^n_k, \mathcal{V}(m))$

Thus for $m' > m$ the sheaf $f_{T_*}F(m')$ is determined by the injection

$$0 \to f_{T_*}K(m') \to \mathcal{O}_T \otimes_k H^0(\mathbb{P}^n_k, \mathcal{V}(m')). \tag{6}$$

Since $m > \mathrm{reg}(K_t)$ for all $t \in T$, by Theorem 4.12 the multiplication

$$f_{T_*}K(m) \otimes H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n}(m'-m)) \to f_{T_*}K(m')$$

is surjective. Theorefore $f_{T_*}K(m')$ for $m' > m$ is determined by $f_{T_*}K(m)$. Hence the map in Eq. (6) and hence $\bigoplus_m f_{T_*}F(m)$ are also completely determined by $f_{T_*}\ker(\varphi)(m)$. Furthermore the graded module $\bigoplus_{m'>m} f_{T_*}F(m')$ over the graded ring

$$\bigoplus_{m'>m} f_{T_*}\mathcal{O}_{\mathbb{P}^n_T}(m) \simeq \mathcal{O}_T \otimes \bigoplus_m H^0(X, \mathcal{O}(m)) \simeq \mathcal{O}_T[x_0, \cdots, x_n]$$

recovers F as a $\mathcal{O}_{\mathbb{P}^n_T}$-module. Therefore, the surjection φ determines an element in $\varphi_m \in \mathcal{G}r(r, V)(T)$, where $r = P(m)$ and $V = H^0(X, \mathcal{V}(m))$, and is uniquely determined by φ_m. In other words we have established an injection of functors

$$\mathcal{Q}uot(\mathcal{V}, P) \hookrightarrow \mathcal{G}r(r, V).$$

In what follows we fix this positive integer m and for the ease of notation we use $\mathbf{G}$ in place of $\mathrm{Gr}(r, V)$.

Recall that on Grassmannian we have the tautological bundle $\mathcal{U}$ with

$$\mathcal{O}_{\mathbf{G}} \otimes V \twoheadrightarrow \mathcal{U}$$

such that the fibre of $\mathcal{U}$ over a point $[V \twoheadrightarrow W] \in \mathbf{G}$ is given by $W^\vee$. Furthermore, any element $[f_{T_*}\varphi(m) \colon \mathcal{O}_T \otimes V \twoheadrightarrow f_{T_*}F(m)] \in \mathcal{G}r(r,V)(T)$ is determined by a map $u \colon T \to \mathbf{G}$ and the pull-back $\mathcal{O}_T \otimes V \twoheadrightarrow u^*\mathcal{U}$. In other words $f_{T_*}F(m) \simeq u^*\mathcal{U}$.

Let $f_{\mathbf{G}} \colon \mathbb{P}^n_{\mathbf{G}} \to \mathbf{G}$ be the pull-back of f. By the previous discussion the surjection $\mathcal{O}_{\mathbf{G}} \otimes H^0(\mathbb{P}^n_k, \mathcal{V}(m)) \twoheadrightarrow \mathcal{U}$ determines a graded $\mathcal{O}_{\mathbb{P}^n_{\mathbf{G}}}$-module U as follows: Let

$$\mathcal{A} \coloneqq \operatorname{Ker}(\mathcal{O}_{\mathbf{G}} \otimes V \twoheadrightarrow \mathcal{U})$$

and let $\widetilde{\mathcal{A}} \coloneqq \bigoplus_{m' \geq 0} \mathcal{A} \cdot H^0(\mathbb{P}^n_k, \mathcal{V}(m+m'))$ be consist of graded pieces given by the image of $\mathcal{A}$, under the multiplication map

$$\mathcal{A} \otimes H^0(\mathbb{P}^n_k, \mathcal{V}(m')) \to H^0(\mathbb{P}^n_k, \mathcal{V}(m+m')).$$

Then we define U to be the $\mathbb{P}^n_G$-module induced by graded module $\bigoplus_{m' \geq 0} H^0(\mathbb{P}^n_k, \mathcal{V}(m+m'))/\widetilde{\mathcal{A}}$.

Let $\mathbf{G}_P \subset \mathbf{G}$ be a locally closed subset on which U has Hilbert polynomial P. Recall that $\mathbf{G}_P$ satisfies the following universal property: Let $[\varphi \colon \mathcal{V} \twoheadrightarrow F] \in \mathcal{Q}uot(\mathcal{V}, P)(T)$ for some scheme T, such that F is T-flat and $P(F_t) = P$. Let $u \colon T \to \mathbf{G}$ be the moduli map such that $u^*\mathcal{U} = f_{T_*}F(m)$. Therefore as graded module and hence as $\mathcal{O}_{\mathbb{P}^n_T}$-modules $u^*_{\mathbb{P}}U \simeq F$ where $u_{\mathbb{P}} \colon \mathbb{P}^n_T \to \mathbb{P}^n_{\mathbf{G}}$ is the pullback of u. Since F is flat with fibre-wise Hilbert polynomial P, the universal property of $\mathbf{G}_P$ is that $u \colon T \to \mathbf{G}$ factors through $\mathbf{G}_P$.

We have established that $\mathbf{G}_P$ represents $\mathcal{Q}out_{\mathbb{P}^n_k/k}(\mathcal{V}, P)$ and hence will be denoted by $\operatorname{Quot}_{\mathbb{P}^n_k/k}(\mathcal{V}, P)$. Via the Plücker emmbedding $\mathbf{G} \subset \mathbb{P}_k(\wedge^r V)$ we further observe that $\operatorname{Quot}_{\mathbb{P}^n_k/k}(\mathcal{V}, P)$ is a locally closed subset of the projective space and hence is separated and of finite type.

For properness we need to check the valuative criterion, i.e. we need to show that given R a dvr over k with field of fractions K, then the restriction map

$$\mathcal{Q}uot_{\mathbb{P}^n_k/k}(\mathcal{V}, P)(R) \to \mathcal{Q}uot_{\mathbb{P}^n_k/k}(\mathcal{V}, P)(K)$$

is bijective. Let $[\varphi \colon \mathcal{V}_K \twoheadrightarrow F] \in \mathcal{Q}uot_{\mathbb{P}^n_k/k}(\mathcal{V}, P)(K)$. Let $j \colon \operatorname{Spec} K \to \operatorname{Spec} R$ be the inclusion. First note that $j_{\mathbb{P}_*}\mathcal{V}_K \simeq \mathcal{V}_R$. Let $F_R \coloneqq \operatorname{im}(\mathcal{V}_R \to j_{\mathbb{P}_*}F)$. Since on a dvr flatness is equivalent to torsion freeness F_R is R-flat. Furthermore F_R is unique since we know that $\operatorname{Quot}_{\mathbb{P}^n_k/k}(\mathcal{V}, P)$ is separated.

Step 2. *The case $X = \mathbb{P}^n_S$.*

Since $\mathcal{V}$ is coherent, there is a surjective map $q \colon \mathcal{O}(-a)^{\oplus k} \twoheadrightarrow \mathcal{V}$ for some integers a and k. Define the injective map

$$\mathcal{Q}uot_{X/S}(\mathcal{V}, P) \hookrightarrow \mathcal{Q}uot_{X/S}(\mathcal{O}(-a)^{\oplus k}, P)$$

by sending $[\varphi \colon \mathcal{V}_T \twoheadrightarrow F] \mapsto \varphi \circ q$. The functor on the right hand side is represented by $S \times \operatorname{Quot}_{\mathbb{P}^n_k/k}(\mathcal{O}(-a)^{\oplus k}, P)$.

For the ease of notation, let $\mathbf{Q} \coloneqq S \times \operatorname{Quot}_{\mathbb{P}^n_k/k}(\mathcal{O}(-a)^{\oplus k}, P)$. Note that $\mathcal{O}_{\mathbb{P}^n_T}(-a)^{\oplus m} \twoheadrightarrow F$ factors through $\mathcal{V}_T$ if and only if the composition $\ker(q) \to F$ is zero. Applying this to $T = \mathbf{Q}$, we obtain a closed subscheme $\iota \colon \mathbf{Q}' \subset \mathbf{Q}$ such that any element $[\varphi \colon \mathcal{V}_T \twoheadrightarrow F]$ determines a morphism $u \colon T \to \mathbf{Q}$ such that u factors through the inclusion ι. Hence $\mathbf{Q}' \simeq \operatorname{Quot}_{X/S}(\mathcal{V}, P)$ represents $\mathcal{Q}uot_{X/S}(\mathcal{V}, P)$. This is true more generally, see Exercise 13.

Step 3. *The general case.*

Since $f \colon X \to S$ is projective, there is a closed immersion $\iota \colon X \hookrightarrow \mathbb{P}^n_S$ compatible with the maps to S. We replace X by $\mathbb{P}^n_S$ by replacing $\mathcal{V}$ by $\iota_*\mathcal{V}$. □

Exercise 13. Let $\phi \colon E \to G$ be a surjective homomorphism of coherent sheaves on X. The corresponding natural transformation $\mathcal{Q}uot_{X/S}(G, P) \to \mathcal{Q}uot_{X/S}(E, P)$ induced a closed embedding. (See [FGI+05, Nitsure, Lemma 5.17(ii)])

4.5. **Applications.**

4.5.1. *Other representability.* Quot schemes are used to show representability of an array of other moduli problems. The Hilbert functor is one of the most well-known amongst them.

$$\mathcal{H}ilb_{\mathbb{P}^n_k} \colon (\mathbf{Sch}/k)^{\mathrm{op}} \to \mathbf{Sets}$$

defined by $T \mapsto \{Y \subset \mathbb{P}^n_T | Y \text{ is flat over } T\}$. Note that this functor is isomorphic to $\mathcal{Q}uot_{\mathbb{P}^n_k/k}(\mathcal{O}_{\mathbb{P}^n_k}, P)$ where P is the Hilbert polynomial of Y.

Needless to say Grassmannian is also a special case of the Quot scheme; namely when $X = S$. Here flatness is equivalent to locally freeness.

4.5.2. *Openness of various properties of sheaves.* These will be discussed once we have discussed the notion of Gieseker stability. For now we state the result

Theorem 4.15 ([HL10, Prop. 2.3.1])**.** *The following properties of coherent sheaves are open in flat families: simple, of pure dimension, semistable, geometrically stable.*

4.5.3. *HN-filtration in family.*

Theorem 4.16. *Let $f \colon X \to S$ be a projective morphism of k-schemes of finite type. Let F be an S-flat coherent sheaf on X. Then there is an open subset $U \subset S$, such that the graded pieces of the Harder–Narasimhan filtration of F are U-flat.*

The proof uses the openness of stability. The result is in fact much stronger. We will come back to it later.

5. Semistable sheaves

From now on let X be a connected projective scheme over an algebraically closed field k.

Definition 5.1. Pure coherent sheaves are the ones whose all non-trivial coherent subsheaf $F \subseteq E$ has support of the same dimension d as that of E. Support is defined by the closed set $\mathrm{Supp}(E) = \{x \in X | E_x \neq 0\}$. We say the coherent sheaf E is pure of dimension d.

The Euler characteristic of a coherent sheaf E is given by $\chi(E) := \sum_i (-)^i h^i(X,E)$ where $h^i = \dim H^i$. Recall that $P(E,t)$ with respect to the polarisation $\mathcal{O}_X(1)$ is given by the numerical polyomial that satisfies

$$P(E,m) = \chi(E \otimes \mathcal{O}_X(m)) = \sum_{k=0}^{d} \alpha_i(E) \frac{m^k}{k!}$$

for $m \gg 0$. The degree of the Hilbert polynomial is the dimension of the support of E we denote this by d[14]. Furthermore $\alpha_d(E) > 0$ whenever $E \neq 0$.

Example 5.2. Let $X \hookrightarrow \mathbb{P}^n$ be a projective variety with polarisation $\mathcal{O}_X(1)$ and degree d. Then $P(\mathcal{O}_X, m) = d\frac{m^n}{n!} + O(m^{n-1})$ where $n = \dim X$.

The reduced Hilbert polynomial of E is defined by $p(E,t) := \frac{P(E,t)}{\alpha_d}$.

Definition 5.3 ((semi-)stability)**.** A pure coherent sheaf E of dimension d is said to be (semi-)stable if for any proper subsheaf (equivalently, any quotient $E \twoheadrightarrow G$ with G pure of dimension d) $F \subset E$ one has $p(F,t)$ (resp. $\leq$) $< p(E,t)$ (equivalently, $p(E,t)$ (resp. $\leq$) $< p(G,t)$) in the lexicographic order of coefficients.

[14] not to be confused with degree

Remark 5.4. Moritz asked whether in the definition we could get away with locally free subsheaves instead of coherent subsheaves. If E is not locally free itself this simplification may be problematic. For instance, let $p \in A$ be a closed point in an abelian surface A and let $E = \mathcal{L}^{-2} \oplus \mathfrak{m}_p$ where $\mathcal{L}$ is an ample line bundle on A and $\mathfrak{m}_p$ is the defining ideal of the point p. Then $p(E,t) = \frac{(t-2)^2+t^2}{2} - 1 = t^2 - t + 1$. Therefore $\mathfrak{m}_p$ destabilises it and $\mathcal{L}^{-2}$ does not.

That said, it is a fairly common occurrence if E is locally free itself. Take, for example, curves. However, it is worth considering if this simplification could be imposed whenver E itself is locally free. Let $F \subset E$. Since F is torsion free it injects into its reflexive Hull, i.e. $F \hookrightarrow F^{\vee\vee}$ and the quotient is supported on a subset of codimension ≥ 2. Thus $p(F,t) \leq p(F^{\vee\vee}, t)$. Furthermore, $F^{\vee\vee} \subset E$. Hence we may assume F is reflexive itself. Therefore on surfaces F is locally free[15]. In higher dimension we can therefore always assume F is reflexive.

*Find an example of a locally free sheaf that is destabilised by a reflexive coherent sub-sheaf and not by any locally free subsheaf.

Exercise 14. Show that the "equivalently" part of the definition is actually an equivalent definition.

5.0.1. *Properties.*

Exercise 15. Show that if $f\colon F \to G$ is a non-trivial morphism of pure semi-stable sheaves then $p(F) \leq p(G)$. In the case of equality, additionally we have f is injective if F is stable and surjective if G is stable. Thus is $F = G$, $f = 0$ or an isomorphism.

The next result concerns automorphisms of stable sheaves. In moduli problem automorphisms of object is another obstacle in the path of obtaining a nice moduli space.

Theorem 5.5. *Let E be a stable sheaf. Then* $\mathrm{End}(E)$ *is a finite dimensional associative algebra over k. In particular, if k is algebraically closed,* $\mathrm{End}(E) \simeq k$*, in other words E is simple.*

Proof. The first part follows from noting that $\mathrm{End}(E) = \mathrm{Iso}(E)$. Let $\varphi \in \mathrm{End}(E)$ such that it is not multiplication by an element in k. Since φ is of finite dimension, i.e. $\varphi^\ell = \mathrm{id}$ for some $\ell \in \mathbb{Z}_{>0}$, the algebra $k[\varphi]$ is a finite extension of k. Hence $\mathrm{End}(E) = k$ when $k = \overline{k}$. ☐

5.1. **Stability on K3 surfaces.**

Definition 5.6. A simply connected (i.e $\pi_1(S) = 0$) compact complex manifold of dimension 2 with trivial canonical bundle ($\Rightarrow \omega_S \simeq \mathcal{O}_S$) are called K3 surfaces.

Note that only surfaces with trivial canonical bundle are K3 surfaces or abelian surfaces. The latter are not simply connected. By the Riemann–Roch for surfaces $\chi(\mathcal{L}) = \frac{(-K_S + c_1\mathcal{L})\cdot c_1(\mathcal{L})}{2} + \chi(\mathcal{O}_X) = \frac{c_1(\mathcal{L})^2}{2} + 2$. In general for any coherent sheaf E we have $\chi(E) = \int_X \mathrm{ch}(E)\cdot \mathrm{td}(X)$.

For a coherent sheaf E of rank r and Chern classes c_1, c_2 on a smooth K3 surface S with polarisation $H = c_1(\mathcal{O}_S(1))$ is given by

$$P(E,m) = = \frac{rH^2}{2}m^2 + (c_1(E)\cdot H)m + 2\,\mathrm{rk}(E) + \frac{(c_1(E)^2 - 2c_2(E))}{2}$$

The reduced Hilbert polynomial $p(E,m)$ is given by

$$p(E,m) = m^2 + \frac{2(c_1(E)\cdot H)}{rH^2}m + \alpha_0(E). \tag{7}$$

The semistability condition thus in this case translates to the following

[15]this fact will feature in an exercise later on.

Definition 5.7 (Stability on K3 surfaces)**.** A torsion free coherent sheaf E is Gieseker stable with respect to a polarisation H if for all coherent subsheaf $F \subsetneq E$

$$\frac{2(c_1(F) \cdot H)}{\mathrm{rk}(F)H^2} < \frac{2(c_1(E) \cdot H)}{\mathrm{rk}(E)H^2}$$

or,

$$\frac{2(c_1(F) \cdot H)}{\mathrm{rk}(F)H^2} = \frac{2(c_1(E) \cdot H)}{\mathrm{rk}(E)H^2} \text{ and } \alpha_0(F) < \alpha_0(E)$$

A torsion free coherent sheaf E is said to be μ-stable with respect to a polarisation H if for all torsion free coherent subsheaf $F \subsetneq E$ with $\mathrm{rk}\, F < \mathrm{rk}\, E$ we have

$$\frac{(c_1(F) \cdot H)}{\mathrm{rk}(F)} < \frac{(c_1(E) \cdot H)}{\mathrm{rk}(E)}.$$

Thus on surfaces for any torsion free we have μ-stable $\Rightarrow$ Gieseker stable, where the slope is given by $\mu(E) \coloneqq \frac{(c_1(E) \cdot H)}{\mathrm{rk}(E)}$. Similarly Gieseker semistable sheaf is μ-semistable.

5.2. **Examples.**

Example 5.8.
1. Skyscrapper sheaf F at a closed point s is always semistable. Indeed, $P(F, m) =$ const. and thus $p(F, m) = 1$.
2. Let E be a sheaf supported along a curve $C \in |H|$. Since stability of E is the same as that of $E|_C$ which is the same as μ-stability of $E|_C$.
3. The cotangent bundle Ω_S of a K3 surface is stable with respect to any polarisation and for a very general non-hyperelliptic curve $C \in |H|$, $\Omega_S|_C$ is stable.
4. $\Omega_{\mathbb{P}^n}$ is stable.
5. for an abelian variety A of dimension n, Ω_A is not stable since $\Omega_A \simeq \mathcal{O}_A^{\oplus n}$. Then $\mathcal{O}_A$ destabilises Ω_A. Indeed, $p(\mathcal{O}_A, t) = t^n = p(\Omega_A, t)$. It is however polystable, i.e. direct sum of semistable sheaves of the same reduced Hilbert polynomial.

5.3. **HN and JH filtration.** As we have seen in the case of curves stability gives rise to two very useful filtrations; namely the *Harder–Narasimhan filtration* by semi-stable quotients and *the Jordan–Hölder*-filtration for semistable sheaves by stable quotients. The latter induces a notion of S-equivalence given by isomorphism of the direct sum of the graded pieces. Our moduli functor will not distinguish sheaves that are S-equivalent to each other.

5.3.1. *Harder–Narasimhan filtration.*

Definition 5.9. Let E be a pure sheaf of dimension $d$on X. A Harder-Narasimhan (HN) filtration for length ℓ for E is an increasing filtration by pure subsheaves of dimension d

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_\ell = E$$

such that $gr_i E \coloneqq E_i/E_{i-1}$ is a semistable sheaf of dimension d and with reduced Hilbert polynomial p_i satisfying

$$p_{\max}(E) \coloneqq p_1 > \cdots p_\ell =: p_{\min}(E)$$

in the lexicographic order of coefficients.

Needless to say, if E is semistable the filtration is of length 0 and $p_{\max}(E) = p_{\min}(E)$. Two concerns that should be addressed immediately; existence and uniqueness. We do that next.

Theorem 5.10. *Every pure sheaf E has a unique Harder-Narasimhan filtration.*

Proof. For any subsheaves $F_1, F_2 \subset E$ define $F_1 \leq F_2$ if $F_1 \subset F_2$ and $p(F_1) \leq p(F_2)$. Note that any chain of $\leq$ will have a maximal element. We let E_1 to be the maximal with respect to $\leq$ that also has the minimal $\alpha_d(E_1)$ amongst all such $\leq$-maximal sheaves. Suppose $G \subset E$ be a sheaf such that $G \neq E_1$ and $p(G) > p(E_1)$. We will show that such G cannot exist. If $G \subset E_1$ we replace G by the maximal subsheaf of E_1 which has reduced Hilbert polynomial larger than $p(E_1)$. Let G' be the $\leq$-maximal element of the chain of inclusion that contains G. Then we obtain, $p(G') > p(G) > p(E_1)$. Recall that if $G' \subset E_1$, the Hilbert polynomial, since it is defined in terms of global sections it always satisfies $P(G', m) \leq P(E_1, m)$. Thus in order for $p(G') > p(E_1)$, we need $\alpha_d(G') < \alpha_d(E_1)$. This contradicts the minimiality of $\alpha_d(E_1)$. Thus $G' \not\subset E_1$. In this case, $E_1 \subset E_1 + G'$. By the maximality of E_1, we obtain $p(E_1) \geq p(E_1 + G')$. Now consider the short exact sequence

$$0 \to E_1 \cap G' \to E_1 \oplus G' \to E_1 + G' \to 0$$

For Hilbert polynomials and the leading terms one has

$$\alpha_d(E_1)p(E_1) + \alpha_d(G')p(G') = \alpha_d(E_1 \cap G')p(E_1 \cap G') + \alpha_d(E_1 + G')p(E_1 + G') \tag{8}$$

and

$$\alpha_d(E_1) + \alpha_d(G') = \alpha_d(E_1 \cap G') + \alpha_d(E_1 + G')$$

Replacing $\alpha_d(E_1)$ by $\alpha_d(E_1 \cap G') + \alpha_d(E_1 + G') - \alpha_d(G')$ in Equation 8 we obtain

$$\alpha_d(E_1\cap G')(p(G') - p(E_1\cap G')) = (\alpha_d(G') - \alpha_d(E_1\cap G'))(p(E_1) - p(G')) + \alpha_d(E_1+G')(p(E_1+G') - p(E_1)). \tag{9}$$

Since $p(E_1) > p(E_1 + G')$ it must be the case that $\alpha_d(G') - \alpha_d(E_1 \cap G') = \alpha_d(E_1 + G') - \alpha_d(E_1) > 0$. Thus, the right hand side of Equation 9 must be less than 0. Hence we obtain $p(G) < p(G') < p(E_1 \cap G')$. This is a contradiction since $E_1 \cap G' \subset E_1$ and E_1 was $\leq$-maximal for its chain. But this contradicts maximality of G in E_1 since $E_1 \cap G' \subset E_1$ and has reduce Hilbert polynomial bigger than G.

Now if $G \not\subset E_1$ to begin with, then $E_1 \subset E_1 + G$. By the similar argument as before we obtain $p(G) < p(E_1 \cap G)$. Replacing G by $E_1 \cap G$ we may assume that $G \subset E_1$. Thus we may assume $G \subset E_1$ and proceed as above.

Note that since for any $G \subset E_1$ we have $p(G) \leq p(E_1)$, we have also shown that E_1 is semistable. This constitutes the first step in constructing the HN filtration of E. In order to construct the next step we proceed inductively on E/E_1. Let

$$0 \subset G_1 \subset \cdots \subset G_{\ell-1} = E/E_1$$

be a HN filtration of E/E_1. Then $E_i = E_1 + G_i$ gives a filtration of E with quotients $E_i/E_{i-1} \simeq G_i/G_{i-1}$.

For the uniqueness, let $\{E'_i\}$ be different HN filtration of E such that $E_1 \neq E'_1$. Since $p(E_1) \geq p(E'_1)$ and E'_1 is semistable, either $E_1 \subset E'_1$ and $p(E_1) = p(E'_1)$ or $p(E_1) > p(E'_1)$ and thus $E_1 \not\subset E'_1$. The former threatens the $\leq$-maximality of E_1, since it implies $E'_1 > E_1$. For the latter, let $E_1 \subset E'_j$ for some j. Now consider the composition $E_1 \to E'_j \to E'_j/E'_{j-1}$. Since both E_1 and E'_j/E'_{j-1} are semistable and the composition gives a non-trivial homomorphism, we conclude from Exercise 15 $p(E'_1) < p(E_1) \leq p(E'_j/E'_{j-1})$. But since $\{E'_i\}$ constitutes a HN filtration it must be the case that $p(E'_1) \geq p(E'_j/E'_{j-1})$. □

A simple example of Harder–Narasimhan filtration is splitting of any vector bundle on $\mathbb{P}^1$ as direct sum of line bundles.

5.3.2. *Jordan–Hölder filtration and S-equivalence.*

Definition 5.11. Let E be a semistable sheaf of dimension d on X. A Jordan-Hlder filtration of E is a filtration by subsheaves

$$0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E$$

such that the graded pieces $gr^i(E) := E_i/E_{i-1}$ are stable with reduced Hilbert polynomial $p(E)$.

We have already seen in case of curves that Jordan–Hölder filtration is not unique. But we have

Theorem 5.12. *JH-filtration exists and the graded module $gr(E) := \oplus_i gr_i(E)$ is unique.*

Proof. The category of semistable sheaves of a fixed reduced Hilbert polynomial is abelian. First note that when $f\colon E \to F$ is an injective morphism of semistable sheaves with $p(E) = p(F)$, the quotient F/G if non-zero must be pure. This is because if $\dim(\mathrm{Supp}(F/G)) < d$, the leading terms in the Hilbert polynomial of E and F are equal, i.e. $\alpha_d(E) = \alpha_d(F)$. But since $F \subsetneq G$, $P(F) < P(G)$. Therefore the condition $p(E) = p(F)$ cannot be satisfied.

Note, let $I = \mathrm{im}(f)$. Since any quotient of I is a quotient of E and $p(I) \leq p(F) = p(E)$, we obtain that I is also semistable. Now since E is semistable $p(E) \leq p(I)$ hence it must be that $p(E) = p(I)$. A similar analysis reveals that the kernel must also be semistable with the same reduced Hilbert polynomial.

Now let $E' \subset E$ such that $p(E') = p(E)$. Note that E' must be semistable. If it is not stable, there must $E'' \subset E'$ such that $p(E') = p(E'')$. This descending chain of pure sheaves with pure quotients must terminate. Indeed, generically the rank of such a chain must decrease. Hence we have found $E_1 \subset E$ such that E_1 is stable. We do the same with the quotient E/E_1 which is semistable from the above discussion. Continuing this way we end up with a finite chain. [16]

We show the uniqueness by induction on the leading coefficient of E. Let $\{E_i\}$ and $\{E'_i\}$ be two JH-filtrations. Say j is the smallest such that $E_1 \subset E'_j \twoheadrightarrow E'_j/E'_{j-1}$. Since both E_1 and E'_j/E'_{j-1} are stable, the non-trivial composition must be an isomorphism. Therefore, $E'_j \simeq E_1 \oplus E'_{j-1}$. We have the following ses

$$0 \to E'_{j-1} \to E/E_1 \to E/E'_j \to 0$$

Hence, Along with $\{E_i/E_1\}$ we can also define $F'_i := E'_i$ for $i < j$ and $F'_i := E'_i/E'_j$ for $i > j$. Both are JH-filtration for F.

Note that E/E_1 is semistable with $p(E) = p(E)$. Since $P(E) > P(F)$ it must be the case that $\alpha_d(F) < \alpha_d(E)$. Hence, by induction we are done. □

Example 5.13. Any extension of line bundles of the same reduced Hilbert polynomial p, extends to a semistable bundle of Hilbert polynomial p.

Finally, we show that semistability is open in flat families.

Theorem 5.14. *Let $f\colon X \to S$ be a projective morphism of noetherian schemes such that F is an S-flat coherent sheaf on X. If F_{s_0} is semistable for some $s_0 \in S$, there exists an open neghbourhood U of s_0 such that for all $s \in U$, F_s is semistable.*

Sketch. The idea is to argue that the set where F_s is not semistable is closed in S. Since it uses notions that we have not introduced we only sketch the proof.

Consider such an F_s. By the quotient definition of semistability there exists a coherent sheaf G with $F_s \twoheadrightarrow G$ and $p(F_s) > p(G)$. Now consider the set $A := \{P | P = P(G) \text{ for such a } G\}$. Assume that A is finite, then the point s lies in the union of the image of the projective map $\mathrm{Quot}_{X/S}(F, P) \to S$ for $P \in A$, which is a closed subscheme of S. Indeed, the projection $[F_s \twoheadrightarrow G] \in \mathcal{Q}uot_{X/S}(F, P)(k) = \mathrm{Quot}_{X/S}(F, P)(k)$.

Showing the set A is finite requires a boundedness result due to Groethendieck (see e.g. [HL10, Lem. 1.7.9]). □

Definition 5.15 (S-equivalence)**.** Two semistable coherent sheaves E_1 and E_2 are said to be S-equivalent if $gr(E_1) \simeq gr(E_2)$.

In the moduli space that will be dealt with next S-equivalent semistable sheaves get identified.

[16] For the category theory minded people, note that all this is saying is that the category of semistable sheaves with a fixed reduced Hilbert polynomial is artinian and noetherian. This is why composition series or Jordan–Hölder series exists.

6. Moduli space of semistable sheaves

6.1. **The set-up.** Fix a polynomial $P \in \mathbb{Q}[t]$ and define

$$\mathcal{M}_P \colon (\mathbf{Sch}/k)^{\mathrm{op}} \to \mathbf{Sets}$$

by sending

$$S \mapsto \{F \in \mathrm{Coh}(X \times S) | P(F_s) = P \text{ and } F_s \text{ is semistable for all } s \in S\}/\sim .$$

Here $F \sim F'$ if for $p \colon X \times S \to S$, we have $F \simeq F' \otimes p^*L$ for some line bundle L on S.

Recall that the notion of semistability was introduced in order to get a bounded moduli (see Example **??**).

Remark 6.1. For a moment lets ignore the twisting by the line bundle and consider the functor $\mathcal{M}'_P \colon (\mathbf{Sch}/k)^{\mathrm{op}} \to \mathbf{Sets}$

$$S \mapsto \{F \in \mathrm{Coh}(X \times S) | P(F_s) = P \text{ and } F_s \text{ is semistable for all } s \in S\}/\sim .$$

Here $F \sim F'$ if $F \simeq F$.

From the point of view of corepresentability it does not change anything. By this we mean a scheme M corepresents $\mathcal{M}$ if and only if it corepresents $\mathcal{M}'$ [17] Indeed, $\mathcal{M} \simeq \mathcal{M}'/\sim$ where $F \sim F'$ if for $p \colon X \times S \to S$, we have $F \simeq F' \otimes p^*L$ for some line bundle L on S.

Max gave the following nice argument: if M corepresents $\mathcal{M}'$, for any $T \in \mathbf{Sch}$ and elements $F, F \otimes p^*L \in \mathcal{M}'(T)$ we obtain maps $\varphi, \varphi_L \colon T \to M$. We want to show that these two maps coincide. Consider a trivialisation of L given by open sets $\{T_i\}$ covering T. On each of these open sets we have $F|_{T_i} \simeq F'|_{T_i}$, which induce maps $\varphi_i \colon T_i \to M$. Since $\mathcal{M}' \to h_M$ is functorial, we obtain $\varphi_i = \varphi|_{T_i} = \varphi_L|_{T_i}$. Hence $\varphi = \varphi_L$.

Theorem 6.2. *In the presence of non-isomorphic S-equivalent points, $\mathcal{M}$ does not admit a coarse moduli space.*

Proof. We have seen this before for curves. Formally, let F be a non-trivial extension of two semistable sheaves F' and F'', then consider the trivial family $\mathcal{F} \simeq \mathcal{O}_{\mathbb{A}^1} \otimes F$ on $\mathbb{A}^1 \times X$ given by $\mathcal{F}|_{X_t} \simeq F$ for all $t \in \mathbb{A}^1$ or consider the family given by the $\mathbb{A}^1$-flat sheaf [Lan83] $\widetilde{\mathcal{F}}$ induced by the affine line passing through the class $[F] \in \mathrm{Ext}^1(F', F'')$. In the latter case we have $\widetilde{\mathcal{F}}_t \simeq F$ but $\widetilde{\mathcal{F}}_0 \simeq F' \oplus F''$.

Let M be a separated scheme that represents $\mathcal{M}_P$ as a coarse moduli. The map $\mathcal{M}_P(\mathbb{A}^1) \to \mathrm{Hom}(\mathbb{A}^1, M)$ would have sent $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ to the same point. Indeed, the map $\mathcal{M}_P \to h_M$ should be functorial with the inclusion $\mathbb{A}^1 \backslash \{0\} \hookrightarrow \mathbb{A}^1$ and the elements in the $h_M(\mathbb{A}^1 \backslash \{0\})$ sends $\mathcal{F}|_{\mathbb{A}^1 \backslash \{0\}}$ and $\widetilde{\mathcal{F}}|_{\mathbb{A}^1 \backslash \{0\}}$ to the same point $m \in M$. These two maps must extended $\mathbb{A}^1$ would still map everything to $m \in M$.

Now consider the closed immersion $\mathrm{Spec}\, k = \{0\} \hookrightarrow \mathbb{A}^1$. If M represented $\mathcal{M}$ as a coarse moduli, $\mathcal{M}(\mathrm{Spec}\, k) = \mathrm{Hom}(\mathrm{Spec}\, k, M)$. But in this case we see that F and $F' \oplus F''$ map to the same point in $h_M(\mathrm{Spec}\, k)$. Hence M cannot be a coarse moduli. ☐

The goal of the next few lectures would be show that

Theorem 6.3. *There exists a projective scheme M universally corepresenting $\mathcal{M}$.*

Here is a brief outline of how we will construct M.

Step 1. *The set of semistable sheaves with a fixed Hilbert polynomial on X is bounded, i.e. there exists a scheme of finite type S and a sheaf $\mathcal{F}$ on $X \times S$ such that for any semistable sheaf F on X, there exists $s \in S$ such that $\mathcal{F}_s \simeq F$.*

[17] we drop P when it is clear from the context.

Step 2. *Hence there exists a uniform m such that any semistable sheaf on X of Hilbert polynomial P is m-regular. Therefore, $P(m) = h^0(X, F(m))$ and $F(m)$ is globally generated, i.e. $[\mathcal{H} := \mathcal{O}_X \otimes H^0(X, F(m))(-m) \twoheadrightarrow F] \in \mathrm{Quot}_{X/k}(\mathcal{H}, P)$. The locus $R \subset \mathrm{Quot}_{X/k}(\mathcal{H}, P)$ of semistable sheaves F on X such that $h^0(X, F(m)) = P(m)$ is open. This is by Theorem 5.14.*

Step 3. *Use notation $V := k^{\oplus P(m)}$ and $\mathbf{Q} := \mathrm{Quot}_{X/k}(\mathcal{H}, P(m))$. The linear group scheme $GL(V)$ acts on R by simply changing the basis of V. Since the isomorphism class of semistable sheaf will ignore this it makes sense to consider the quotient $R(k)/GL(V)(k)$. In this step when this quotient is a "nice" scheme, this scheme universally corepresents $\mathcal{M}'$. See Lemma 6.10.*

Step 4. *Now we identify the semistable points for GIT with Gieseker semistable sheaves. The idea is to interpret both of them numerically. To this end, we need the Hilbert–Mumford criterion for GIT semistability (see* **??***) and Le-Potier's equivalent numerical interpretation of Gieseker semistability (see* **??***).*

Step 5. *Employ results from geometric invariant theory (GIT) to construct the GIT-quotient.*

This is the plan: We will not prove Step 1 but extensively use it. We have already seen Step 2. Before going into the main crux of the construction, namely Step 3 and Step 4, we start with a short discussion on general ideas from GIT and the Hilbert–Mumford criterion. Finally, while implementing Step 4 we will focus more on how to employ this criterion to interpret GIT-semistability numerically. We will state and use in Step 5, the criterion of Le-Potier, however we will not see a proof. For a complete treatment of the steps above consult [HL10, §3,4].

6.2. **Geometric Invariant Theory.** My favourite reference for this part in Mukai's book [Muk03]. The more classical references are [MFK94, New78].

GIT concerns with understanding the orbit space of group actions. However, unlike actions of discrete sets, we would like to give a continuous topology on the orbit space. Turns out it is too much to hope for. Here's an example, consider the action of the multiplicative affine group scheme $\mathbb{G}_m := \mathrm{Spec}\, \mathbb{C}[t, t^{-1}]$ on the affine plane $\mathbb{A}^2_{\mathbb{C}}$ given by

$$t \mapsto [(x, y) \mapsto (tx, t^{-1}y)].$$

The following are the orbits of this action

- The origin $(0, 0)$.
- The curves $(xy = c)$ for $c \in \mathbb{C}$.
- The x-axis$\setminus\{(0, 0)\}$.
- The y-axis$\setminus\{(0, 0)\}$.

It is not difficult to see immediately that no topological space Y with a continuous map $\mathbb{A}^2 \to Y$ can be the orbit space. Indeed, the origin lies on the limits of the last two orbits and therefore we should not be able to distinguish them in Y.

One can immediately propose the following work-around for this setback; just don't distinguish the orbits if they share points in the limits. This idea is indeed useful and we will need to refer to it a lot.

Definition 6.4 (Closure equivalent orbits)**.** Two orbits O and O' are said to be closure equivalent if $\overline{O} \cap \overline{O'} \neq \emptyset$.

With this definition, we let $Y = \mathrm{Spec}\, \mathbb{C}[xy] \simeq \mathbb{A}^1$. In general, let $X = \mathrm{Spec}\, R$ be any affine algebraic variety, and G is a linearly reductive group (e.g. $\mathbb{G}_m$ GL_n, SL_n etc.) R^G is a finitely generated and $X/G = \mathrm{Spec}\, R^G$.

Direct projectivisation of this construction does not quite work as expected. Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ by $(x, y) \mapsto (tx, ty)$. The orbits are lines passing through the origin minus the origin and the origin itself. Thus from the logic above the quotient should be $\mathrm{Spec}\, k$. This is a bit strange, since we should expect that quotienting a 2-dimensional variety by a one dimensional group scheme should give us something one dimensional. In this situation the optimal way to take quotient would be to ignore the origin and consider the closure equivalent orbits in the complement. On one hand, the orbits are already closed in $\mathbb{A}^2 \setminus \{(0,0)\}$, on the other hand $\mathbb{A}^2 \setminus \{(0,0)\}$ is not affine anymore. The induced action of $\mathbb{G}_m$, on $k[x]$ and $k[y]$ glue outside $(0,0)$ and the quotient is $\mathbb{P}^1$. This is infact a protoype example that we will keep in mind.

More generally, the action of $\mathbb{G}_m$ on $X = \operatorname{Spec} R$ always splits $R = \oplus R_i^G$ such that $v \mapsto t^i v$ for $v \in R_i^G \subset R$ under this action. Thus $\operatorname{Spec} R_0$ is the fixed part and $\operatorname{Proj} \oplus R_i^G$ is the projective quotient which is denoted by $X /\!\!/ \mathbb{G}_m$. The set $X^{\mathrm{ss}} := X \setminus Z((\oplus_{m>0} R_m^G) \cdot R)$ is said to be the set of semistable points. Recall that the zero set defined by the irrelevant ideal $(\oplus_{m>0} R_m^G) \cdot R$ is in this case precisely $\operatorname{Spec} R_0$.

Even more generally given a projective scheme X and an action of a reductive group G (e.g. GL_n) on it, the goal is to find a G-linearised ample line bundle on it. The linearisation induces an action of G on $\mathbb{P}^N$ where L embeds $X \hookrightarrow \mathbb{P}^N$. This induces further actions on $\mathbb{A}^{N+1}$ and on the subvariety given by the affine cone $\operatorname{Cone}_L(X) := \operatorname{Spec} R = \operatorname{Spec} \oplus H^0(X, L^{\otimes n})$. This affine action look very similar to our prototype example above, which roughly the so-called "ray type" action. The projective quotient is then given by $\operatorname{Proj} R^G$ and is considered to be the GIT quotient of X. To make things more precise we recall

Definition 6.5 (G-linearised line bundle)**.** Let G be an algebraic k-group scheme acting on a finite type k-scheme X. Let $\sigma\colon X \times G \to X$ be the group action. A G-linearisation of a line bundle L is an isomorphism

$$\Phi\colon \sigma^* L \to p_X^* L,$$

where $p_X\colon X \times G \to X$ is the projection, together with the following cocycle condition that the isomorphism Φ restricted to the point (x, gh) is compatible with (x, g) and (xg, h) in the sense that $\Phi_{x,g} \circ \Phi_{xg,h}\colon L_{xgh} \to L_x$ is the same as $\Phi_{x,gh}$.

Globally using the multiplication map $m\colon G \times G \to G$ and the first two coordinate projection $p_{12}\colon X \times G \times G \to X \times G$, this translates to

$$(\mathrm{id}_X \times \sigma)^* \Phi = p_{12}^* (\sigma \times \mathrm{id}_G)^* \Phi$$

Given a G-linear ample line bundle L on a projective scheme X, we let $R = \bigoplus_n H^0(X, L^{\otimes n})$. From our discussion before, the locus that we should ignore is the zero of the irrelevant ideal $(\bigoplus_{m>0} R_m^G) \cdot R \subset R$ where R_m^G consists of the sections $s \in R$ such that under the action of t, $s \mapsto t^m s$ as before. Paraphrasing, we have the following definition

Definition 6.6 (GIT (semi)stability)**.** A point $x \in X$ is called semistable with respect to a G-linearized ample line bundle L if there is an invariant global section $s \in H^0(X, L^{\otimes n})$ for some n, with $s(x) \neq 0$. We denote this locus by $X^{\mathrm{ss}}(L)$.

Furthermore, $x \in X^{\mathrm{ss}}(L)$ is said to be stable if in addition the stabilizer G_x is finite and the G-orbit of x is closed in the open set $X^{\mathrm{ss}}(L)$ of all semistable points in X.

Altogether, we have the following theorem

Theorem 6.7. *Let G be a reductive group. Given a G-linear ample line bundle L on a projective scheme X, there is a projective scheme Y and a morphism $p\colon X^{\mathrm{ss}}(L) \to Y$ such that Y is "acceptable" as a quotient in the sense that p satisfies the following Y universally corepresents the functor $X^{\mathrm{ss}}(L)/G\colon (\mathbf{Sch}/k)^{\mathrm{op}} \to \mathbf{Sets}$ by sending $S \mapsto X^{\mathrm{ss}}(L)(S)/G(S)$.*

Furthermore, p is surjective, open, affine, G-equivariant for the trivial action of G on Y, i.e. closure equivalent orbits in $X^{\mathrm{ss}}(L)$ maps to the same point in Y. Also, for every affine open $U = \operatorname{Spec} S \subset Y$ such that $p^{-1}(U) = \operatorname{Spec} R$, then p induces an isomorphism of $S \simeq R^G$, and if W and W' are two disjoint closed invariant closed subset in X, then $p(W) \cap p(W') = \emptyset$. In other words, p is a universal good quotient for the G-action on X.

Moreover, there is an open subset $Y^{\mathrm{s}} \subseteq Y$ *such that the GIT-stable points* $X^s(L) = p^{-1}(Y^{\mathrm{s}})$ *and such that* $p\colon X^{\mathrm{s}}(L) \to Y^{\mathrm{s}}$ *satisfies furthermore that the geometric fibres of* p *are the orbits of geometric points of* X. *This also works similarly under base change; said differently* $p|_{X^{\mathrm{s}}}$ *is a universal geometric quotient. Note that in this case* Y^{s} *is a true orbit space.*

Finally, there is a positive integer m *and a very ample line bundle* M *on* Y *such that* $L^{\otimes m}|_{X^{\mathrm{ss}}(L)} \simeq p^*(M)$.

We denote Y by $X /\!\!/_L G$.

The Definition 6.6 is not convenient to work with, especially for deciding whether a point is semistable or not. Besides, we need a criterion for general reductive groups, most important for us is GL_n. The set-up in the general case is reduced to $\mathbb{G}_m$ via one-parameter subgroup given by any non-trivial group homomorphism $\lambda\colon \mathbb{G}_m \to G$. The action $\sigma\colon G \times X \to X$, induces $\sigma_\lambda(t,x) \mapsto \sigma(\lambda(t),x)$ which in turn extends to give a map

$$\widetilde{\sigma_\lambda}\colon \mathbb{A}^1 \times X \to X$$

where $\widetilde{\sigma_\lambda}(0,x) = \lim_{t\to 0} \sigma_\lambda(t,x) \in X$. The Hilbert–Mumford criterion roughly speaking says that x is semistable if and only if for all such λ this limit either does not exist or if it does it does not lie in the zero set of the irrelevant ideal. In order to translate this to the G-linearised line bundle L we need a few more notation.

Lets denote this limit point by $\mathbf{0}_{x,\lambda} \in X$. Note that $\sigma(g,\mathbf{0}_\lambda) = \mathbf{0}_\lambda$ for all $g \in \lambda(\mathbb{G}_m)$. In particular for any $v \in L \otimes \kappa(\mathbf{0}_{x,\lambda})$, $v \mapsto t^r v$ under the action of $\mathbb{G}_m$. We denote this power of t by $\mu_L(x,\lambda)$.

Theorem 6.8 (Hilbert–Mumford, [HL10, Thm. 4.2.11])**.** *A point* $x \in X^{\mathrm{ss}}(L)$ *if and only if for all non-trivial one-parameter subgroups* $\lambda\colon \mathbb{G}_m \to G$, *one has*

$$\mu_L(x,\lambda) \geq 0.$$

Furthermore $x \in X^{\mathrm{s}}(L)$ *if and only if the strict inequality holds for all non-trivial* λ.

One final result from this general theory that we need is called Luna's étale slice theorem [HL10, Thm. 4.2.12]. This result shows that the local structure of a GIT quotient is again a GIT quotient. This will later on help us understand local structure of $X^{\mathrm{s}}(L)$ in our particular situation of moduli of semistable sheaves.

Theorem 6.9. *Let* G *and* X *be as before. Let* $p\colon X \to X /\!\!/ G$ *be a good quotient (e.g.* X *is affine). Let* $x \in X$ *be a point with a closed* G*-orbit and therefore reductive stabilizer* G_x *. Then there is a* G_x*-invariant locally closed subscheme* $S \subset X$ *through* x *such that the map* $S \times^{G_x} G \to X$ *is étale and induces an tale morphism* $S /\!\!/ G_x \to X /\!\!/ G$, *and the diagram*

$$\begin{tikzcd} S \times^{G_x} G \arrow[r] \arrow[d] & X \arrow[d] \\ S /\!\!/ G_x \arrow[r] & X /\!\!/ G \end{tikzcd}$$

is Cartesian.

6.3. **The construction.** We first connect the functor $R/\operatorname{GL}(V)$ with $\mathcal{M}'$. This requires no GIT. With notations from the previous section we have

Lemma 6.10. *If* $R \to M$ *is a categorical quotient for with* M *a scheme of finite type, then* M *corepresents* $\mathcal{M}'$.

Conversely, if M *corepresents* $\mathcal{M}'$ *there exists a quotient map* $R \to M$ *which is a categorical quotient.*

Proof Sketch. Let S be a Noetherian k-scheme. The idea is to produce maps $\mathcal{M}' \to R/GL_r$ and also in the other direction. The other direction is rather easy. Note that there is a map from $R \to S$. Indeed, since R was chosen to parametrise semistable quotients of $\mathcal{H}$, the universal quotient restricts to R to give an universal quotient $\mathcal{H} \otimes \mathcal{O}_R \twoheadrightarrow U$ such that for any map $S \to R$ we obtain a quotient $\mathcal{H} \twoheadrightarrow F$ on $X \times S$ with F_s semistable for all $s \in S$. On $X \times S$ the $GL(V)$-action does not change the quotient F and hence we have a map from $R/GL(V)$.

For the converse, let $\mathcal{O}_X(1)$ be the polarisation of X, and let $\mathcal{F}$ be a flat family of sheaves with fibrewise Hilbert polynomial P. By Lemma 4.13 we can choose an integer m such that $\mathcal{F}_s$ is m-regular for all $s \in S$ and $p_*\mathcal{F}(m)$ is locally free of rank $r = P(m)$.

Let $\mathbb{R}(p_*\mathcal{F}(m))$ denote the frame bundle for $p_*\mathcal{F}(m)$. In other words, let $\mathbb{R}(p_*\mathcal{F}(m))$ be the open affine in $\operatorname{Spec} S^\bullet(Hom(\mathcal{O}_S^{\oplus r}, p_*\mathcal{F}(m)))$ defined by the isomorphisms. The map $\pi\colon \mathbb{R}(p_*\mathcal{F}(m)) \to S$ with fibre over $s \in S$ given by the set of isomorphisms $\kappa(s)^{\oplus r} \simeq p_*\mathcal{F}(m) \otimes \kappa(s)$. It is known that π is a categorical quotient under the action of GL_r on $\mathbb{R}(p_*\mathcal{F}(m))$. In fact more is true, it is known that $\mathbb{R}(p_*\mathcal{F}(m))$ is a principle GL_r bundle, this means they are in particular Zariski locally trivial with fibres isomorphic to GL_r.

On $\mathbb{R}(p_*\mathcal{F}(m))$ there is a natural element of $\operatorname{Quot}_{X/k}(\mathcal{H}, P)$. Indeed, $\pi^*(p_*F(m))$ is a trivial bundle on $\mathbb{R}(p_*\mathcal{F}(m))$, in other words we have an isomorphism $\mathcal{O}_{\mathbb{R}(F)}^{\oplus r} \to \pi^*(p_*F(m))$. Since π is flat, letting $\pi_X\colon \mathbb{R}(F) \times_S X \to X$ be the projection we have by flat base change we have $\pi^*(p_*F(m)) = p_*\pi_X^*F(m)$ and hence the surjection

$$\mathcal{O}_{\mathbb{R}(F)} \otimes \mathcal{H} \twoheadrightarrow \pi_X^* F$$

on $X \times \mathbb{R}(F)$. This induces an element a map $\rho\colon \mathbb{R}(F) \to \operatorname{Quot}_{X/k}(\mathcal{H}, P)$.

Note that if $\mathcal{F}_s$ is in addition semistable, image $\rho(\mathbb{R}(F)) \subset R$. Furthermore ρ is equivariant with respect to the action of GL_r. This gives a map

$$\mathcal{M}'(S) \to R(S)/GL_r(S)$$

□

6.4. **GL_r-linearised bundle.** In order to construct R/GL_r we let $\overline{R} \subset \operatorname{Quot}_{X/k}(\mathcal{H}, P)$ to be the closure of R in the Quot scheme and apply GIT on $\overline{R}$. The GIT-semistable points with respect to the forthcoming very ample line bundle are precisely R. Thus GIT-ss recovers the usual semistable locus in the Quot scheme, in other words we will argue below $\overline{R}^{\text{GIT-}ss} = R$. As a consequence we obtain a projective scheme M that is a categorical qoutient of R under the GL_r action and hence by Lemma 6.10 corepresents $\mathcal{M}'$.

Let $f\colon X \to \operatorname{Spec} k$ be the structure map. The GL_r-linearised line bundle is constructed as follows: Recall that for any coherent sheaf $\mathcal{H}$, $\iota\colon \operatorname{Quot}(\mathcal{H}, P) \hookrightarrow \operatorname{Gr}_k(f_*\mathcal{H}(\ell), P(\ell))$ is a closed immersion for $\ell \gg 0$. Furthermore $\operatorname{Gr}_k(f_*\mathcal{H}(\ell), P(\ell)) \hookrightarrow \mathbb{P}(\bigwedge^{P(\ell)} f_*\mathcal{H}(\ell))$ is determined by the ample line bundle $\det\mathcal{U}$ where $\mathcal{U}$ is the universal quotient bundle on Gr. Let $\rho\colon \mathcal{H} \twoheadrightarrow U$ be the universal quotient on the Quot scheme. Then from the construction of ι, we obtain $\iota^* \det\mathcal{U} \simeq \det f_*U(\ell)$. One can then argue that $L_\ell \coloneqq \det f_*U(\ell)$ is GL_r-linearised (see [HL10, p. 101] for details).

6.5. **GIT-(semi)stability vs. Gieseker (semi)stability.** The following theorem of Le Potier translates Gieseker (semi)stability to a numerical criterion.

Theorem 6.11. *Let $p(t)$ be a polynomial of degree d and let $\alpha \in \mathbb{Z}_{>0}$. Then a coherent sheaf F with Hilbert polynomial $\alpha p(t)$ is (semi)stable if and only if $h^0(X, F(m)) \geq \alpha p(m)$ and any coherent subsheaf $F' \subset F$ with $0 < \alpha' < \alpha$ as the leading coefficient in its Hilbert polynomial satisfies*

$$\alpha' p(m)(\geq) > h^0(X, F'(m)) \text{ for } m \gg 0. \tag{10}$$

Furthermore if $\alpha' p(m) = h^0(X, F'(m))$ F' destabilises F, i.e. $p(F') > p(F)$.

For a proof see [HL10], but lets do a quick sanity check. For $m \gg 0$ we have $\alpha' p(m) \geq h^0(X, F'(m)) = \alpha' p'(m)$, which implies $p(m) \geq p'(m)$, this is in the right direction for semistability.

We will now convert Hilbert–Mumford criterion into a numerical criterion. For this let $[\rho\colon \mathcal{H} \twoheadrightarrow F] \in \overline{R}$ such that GL_r acts on it. Recall that $\mathcal{H} \simeq \mathcal{O}_X(-m) \otimes V$, $V = k^{\oplus r}$.

Let $\lambda\colon \mathbb{G}_m \to GL_r$ be a one-parameter subgroup. We want to find its limit. Let $V = \bigoplus_m V_m$ be the decomposition under the action of $\mathbb{G}_m$, i.e. letting $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$, $t \cdot v \mapsto t^m v$ if $v \in V_m$. We let

$$F_{\leq k} \coloneqq \rho(\bigoplus_{m \leq k} V_m).$$

Let $F_k \coloneqq F_{\leq k}/F_{\leq k-1}$ and hence $\rho_k\colon \mathcal{O}_X(-m) \otimes V_k \twoheadrightarrow F_k$ are surjections for all k. Let

$$\rho'\colon \bigoplus V_k \otimes \mathcal{O}_X(-m) \twoheadrightarrow \bigoplus F_k.$$

[HL10, Lemma 4.4.3] shows that the $\mathbb{A}^1$-flat sheaf given by the extension of $\mathcal{O}_X[t, t^{-1}]$-module

$$\mathcal{F} \coloneqq \oplus_k F_{\leq k} \otimes t^k$$

on $X \times \mathbb{A}^1$ together with a surjection $\overline{\rho}\colon \mathcal{H} \otimes k[t] \twoheadrightarrow \mathcal{F}$ satisfies $\overline{\rho}|_t = \rho \circ \lambda(t)$ and $\overline{\rho}|_0 = \rho'$. Therefore

$$\lim_{t \to 0} [\rho] \cdot \lambda(t) = \rho' \in \overline{R}$$

in the GIT-sense. Hence the fibre of the line bundle L_ℓ at this limit point acquires an action of $\mathbb{G}_m$. The weight of this action is calculated in [HL10, Lemma 4.4.4] to be $\mu_L([\rho'], \lambda) = -\sum_n nP(F_n, \ell)$. We have

Lemma 6.12. *A point $\rho \in \overline{R}$ is GIT-semistable if and only if $\mu_L([\rho], \lambda) > 0$ if and only if $\dim V \cdot P(F_{\leq n}, \ell) \geq (\dim V_{\leq n}) \cdot P(F, \ell)$ for $\ell \gg 0$ if and only if for all coherent subsheaf $0 \neq F' \subset F$ of dimension d*

$$\dim V \cdot P(F', \ell) \geq (\dim V') \cdot P(F, \ell) \tag{11}$$

where $V' \coloneqq H^0(X, F')$ and $\ell \gg 0$.

The first equivalence is Hilbert–Mumford criterion, the second is unwrapping the expression $\sum_n nP(F_n, \ell)$ and the third requires some argument which sketched in [HL10, Lemma 4.4.5 and 4.4.6].

Last, but certainly not least, it remains to prove that $\overline{R}^{\text{GIT-}ss} = R$ using the numerical criterion in Eq. (11) and Eq. (10). To this end, let's start by assuming that F is semistable. Since F is m-regular, $\dim V = \alpha_d(F) p(m)$. Then by Eq. (10) we have for any subsheaf $F' \subset F$ with $0 < \alpha_d(F') < \alpha_d(F)$ the following numerical criterion

$$\dim V \cdot \alpha_d(F') = \alpha_d(F) p(m) \cdot \alpha_d(F') \geq h^0(X, F'(m)) \cdot \alpha_d(F) = \dim V' \cdot \alpha_d(F).$$

The left hand side is the leading coefficient of the polynomial $\dim V \cdot P(F')$ and the right hand side is the leading coefficient of the polynomial $\dim V' \cdot P(F)$. Hence in the lexicographic order we get Eq. (11).

The converse is a similar argument. Furthermore, by carefully distinguishing between the inequalities and strict inequalities, we can infer that $[\rho\colon \mathcal{H} \twoheadrightarrow F] \in \overline{R}^{\text{GIT-}s} \subset R$ if and only if F is Gieseker stable.

6.6. **An example on K3 surface.** Recall from Definition 5.7 that on a K3 surface S we have the following implications

$$\mu\text{-stability} \;\Rightarrow\; \text{stability} \;\Rightarrow\; \text{semistability} \;\Rightarrow \mu\text{-semistability}.$$

Exercise 16. Show that F is μ-(semi)stable if and only if $F^\vee$ is μ-(semi)stable.

Let $S \subset \mathbb{P}^3$ be a general quartic surface. Let $H = c_1(\mathcal{O}_S(1))$. Since $\omega_S \simeq \mathcal{O}_S(4-3-1) = \mathcal{O}_S$, S is a K3 surface [18]. In this case, the rank of the Picard group $\operatorname{rk}\operatorname{Pic}(S) = 1$ and is generated by H. Consider the Hilbert polynomial $P(t) = 4t^2 - 4t + 3$. Any coherent sheaf E satisying $P(E,t) = P(E)$ must have the following invariants (see Eq. (7)): $\operatorname{rk} E = 2$, $c_1(E) = -H$ and $c_2(E) = 3$, since $H^2 = 4$ and $c_1(E)$ must be a multiple of H. In order to keep track of these invariants, we rename $M_S(P)$ as $M_S(2,-H,3)$. If the moduli space $M_S(2,-H,c_2)$ is non-empty then it is smooth of dimension $4c_2 - 10$. This fact will be revisited later on.

Furthermore, all notions of stability coincide with respect to these invariants. Indeed, let F be μ-semistable of rank 2 and $c_1(F) = -H$. If F is not stable, there exists a rank one torsion free sheaf $K \hookrightarrow F$ such that $c_1(K)\cdot H = \frac{c_1(F)\cdot H}{\operatorname{rk} F} = -2$. But this is impossible since the $c_1(K)$ must be an integral multiple of H and $H^2 = 4$.

Additionally, we need the following facts about chern classes [**?**] in what follows

- Let $0 \to F' \to F \to F'' \to 0$ be a short exact sequence. Then $c_1(F) = c_1(F') + c_1(F'')$ and $c_2(F) = c_2(F') + c_1(F)c_1(F'') + c_2(F'')$.
- For any line bundle L, $c_1(F \otimes L) = c_1(F) + \operatorname{rk}(F)c_1(L)$. $c_2(F\otimes L) = c_2(F) + (\operatorname{rk}(F) - 1)c_1(F)c_1(L) + \binom{\operatorname{rk}(F)}{2}c_1(L)^2$.
- Let $x \in S$ be a closed point and let $\mathfrak{m}_x$ be its maximal ideal. Then $c_1(\mathfrak{m}_x(1)) = H$ and $c_2(\mathfrak{m}_x) = 1$. Thus $c_2(\kappa(x)) = -1$.

Then we have the following

Proposition 6.13. *$M_S(2,-H,3) \simeq S$.*

Proof. We know that if $M \coloneqq M_S(2,-H,3) \neq \emptyset$ $\dim M = 2$.

It is enough to check bijection on closed points on S and M. Since both are normal schemes by

We first construct a map $S \to M$, i.e. given $x \in S$, we will construct a μ-stable sheaf F_x on S such that they are all fibres of an S-flat family of sheaves on $S \times S$. To this end, consider the evaluation map

$$H^0(S,\mathfrak{m}_x(1)) \otimes \mathcal{O}_S \xrightarrow{\mathrm{ev}_x} \mathfrak{m}_x(1) \tag{12}$$

and let $F_x \coloneqq \ker(\mathrm{ev}_x)$. Note that $h^0(S,\mathfrak{m}_x(1)) = 3$ since it is the kernel of the surjective evaluation map $H^0(S,\mathcal{O}_S(1)) \twoheadrightarrow \kappa(x)$. Note also that ev_x is surjective everywhere except possibly at x. At x, the map fits in the following short exact sequence

$$0 \to H^0(S,\mathfrak{m}_x^2(1)) \to H^0(S,\mathfrak{m}_x(1)) \overset{\mathrm{ev}_x(x)}{\to} \mathfrak{m}_x/\mathfrak{m}_x^2 \to 0.$$

Indeed, the vanishing of $H^1(S,\mathfrak{m}_x^2(1)) = 0$ follows from the fact that $\mathcal{O}_S(1)$ is very ample and hence the evaluation map $H^0(S,\mathcal{O}_S(1)) \twoheadrightarrow \mathcal{O}_S/\mathfrak{m}_x^2$ is also surjective.

All in all, we established that the generic rank of F_x is 2 and $c_1(F_x) = -H$. Furthermore F_x is μ-stable since any rank 1 torsion free coherent sheaf $K \hookrightarrow F_x$ must satisfy $c_1(K) = -kH$ for $k \geq 0$. Indeed otherwise there are no non-trivial map $K \hookrightarrow F_x \hookrightarrow \mathcal{O}_S^{\oplus 3}$. Furthermore $k \neq 0$ since in that case $K \simeq \mathcal{I}_V$, ideal sheaf of some 0-dimensional subscheme $V \subset S$. But then $K \hookrightarrow F$ would factor through $K \hookrightarrow \mathcal{O}_S \to F$ which is impossible since F has no global sections, which can be seen by taking global section in Eq. (12). Therefore $\mu(K) < \mu(F)$ for all $0 \neq K \subset F$. Finally we argue that F_x is locally free since $F_x^{\vee\vee}$ is also μ-stable with $c_1(F^{\vee\vee}) = -H$ and $c_2(F^{\vee\vee}) \leq 3$. The last inequality can be seen from the fact that $F_x^{\vee\vee}/F_x$ is supported only at points. Therefore $F^{\vee\vee} \in M_S(2,-H,c_2) \neq \emptyset$. But then $4c_2(F_x^{\vee\vee}) - 10 \geq 0$. Hence $c_2(F_x^{\vee\vee}) = 3$ and $F_x \simeq F_x^{\vee\vee}$.

To find a global S-flat sheaf on $S \times S$, lets denote the projections by $p, q\colon S \times S \to S$ and $\Delta \hookrightarrow S \times S$ the diagonal. Then define $\mathcal{F}$ to be the kernel of

$$p^*(p_*\mathcal{I}_\Delta \otimes q^*\mathcal{O}_S(1)) \twoheadrightarrow \mathcal{I}_\Delta \otimes q^*\mathcal{O}_S(1).$$

It is not difficult to see that is $x \neq x'$ we have $F_x \not\simeq F_{x'}$ and the map $S \to M$ is given by $x \mapsto [F_x]$.

[18] argue that S is simply connected using Lefschetz hyperplane theorem for fundamental groups.

Now for the map in the other direction, let $F \in M$ be a μ-stable sheaf. As argued above we know that F is locally free and $H^0(S,F) = 0$. Now note that $\chi(F) = P(1) = 3$. Hence $H^2(S,F) = Hom(F,\mathcal{O}_S) \geq 3$. Considering 3 linearly independent cosections $F \to \mathcal{O}_S$, we claim that there is a short exact sequence

$$0 \to F \xrightarrow{\varphi} \mathcal{O}_S^{\oplus 3} \to \mathfrak{m}_x(1) \to 0.$$

Assuming the claim note that $H^0(S,\mathcal{O}_S)^{\oplus 3} \simeq H^0(S,\mathfrak{m}_x(1))$ and hence $h^1(S,F) = 0$. Thus F is determined by φ upto isomorphism.

To see the short exact sequence above, note that if φ was not injective, $\mathrm{im}(\varphi)$ is a generic rank 1 torsion free sheaf and hence is isomorphic to $\mathcal{I}_V(a)$ for some dimension 0 subscheme $V \subset S$ and $a \in \mathbb{Z}_{\leq 0}$. The torsion freeness and the condition on the twist a comes from the fact that $\mathrm{im}(\varphi) \subset \mathcal{O}_S^{\oplus 3}$. But then φ factors through $\mathcal{I}_V \hookrightarrow \mathcal{O}_S$ contradicting linear independence of the choice of cosections.

A similar argument shows that $Q := \mathrm{Coker}(\varphi)$ is torsion free. If not consider the saturation $F' \supsetneq F$, i.e. the quotient $Q' := \mathcal{O}_S^{\oplus 3}/F'$ is torsion free. We have $\mathcal{O}_S(-H) = \det(F) \subsetneq \det(F') \subset \bigwedge^2 \mathcal{O}_S^{\oplus 3}$. Hence $\det(F') = \mathcal{O}_S$ and $c_1(Q) = 0$. Since Q is torsion free and there are no maps from $\mathcal{O}_S \to \mathcal{I}_V$ for any ideal sheaf, it must be the case that $Q \simeq \mathcal{O}_S$ and $F' \simeq \mathcal{O}_S^{\oplus 2}$, again contradicting linear indepence of choice of cosections.

Since $c_1(Q) = \mathcal{O}_S(H)$ and $c_2(Q) + c_2(F) - H^2 = 3c_2(\mathcal{O}_S) = 0$, i.e. $c_2(Q) = 1$, hence it must be the case that $Q = \mathcal{I}_x(1)$ for some point $x \in S$. □

Exercise 17. On a K3 surface defined by a smooth quartic, show that $h^0(S,\mathcal{O}_S(1)) = 4$.

Exercise 18. Let $\mathcal{F}$ be a reflexive coherent sheaf on a smooth surface S then $\mathcal{F}$ is locally free.

Hint: $\mathcal{F}$ is projective if and only if $\mathcal{F}$ is locally free (Check locally). To check projectivity use the definition of projective dimension in terms of Ext-groups.

6.7. **Local structure and smoothness criterion.** GIT construction usually results in normal singularities. However from the point of view of birational geometry and minimal model program they are not terrible; namely the moduli spaces have the so called kawamata log terminal singularities ([BGLM21, Corollary 2]). In this section we will study the local structure in order to find out when they are actually smooth, with a view toward applying the theory to the case of K3 surfaces.

For this section unless mentioned otherwise let's assume that X is a projective scheme over a fiel $k = \overline{k}$ with $\mathrm{Char}(k) = 0$.

By [Har77, Prop 5.2A] that for a noetherian local ring $(A,\mathfrak{m})$ and $A/\mathfrak{m} = k$ we have $\dim_k \dfrac{\mathfrak{m}}{\mathfrak{m}^2} \geq \dim A$. On the other hand $\mathfrak{m}/\mathfrak{m}^2$ is the tangent space of $\mathrm{Spec}\, A$ at the point $x = Z(\mathfrak{m})$.

Let $D = k[\epsilon]/\epsilon^2$ denote the ring of dual numbers. Then by [Har77, Exc. II.2.8] we have

$$\mathfrak{m}/\mathfrak{m}^2 \overset{1:1}{\leftrightarrow} \{f \in \mathrm{Hom}(A,D) | (\epsilon) = f(\mathfrak{m})\}.$$

The description on the right hand side lets us define a notion of tangent spaces for any moduli functor. Needless to say if $\mathcal{M}$ is representable by a scheme M, for any $p \in M$, we have

$$T_pM = \{f \in \mathrm{Hom}(\mathrm{Spec}\, D, M) | f(0) = p\}.$$

Let X be a projective scheme over an algebraically closed field $k = \overline{k}$. Let $[F] \in \mathcal{M}_P(\mathrm{Spec}\, k)$, i.e. $F \in \mathrm{Coh}(X)$ is semistable with Hilbert polynomial P. If M represents $\mathcal{M}$ we have

$$T_{[F]}M_P := \{F' \in \mathrm{Coh}(X \times \mathrm{Spec}\, D) | F' \otimes_D k \simeq F\}.$$

Lemma 6.14. *Let F be a coherent sheaf on X .*

$$\{F' \in \mathrm{Coh}(X \times \mathrm{Spec}\, D) | F' \otimes_D k \simeq F\} \simeq \mathrm{Ext}^1_X(F,F)$$

Proof. Consider the exact sequence of $\mathcal{O}_{X\times \operatorname{Spec} D}$-algebras that is split as a sequence of $\mathcal{O}_X$-modules

$$0 \to k\langle\epsilon\rangle \to D \to k \to 0.$$

Let $F' \in \operatorname{Coh}(X \times \operatorname{Spec} D)$ be considered as a flat D-module, then tensoring the above short exact sequence with F' we obtain

$$0 \to F \to F' \to F \to 0$$

Hence F' is an extension of F by F.

Conversely given such an extension

$$0 \to F \xrightarrow{f} F' \xrightarrow{g} F \to 0$$

we put on F' the structure of a D-module by defining $\epsilon = f\circ g$. Note that the map $\widetilde{g}\colon F'\otimes_D D/\epsilon \to F$ given by sending $m\otimes \overline{a} \mapsto \overline{a}g(m)$ is an isomorphism where $\overline{a} \in k \simeq D/\epsilon$. Indeed, $\widetilde{g}$ is surjective since g is surjective. Furthermore if $g(Fm_1') = g(m_2')$ then there exists an $m \in F$ such that $f(m) = m_1' - m_2'$. On the other hand since g is surjective there exists $m' \in F'$ such that $g(m') = m$. Hence, for such m_1' and m_2', we obtain

$$(m_1' - m_2') \otimes \overline{a} = f \circ g(m') \otimes \overline{a} = \epsilon(m') \otimes \overline{a} = m' \otimes \epsilon\overline{a} = 0$$

□

Remark 6.15. Note that by Lemma 2.4, a D-module F' is D-flat if $F \simeq F'\otimes_D k$ is k-flat (which is obvious here) and if $\epsilon F' \simeq \epsilon \otimes F'$. To see the latter we use Nakayama lemma on $F'/\epsilon F' \simeq F$ to conclude that F' as a D-module is generated by same number of elements as F as k-vector space; lets call this generators of F' $\{f_i\}_{i\leq i\leq r}$. Thus as k-vector space F' is generated by $(f_1, \cdots, f_r, \epsilon f_1, \cdots, \epsilon f_r)$. Hence the map $(\epsilon)\otimes F' \to (\epsilon)F'$ given by $\epsilon \otimes f_i \to \epsilon f_i$ defined on the generators shows that the map is injective. Hence the lemma above implies $T_{[F]}\mathcal{M} \simeq \operatorname{Ext}^1_X(F,F)$.

6.7.1. *Small deformation.* In order to understand local structure of the moduli beyond the tangent space we need to allow further deformation. For instance what happens if we want to extend $F_\epsilon \in \mathcal{M}(\operatorname{Spec} D)$ to $\mathcal{M}(\operatorname{Spec} k[\epsilon]/(\epsilon^n))$ for arbitary n? Or more generally to $\mathcal{M}(\operatorname{Spec} A)$ for local Artin k-algebras $(A, \mathfrak{m})$?

Recall that Artin rings are by definition rings in which descending chain conditions on ideals are satisfied. One can show that a ring is artinian if and only if it noetherian of dimension 0. Hence on a local Artin algebra $(A, \mathfrak{m})$ we have $\mathfrak{m}^N = 0$ for some $N \gg 0$. (Exercise!). Finite dimensional local algebras over k are examples of local Artin algebras.

Denote by $\mathbf{Art}_k$ the category of local Artinian algebras with morphisms given by local homomorphism of rings. Denote by

$$\mathcal{D}_F\colon \mathbf{Art}_k \to \mathbf{Sets}$$

the deformation functor given by

$$\mathcal{D}_F(A) \coloneqq \{F' \in \operatorname{Coh}(X \times \operatorname{Spec} A) | F' \text{ is } A \text{ flat and } F' \otimes_A A/\mathfrak{m} \to F \text{ is an isomorphism}\}/\sim.$$

The equivalence $F_1' \sim F_2'$ if there exists an isomorphism $\varphi\colon F_1' \simeq F_2'$ such that it commutes with the respective restrictions to k. We will show that this functor is representable by $\widehat{\mathcal{O}}_{M,[F]}$. In order to establish this we need some more information about the functor $\mathcal{D}_F$ (see Theorem 6.21). For instance, what is the image of $\mathcal{D}_F(\sigma)$ for a local homomorphism of Artinian k-algebras $\sigma\colon B \to A$. Since $\sigma(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$ such morphism always factors through $B/\mathfrak{m}_B^n$ for some n and therefore we assume that $\ker(\sigma)\cdot \mathfrak{m}_B = 0$. Such morphisms are called *small extensions*. We have the following

Theorem 6.16. *Let $0 \to \mathfrak{a} \to B \xrightarrow{\sigma} A \to 0$ be small extension. Let F be a stable sheaf. We have*

(1) The non-trivial fibres of $\mathcal{D}_F(\sigma)$ are non-canonically isomorphic to the vector space $\operatorname{Ext}^1(F,F)\otimes \mathfrak{a}$.

(2) $\operatorname{im}(\mathcal{D}_F(\sigma)) = \mathfrak{o}_{\sigma,F}^{-1}(0)$ where $\mathfrak{o}_{\sigma,F}\colon \mathcal{D}_F(A) \to \operatorname{Ext}^2(F,F)\otimes I$ is the map that detects obstruction to lifting $F_A \in \mathcal{D}_F(A)$.

Proof Sketch. For since $\mathfrak{a}^n = 0$, $\mathfrak{a} \simeq \mathfrak{a}/\mathfrak{a}^n$ is also an $A \simeq B/\mathfrak{a}$-module. Moreover since $\mathfrak{a}\mathfrak{m} = 0$, $\mathfrak{a}$ can also be seen as a finite dimensional k-vector space. Thus $F_B \otimes_B \mathfrak{a} \simeq F_A \otimes_A \mathfrak{a} \simeq F_A \otimes_A \mathfrak{a}/\mathfrak{a}^n \simeq F \otimes_k \mathfrak{a}$.

The fibres of $\mathcal{D}_F(A)$, determines the number of ways one can extend a deformation $F_A \in \mathcal{D}_F(A)$ of F to $F_B \in \mathcal{D}_F(B)$. The idea is similar to Lemma 6.14, where it is easier to see that extension F_B produces an element in $\mathrm{Ext}^1(F_A, F_B \otimes_B \mathfrak{a})$. Since $\mathfrak{a}$ is a finite dimensional k-vector space, we have $\mathrm{Ext}^1(F_A, F_B \otimes_B \mathfrak{a}) \simeq \mathrm{Ext}^1(F_A, F_A \otimes_A \mathfrak{a}) \simeq \mathrm{Ext}^1(F, F\otimes)_k \mathfrak{a}$.

For (2) note that the image determines the deformation F_A of F that admits extension to B. We will use the following exact sequence that arises out of a spectral sequence [19]

$$\mathrm{Ext}^1_{X_A}(F_A, F_A \otimes \mathfrak{a}) \to \mathrm{Ext}^1_{X_B}(F_A, F_A \otimes \mathfrak{a}) \to \mathrm{Hom}_{X_A}(F_A \otimes \mathfrak{a}, F_A \otimes \mathfrak{a}) \xrightarrow{\eta} \mathrm{Ext}^2_{X_A}(F_A, F_A \otimes \mathfrak{a}).$$

Thus we want to know when is an extension F_B of F_A and $F_A \otimes \mathfrak{a}$ lies in $\mathrm{Ext}^1_{X_A}(F_A, F_A \otimes \mathfrak{a})$. This is equivalent to saying F_B maps to $\mathrm{id} \in \mathrm{Hom}_{X_A}(F_A \otimes \mathfrak{a}, F_A \otimes \mathfrak{a})$. Define $\mathfrak{o}_{\sigma,F}(F_A) = \eta(\mathrm{id})$. Hence, F_B restricts F_A if and only if $\mathfrak{o}_{\sigma,F}(F_A) = 0$. ☐

Coming back to our moduli problem, let $M := M^{\mathrm{s}}_P(X)$ represent the moduli functor $\mathcal{M}^{\mathrm{s}}_P(X)$. By Theorem 6.21 and the lemma above we have the following dimension estimate.

Theorem 6.17 (Dimension estimate)**.** *For any stable coherent sheaf F*

$$\dim \mathrm{Ext}^1(F, F) \geq \dim_{[F]} M \geq \dim \mathrm{Ext}^1(F, F) - \dim \mathrm{Ext}^2(F, F)$$

Proof. By the structure of local k-algebra $R := \widehat{\mathcal{O}}_{M,[F]} \simeq \frac{k[[t_1,\cdots,t_d]]}{J}$ for some ideal J and $d = \dim \mathfrak{m}/\mathfrak{m}^2$. By the discussion above we know $T_{[F]}M \simeq \mathfrak{m}/\mathfrak{m}^2 \simeq \mathrm{Ext}^1(F, F)$. Thus we obtain the first inequality.

The second follows from showing J is generated by at least $r := \dim \mathrm{Ext}^2(F, F)$. Let $\mathfrak{n} := (t_1, \cdots, t_d)$. The idea is to estimate $\dim J/\mathfrak{n}J$ and use Nakayama lemma. To do so, note that since R is Noetherian, Artin–Rees lemma applies and implies that $J \cap \mathfrak{n}^n \subset J\mathfrak{n}$ for some n. Let $\mathfrak{a} := J + \mathfrak{n}^n / \mathfrak{n}J + \mathfrak{n}^n \simeq J/\mathfrak{n} \cap J \simeq J/\mathfrak{n}J$, $A := R/\mathfrak{m}_R^n \simeq k[[t_1, \cdots .t_d]]/J + \mathfrak{n}^n$ and $B \simeq k[[t_1, \cdots .t_d]]/\mathfrak{n}J + \mathfrak{n}^n$. Thus we have a small extension

$$0 \to J/\mathfrak{n} \cap J \to B \xrightarrow{\sigma} A \to 0.$$

Since $\mathcal{D}_F$ is pro-representable by R, the surjection $R \to A$ defines as element $F_A \in \mathcal{D}_F(A)$. By Theorem 6.16 the obstruction to lifting this to an element in $\mathcal{D}_F(B)$

$$\mathfrak{o}_{\sigma,F}(F_A) \in \mathrm{Ext}^2(F, F) \otimes J/\mathfrak{n}J$$

We write $\mathfrak{o}_{\sigma,F}(F_A) = \sum_{\alpha=1}^{r} \psi_\alpha \otimes \overline{f}_\alpha \in \mathrm{Ext}^2(F, F) \otimes J/\mathfrak{n}J$, where ψ_α's are a basis of $\mathrm{Ext}^2(F, F)$. Let $f_\alpha \in J$ be lifts of $\overline{f}_\alpha$. Since $\mathfrak{o}_{\sigma,F}(F_A) = 0 \in \mathrm{Ext}^2(F, F) \otimes \mathfrak{a}/(f_1, \cdots, f_r)$, the deformation F_A lifts to $F_{A'} \in \mathcal{D}_F(A')$ for $A' \simeq B/(f_1, \cdots, f_r)$. By prorepresentability the map $R \twoheadrightarrow A$ lifts to $q' \colon R \to A'$. Thus we get the diagram

$$\begin{tikzcd} R \arrow[r, "q"] \arrow[d, "q'"] & A \arrow[d] \\ A' \arrow[r] & A \end{tikzcd}$$

Hence $J \subset \mathfrak{n}J + (f_1, \cdots f_r) + \mathfrak{n}^n \subset J + \mathfrak{n}^n$. Quotienting this by $\mathfrak{n}^n$, we obtain

$$(\mathfrak{n}J + (f_1, \cdots f_r) + \mathfrak{n}^n)/\mathfrak{n}^n \simeq J/J \cap \mathfrak{n}^n.$$

Since $J \cap \mathfrak{n}^n \subset \mathfrak{n}J$, we have a surjection $(\mathfrak{n}J + (f_1, \cdots f_r) + \mathfrak{n}^n)/\mathfrak{n}^n \twoheadrightarrow J/\mathfrak{n}J$. Hence we obtain $\mathfrak{n}J + (f_1, \cdots, f_r) = J$. Thus J is generated by r elements.

[19] The argument is taken from Benjamin Schmidt's notes [**?**].

□

Exercise 19 (Nakayama's lemma). Let $(A, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated module on it. Show that $\mathfrak{m}M = M$ then $M = 0$. More generally show that if for any submodule $N \subset M$ we have $\mathfrak{m}N + N = M$ then $N \simeq M$. Use this to show that M admits a minimal generating set of length r if and only if $M/\mathfrak{m}M$ is an r-dimensional vector space over $A/\mathfrak{m}$.

Remark 6.18 (Tangent space and smoothness criterion). We do not in fact require M to represent the moduli functor. Let M be the moduli space of the functor $\mathcal{M}'_P$, then one can argue via the construction using Quot scheme that there exists a natural isomorphism $T_tM \simeq \operatorname{Ext}^1(F,F)$ for any point $t \in M$ corresponding to a stable sheaf $F \in M(k)$. To this end, let P be the Hilbert polynomial corresponding to the Mukai vector v and recall that M was is a good geometric quotient of a subscheme of the Quot scheme $\operatorname{Quot} := \operatorname{Quot}_S(V \otimes \mathcal{O}_S(-m_0), P)$. Let $[q\colon V \otimes \mathcal{O}_S(-m_0) \to$ $\to E] \in \operatorname{Quot}(k)$ be a preimage of t under this quotient map. Then similar to the situation of the Grassmannian we have $T_q \operatorname{Quot} = \operatorname{Hom}(\ker(q), E)$. Consider the long exact sequence of Ext groups associated to $\operatorname{Hom}(\_, E)$ applied to q

$$0 \to \operatorname{End}(E) \to \operatorname{Hom}(V \otimes \mathcal{O}_S(-m_0), E) \xrightarrow{\alpha} \operatorname{Hom}(K, E) \to \operatorname{Ext}^1(E,E) \to 0.$$

The surjectivity on the right hand side can been seen as follows: by the choice of m_0, E was m_0-regular and hence we had $\operatorname{Ext}^1(V \otimes \mathcal{O}_S(-m_0), E) \simeq H^1(S, E(m_0)) = 0$. Note that $\operatorname{Hom}(V \otimes \mathcal{O}_S(-m_0), E) \simeq H^0(S, E(m_0))^{\oplus \dim V} \simeq M_{n\times n}$. Since $\operatorname{End}(E) \simeq \mathbb{C}$, the $\operatorname{Stab}([q]) = \mathrm{id} \subset PGL(V)$. Hence, the image of α identifies the tangent space of the $PGL(V)$ orbit of $[q]$. Hence $T_tM \simeq \operatorname{Ext}^1(E,E)$. (see [HL10, Prop. 10.11])

Therefore the dimension estimate in Theorem 6.17 can be interpreted as a smoothness criterion; namely whenever $\dim \operatorname{Ext}^2(F,F) = 0$, we have $T_{[F]}M = \dim \operatorname{Ext}^1(F,F) = \dim \widehat{\mathcal{O}}_{M,[F]}$ and hence M is smooth at t.

An immediate application of the theorem is the following:

Example 6.19 (Curves). On a curve of genus g, since $\operatorname{Ext}^2 = 0$ for dimension reasons, by Theorem 6.17 we obtain a smooth projective moduli $M_{r,d}(C)$ of dimension $\dim \operatorname{Ext}^1(E,E) = -\chi(E,E) + \dim \operatorname{Hom}(E,E)$ for a simple vector bundle E. By Riemann–Roch, we can compute that the dimension is $1 + \deg(E^\vee \otimes E) + r^2(g-1) = r^2(g-1) + 1$.

Remark 6.20. When X is a smooth projective scheme, there is a finer dimension estimate. Consider the det: $M \to \operatorname{Pic}_X$ by sending $\mathcal{E} \mapsto \det \mathcal{E}$. Then there is a natural map of functors $\mathcal{D}_F \to \mathcal{D}_{\det F}$. Furthermore for a locally free stable sheaf F there are trace maps

$$\begin{aligned} \operatorname{tr}_1\colon\ \operatorname{Ext}^1(F,F) &\xrightarrow{T_{[F]}\det} \operatorname{Ext}^1(\det F, \det F) \simeq H^1(X, \mathcal{O}_X) \\ \text{and similarly }\ \operatorname{tr}_2\colon\ \operatorname{Ext}^2(F,F) &\to \operatorname{Ext}^2(\det F, \det F) \simeq H^2(X, \mathcal{O}_X). \end{aligned} \tag{13}$$

The first map is nothing but the derivative of det and the second map sends $\mathfrak{o}_{\sigma,F} \mapsto \mathfrak{o}_{\sigma,\det F}$ for any small extension σ (see [HL10, Theorem 4.5.3]). This gives rise to a similar estimate for the fibre $M(\mathcal{L})$ of a point $[\mathcal{L}] \in \operatorname{Pic}_X$; namely

$$\dim \operatorname{Ext}^1_0(F,F) \geq \dim M(\mathcal{L}) \geq \dim \operatorname{Ext}^1_0(F,F) - \dim \operatorname{Ext}^2_0(F,F)$$

where $\operatorname{Ext}^i_0 := \ker \operatorname{tr}_i$.

Note that for K3 surfaces this improves the original dimension estimate, since in this case Pic_X is a discrete set of points but $h^2(X, \mathcal{O}_X) = 1$. Hence whenever $\operatorname{Ext}^2(F,F)$ is one dimensional we have a smooth moduli just like in the curve case. We come back to this in Section 7.

We are now left to show the pro-representability

Theorem 6.21. *Let $M := M^{\mathrm{s}}_P(X)$ corepresent the moduli functor $\mathcal{M}^{\mathrm{s}}_P(X)$. Then, for any stable coherent sheaf F, we have an isomorphism of functors $\mathcal{D}_F \simeq \operatorname{Hom}(\widehat{\mathcal{O}}_{M,[F]}, \_)$.*

Proof. One direction is easier; namely we have for any Artin ring A, a map $\mathcal{D}_F(A) \to \mathrm{Hom}(\widehat{\mathcal{O}}_{M,[F]}, A)$. Indeed, a family $F_A \in \mathcal{D}_F(A)$, defines a morphism $f\colon \operatorname{Spec} A \to M$ which induces a map of the structure sheaves $\mathcal{O}_{M,[F]} \to f_*\mathcal{O}_{\operatorname{Spec} A} \simeq A$. Taking completion we obtain a map $\widehat{\mathcal{O}}_{M,[F]} \to \widehat{A}$. Since A is an Artin ring $A \simeq \widehat{A}$.

To construct an inverse, given a map $\widehat{\mathcal{O}}_{M,[F]} \to A$ any artin ring A, we need to produce a family $F_A \in \mathrm{Coh}(X \times \operatorname{Spec} A)$ such that $F_A \otimes k \simeq F$. If there was a universal family $\mathcal{F} \in \mathrm{Coh}(M \times X)$ such that $\mathcal{F}|_{[F]} \simeq F$ then we would simply pull back this universal family to $\operatorname{Spec} A$.

In general, the idea is to us the GIT quotient construction since on the quot scheme there is a universal quotient. Recall that $p\colon R^{\mathrm{s}} \to R^{\mathrm{s}}/\operatorname{PGL}_r \simeq M^{\mathrm{s}}$ arising out of the GIT is a geometric quotient (see Theorem 6.7) . Hence the fibre over $[F] \in M(k)$ is a closed orbit. Consider a point $[q\colon \mathcal{H} \to F] \in p^{-1}([F])$ in this orbit. By Luna's Etale Slice Theorem Theorem 6.9 there is a subscheme $S \subset R^{\mathrm{s}}$ passing through the $[q] \in R^{\mathrm{s}}$ such that the projection $S/\!/\operatorname{Stab}_{[q]} \to M^{\mathrm{s}}$ is tale near $[F]$. But since F is stable, $\operatorname{Stab}_{[q]} = \mathrm{id} \in \operatorname{PGL}_r$ (see [HL10, Lemma 4.3.2]). Hence $S \to M^{\mathrm{s}}$ is étale, or equivalently[20] $\widehat{\mathcal{O}}_{S,[q]} \simeq \widehat{\mathcal{O}}_{M,[F]}$. Let F denote the restriction of the universal quotient on $S \times X \subset \mathrm{Quot}_{X/k}(\mathcal{H}, P) \times X$. So for any local Artin k-algebra $(A, \mathfrak{m})$, admitting a map $\operatorname{Spec} A \to S$ such that $Z(\mathfrak{m}) \mapsto [q]$, will induce a quotient $[\mathcal{H} \to F_A] \in \mathrm{Quot}_{X/k}(\mathcal{H}, P)(A)$ and hence element in $\mathcal{D}_F(A)$ via pullback of the universal family. This induces a map $\mathcal{O}_{S,[q]} \to \mathcal{D}_F$. □

7. Moduli of sheaves on K3 surfaces

Let S denote a complex projective K3 surface, with an ample line bundle $\mathcal{O}_S(1)$ on it. Let $H \coloneqq c_1(\mathcal{O}_S(1))$ be its first chern class. Recall that $\omega_S \simeq \mathcal{O}_S$ and it admits a unique non-degenerate 2-form $\mathbb{C}\langle\sigma\rangle = H^0(S, \Omega_S^2)$ called the non-degenerate *symplectic* 2-form. Hence given two coherent sheaves E, F on S the Serre duality takes the form

$$\mathrm{Hom}(E, F) \simeq \mathrm{Ext}^2(E, F)^\star.$$

Furthermore, since $\mathrm{Ext}^1(E, E) \simeq \mathrm{Ext}^1(E, E)^\star$, we obtain a non-degenerate pairing

$$\mathrm{Ext}^1(E, E) \times \mathrm{Ext}^1(E, E) \to k.$$

We will see later on that this is the pointwise description of the symplectic structure on the moduli of sheaves on K3 surfaces.

Since for K3 surfaces $\mathrm{Ext}^2(E, E) \simeq \mathrm{End}(E)^\vee$, if E is stable we have $\dim \mathrm{Ext}^2(E, E) = 1$. In fact there exists a trace map $\mathrm{tr}\colon \mathrm{Ext}^2(E, E) \to H^2(S, \mathcal{O}_S)$ is Serre dual to the natural inclusion $H^0(S, \mathcal{O}_S) \to \mathrm{End}(E)$ given by sending $\lambda \mapsto \lambda \cdot I$. Hence is non-trivial. Thus $M_v(S)^{\mathrm{s}}$ is smooth of dimension $\dim \mathrm{Ext}^1(E, E)$.

7.1. **Mukai vectors.**

We have seen that on curves, the only numerical invariants associated to a locally free sheaf are its rank and degree. On surfaces we have rank, and two chern classes c_1 and c_2. Indeed, by the Hirzebruch–Riemann–Roch formula on K3 surfaces we have for any coherent sheaf E on S $\chi(E) \coloneqq \int_S \mathrm{ch}(E)\,\mathrm{td}(X)$. Together with the Chern character formulæ $\mathrm{ch}(E) = \mathrm{rk}(E) + c_1(E) + \mathrm{ch}_2(E)$ and Todd class $\mathrm{td}(X) = 1 + \frac{1}{2}c_1(\mathcal{T}_S) + \frac{1}{12}(c_1(\mathcal{T}_S)^2 + c_2(\mathcal{T}_S)) = 1 + 0 + 2$, we obtain

$$\chi(E) = \mathrm{ch}_2(E) + 2\,\mathrm{rk}(E).$$

Exercise 20. Using HRR show that for S a K3 surface $c_2(S) \coloneqq c_2(\mathcal{T}_S) = 24$.

Thus we replace the Hilbert polynomial by the numerical invariant called the *Mukai vector* associated to E

$$v(E) \coloneqq \mathrm{ch}(E)\sqrt{\mathrm{td}(X)} = (\mathrm{rk}(E), c_1(E), \chi(E) - \mathrm{rk}(E)).$$

[20] here we use that $\mathrm{Char}(k) = 0$

Another useful formulation of the last term is that $\chi(E) - \mathrm{rk}(E)) = \mathrm{ch}_2 + \mathrm{rk}(E) = \frac{1}{2}c_1(E)^2 - c_2(E) + \mathrm{rk}(E)$. For example $v(k(x)) = (0,0,1)$, $v(L) = (1, c_1(L), c_1(L)^2/2+1)$ etc.

Exercise 21. Compute $v_0 := v(\mathcal{O}_S \oplus \mathcal{O}_S)$? Show that $M^{\mathrm{s}}_{v_0}(S) = \emptyset$. Show that $\mathcal{O}_S \oplus \mathcal{O}_S$ is semistable.

We can consider the Mukai vector as an element $v = (v_0, v_2, v_4) \in H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$. Given $F \in \mathrm{Coh}(S)$, $v_0(F) = \mathrm{rk}(F)$, $v_2(F) = c_1(F)$ and $v_4(F) = \frac{1}{2}c_1(E)^2 - c_2(E)$.

Definition 7.1 (The Mukai pairing)**.** There is a non-degenerate pairing on $H^*(S,\mathbb{Z})$ given by

$$\langle v, v' \rangle = v_2 v_2' - v_0 v_4' - v_4' v_0$$

where the product on the right hand side is given by the usual topological cup product on $H^*(S,\mathbb{Z})$.

We also have $\chi(E,F) = -\langle v(E), v(F) \rangle$. Hence in a T-flat family $\mathcal{E}$ on $S \times T$ over a connected scheme T, $\chi(\mathcal{E}_t, \mathcal{E}_t)$ and hence $v(\mathcal{E}_t)$ remains invariant. Conversely $\chi(E(m)) = -\langle v(E), v(\mathcal{O}_S(-m)) \rangle$ and hence $v(E)$ determines the Hilbert polynomial of P. Hence $M_v(S)$ is a projective scheme. Moreover, the stable part is either $M^{\mathrm{s}}_v(S) = \emptyset$ or it is a smooth quasi-projective variety of dimension $\dim M_v(S)^{\mathrm{s}} = 2 + \langle v, v \rangle$.

Remark 7.2 (Dimension is even). Note that the complex dimension of the moduli of stable sheaves on a K3 surface is always even. Let $v(E) = (r, c, s)$. Since $c = c_1(E) \in H^1(S, \Omega^1_S) \cap H^2(S,\mathbb{Z}) =: H^{1,1}(S,\mathbb{Z})$, it is enough to observe that the intersection pairing on $H^{1,1}(S,\mathbb{Z})$ is even. One can in fact identify $\mathrm{Pic}(S) \simeq H^{1,1}(S,\mathbb{Z})$ via the first chern class map[21]. Now for any $\mathcal{L} \in \mathrm{Pic}(S)$, $\chi(\mathcal{L}) = \frac{1}{2}c_1(\mathcal{L})^2 + 2$. Hence $c_1(L)^2$ and hence c^2 is even. This implies $v^2 = c^2 - 2rc$ and hence $\dim M^{\mathrm{s}}_v(S) = v^2 + 2$ are all even numbers.

7.1.1. *The stable locus.* In many ways, for a K3 surface S the stable part $M^{\mathrm{s}} := M^{\mathrm{s}}_v(S)$ is less difficult to deal with. For one, we know it is smooth. Secondly it always admits a *quasi-universal family* (see [HL10, §4.6]). This means there is a coherent sheaf $\mathcal{E} \in \mathrm{Coh}(S \times M^{\mathrm{s}})$ such that for any T-flat stable family $\mathcal{F} \in \mathrm{Coh}(S \times T)$ with Mukai vector v, there is a map $p\colon T \to M^{\mathrm{s}}$ such that $p_S^*\mathcal{E} \simeq \mathcal{F} \otimes p^*V$ where V is a vector bundle on T and $p_S\colon S \times T \to S \times M^s$ is the pullback map.

The family is said to be *universal* if V is a line bundle. In this case M^{s} is a fine moduli. It is therefore useful to know when it is actually projective or equivalently for which choice of Mukai vector there are no strictly semistable sheaves.

Theorem 7.3. *Let $v = (r, c, s) \in H^\bullet(S,\mathbb{Z})$ be such that $gcd(r, c \cdot H, s) = 1$ and $s = \chi - r$, then every semistable sheaf with Mukai vector v is stable, i.e. $M^H_v(S)^{\mathrm{s}}$ is projective.*

Proof. Let E be strictly semistable, i.e. there exists a non-trivial torsion free coherent subsheaf $F \subsetneq E$ such that

$$p(F) = p(E).$$

Let $v(F) = (r', c', s')$ and recall that in this case $r' < r$. This means there exists $m \gg 0$ satisfying

$$r\chi(F(m)) = r'\chi(E(m)). \tag{14}$$

Since $\chi(E(m)) = -\langle v, v(-\mathcal{O}_S(m)) \rangle$, the equation above translates to

$$(0, rc' - r'c, rs' - r's)(1, -mH, m^2H^2/2+1) = -(rc' - r'c)mH - (rs' - r's) = 0.$$

for all $m \gg 0$. Thus $r(c'H) = r'(cH)$ and $rs' = r's$.

Since $gcd(r, cH, s) = 1$, we can write $a_0 r + a_2(cH) + a_4 s = 1$. Multiplying this by r' and using the relations deduced above we obtain

$$r' = a_0 r r' + a_2(c'H)r + a_4 s' r.$$

But then $r | r'$ which is absurd since $r' < r$. □

[21]This is Lefschetz (1,1) theorem, the first instance of the Hodge conjecture.

Remark 7.4 (Universal family). Under the assumptions of the theorem above, it is known that there exists a universal family on this projective moduli of stable sheaves. In other words, there is an M-flat family of sheaves $\mathcal{U}$ on $M \times S$ such that for any T-flat family $\mathcal{E}$ on $T \times X$ of stable sheaves with Mukai vector v, there exists a map $\gamma\colon T \to M$ and a line bundle $\mathcal{L}$ on T such that $\gamma_S^*\mathcal{U} \simeq \mathcal{E} \otimes p^*\mathcal{L}$ where $p\colon T \times S \to T$ is the projection.

Hence when v is a primitive Mukai vector M is a fine moduli. See [HL10, §4.6] for the construction of the universal family.

Non-trivial results of Kushelov ([Kus90] for $v^2 = -2$) and Yoshioka [Yos01] also guarantee that whenever $\dim M \geq 0$, i.e. $v^2 \geq -2$, M is non-empty.

7.2. **The case of** $\dim M_v^{\mathrm{s}} = 0$**.**

This happens when $v^2 = -2$. We have the following

Theorem 7.5. *If the expected dimension of M^{s} is 0, when non-empty it is isomorphic to a single reduced point corresponding to a locally free stable sheaf with Mukai vector v.*

Proof. Let $[E] \in M^{\mathrm{s}}(k)$. If there exists $[F] \neq [E] \in M$ then we have $\hom(F, E) = 0 = \hom(E, F)$. Indeed this follows from the fact that $E \not\simeq F$, E is stable and $v(F) = v(E)$, which means $p(E) = p(F)$ and $\mathrm{rk}(E) = \mathrm{rk}(F)$. These Hom sets cannot be both empty since $\chi(E, E) = -v^2 = 2$. Therefore, $F \simeq E$ is the only point of M.

To see why E is locally free, we consider its double dual $E^{\vee\vee}$. On smooth surfaces reflexive sheaves are locally free. We will show that if E is not reflexive, then we can deform E in a non-trivial family contradicting the fact that $\dim M = 0$.

To do so, let $Q := E^{\vee\vee}/E$ be supported on a 0-dimensional subscheme Z_0 and is of length $\ell \neq 0$. Consider the $\mathrm{Quot}_S(E^{\vee\vee}, \ell)$. By [HL10, Theorem 6.A.1] we know $\mathrm{Quot}_S(E^{\vee\vee}, \ell)$ is an irreducible variety of dimension > 0. Hence, for any non-trivial subscheme T passing through $q_0\colon E^{\vee\vee} \twoheadrightarrow Q$, we can find a non-trivial quotient $[q\colon E^{\vee\vee} \otimes \mathcal{O}_T \twoheadrightarrow \mathcal{Q}]$ on $S \times T$ such that $\mathcal{Q}|_{S_{t_0}} = Q$ for a closed point $t_0 \in T$ corresponding to q_0. Let $\mathcal{E} := \ker(q)$. Note that $\mathcal{E}_{t_0} = E$ and $\mathcal{E}_t \neq E$ for all $t \neq t_0$. By openness of stability there exists an open subscheme $U \subset T$ such that $\mathcal{E}_t$ is stable for all $t \in U$, which is absurd. ☐

7.3. **The case of** $\dim M^{\mathrm{s}}(v) = 2$**.**

For what follows we assume that there exists a universal sheaf $\mathcal{E} \in \mathrm{Coh}(M^{\mathrm{s}} \times S)$ in the sense of Remark 7.4.

In this section we consider the case when $\langle v, v\rangle = 0$. Such Mukai vectors are said to be *isotropic*. We have seen in Section 6.6 that $M_{(2,-H,1)}(S) \simeq S$ when $H^2 = 4$ and hence $\langle (2, -H, 1), (2, -H, 1)\rangle = 0$. A more general statement is the following:

Theorem 7.6. *For an isotropic Mukai vector v on a K3 surface S whenever $M_v^{\mathrm{s}}(S)$ has a projective irreducible component, it is everything; i.e. $M^{\mathrm{s}} = M$. In particular, M is a smooth, irreducible projective surface. Moreover, it is a K3 surface[22].*

Remark 7.7. This theorem together with the Theorem 7.3 implies that when v is an isotropic primitive Mukai vector, the moduli space of stable sheaves M is a K3 surface, and in particular is irreducible. This is true more generally due to results of Göttsche–Huybrechts and O'Grady

We know $M^{\mathrm{s}} := M_v^{\mathrm{s}}(S)$ is symplectic. This will be dealt with in Section 7.4. First lets assume the first part of the theorem, i.e. $M^{\mathrm{s}} = M$, i.e. smooth projective and it is irreducible. This leaves two possibilities that M is either a K3 or an abelian surface. Thus we only need to show that $H^1(M, \mathcal{O}_M) = 0$. Indeed for abelian surfaces $H^1(S, \mathcal{O}_S)$ is 4 dimensional. Before we prove this we need a bit of general theory of Fourier–Mukai transform for derived categories of coherent sheaves.

[22]derived equivalent to S

7.3.1. *Fourier–Mukai transform: a brief overview.* Let S and M be a two smooth projective varieties and let $\mathcal{E} \in \mathrm{Coh}(S \times M)$ be a torsion free coherent sheaf. The Fourier–Mukai transform gives a way to connect the cohomologies of S and M.

Let $p\colon M \times X \to M$ and $q\colon M \times S \to S$ be two projections. The correspondence in cohomologies use the language of bounded derived category of coherent sheaves. Since we do not need any heavy machinery from the theory other than using it as an intermediate step before passing to cohomology, we sketch how to do the latter in more detail here. (see [Huy06, Chapter 2,3] for details).

The advantage of the language of derived categories is that on a smooth noetherian scheme X any coherent sheaf G admits an injective resolution, i.e. an exact complex

$$0 \to I_0 \xrightarrow{d_0} I_1 \to \cdots \to$$

of injective $\mathcal{O}_X$-modules I_i such that $\ker(d_0) = G$. In the bounded derived category of coherent sheaves $\mathbf{D}^b(X)$ any such resolution is identified with each other and with the object G. Here the morphism of complexes $G \to I^\bullet$ is an example of a *quasi-isomorphism*. More precisely, any morphism $f^\bullet\colon E^\bullet \to F^\bullet$ that induces an isomorphism on the level of complex cohomology, i.e. $\mathcal{H}^i(f^\bullet)\colon \mathcal{H}^i(E^\bullet) \xrightarrow{\sim} \mathcal{H}^i(F^\bullet)$ for all i, is said to be a quasi-isomorphism.

Given a proper morphism of noetherian schemes $f\colon X \to Y$, by [Har77, Prop. III.1.2A] $Rf_*G \in \mathbf{D}^b(Y)$ is isomorphic to the complex $f_*I^\bullet$ and $R^if_*G \simeq h^i(f_*I^\bullet)$ in $\mathbf{D}^b(Y)$. A similar statement is true for $R\mathcal{H}om(\_, G)$ functor.

Formally, let $\mathbf{Kom}^b(X)$ denote the category of bounded complexes of coherent sheaves on X, the morphisms of this category are given by term-wise maps of coherent sheaves with commutative square. The *bounded derived category of coherent sheaves* $\mathbf{D}^b(X)$ on X is the category such that there exists a functor $Q\colon \mathbf{Kom}^b(X) \to \mathbf{D}^b(X)$ that sends quasi-isomorphism to isomorphism and satisfies the following (universal) property: If $F\colon \mathbf{Kom}^b(X) \to \mathcal{D}$ be any functor such that F sends quasi-isomorphism to isomorphism, then F factors uniquely (up-to isomorphism) through the derived category $\mathbf{D}^b(X)$.

Via Q the objects of the derived category is easily identified with bounded complexes of coherent sheaves

$$G^\bullet := [G^0 \to G^1 \to \cdots \to G^d]$$

of coherent sheaves considered say in degree 0 to d in this demonstration. In the derived category one identifies the category of coherent sheaves as a full subcategory as follows: any coherent sheaf G with the complex $G^\bullet$ concentrated in degree 0; namely $G^0 = G$ and $G^i = 0$ when $i \neq 0$. It comes with an inherent shift functor $[1]\colon \mathbf{D}^b(X) \to \mathbf{D}^b(X)$ where $G^\bullet[1]^k = G^{\bullet+1}$.

Definition 7.8. The Fourier–Mukai transform with kernel $\mathcal{E} \in \mathrm{Coh}(X \times Y)$

$$\Phi_{\mathcal{E}}\colon \mathbf{D}^b(X) \to \mathbf{D}^b(Y)$$

is defined by sending $G^\bullet \mapsto Rp_*(q^*G^\bullet \otimes \mathcal{E})$ where $p\colon X \times Y \to Y$ and $q\colon X \times Y \to X$ are natural projections.

Example 7.9. Given $\mathcal{E} \in \mathrm{Coh}(S \times M)$ is the universal sheaf. Define $\Phi_{\mathcal{E}}\colon \mathbf{D}^b(M) \to \mathbf{D}^b(S)$. For any closed point $t \in M$ corresponding to a stable sheaf F we have, $\Phi_{\mathcal{E}}(k(t)) = \mathcal{E}|_t \simeq F$.

When X is a smooth projective variety. Given $G^\bullet \in \mathbf{D}^b(X)$, we define the *Mukai vector* $v(G^\bullet) := \sum_i (-1)^i \operatorname{ch}(G^i)\sqrt{\operatorname{td}(X)} \in H^\bullet(X, \mathbb{Q})$. Since the Chern character is additive over short exact sequences we have $v(Rf_*G) = \sum_i (-1)^i v(R^if_*G)$. Hence we also have

Definition 7.10. The cohomological Fourier–Mukai transform with kernel $\mathcal{E} \in \mathrm{Coh}(X \times Y)$

$$f_{\mathcal{E}}\colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$$

is defined by $v \mapsto p_*(q^*v \cup v(\mathcal{E}))$. In particular for $G \in \mathbf{D}^b(X)$, $v(G) \mapsto v(\Phi_{\mathcal{E}}(G))$.

Since the chern classes of coherent sheaves lie in the even cohomologies, $f_{\mathcal{E}}$ maintains the pairity; i.e. $f_{\mathcal{E}}(H^{\text{even}}(S)) \subset H^{\text{even}}(M)$ and $f_{\mathcal{E}}(H^{\text{odd}}(S)) \subset H^{\text{odd}}(M)$.

The aforementioned Fourier–Mukai transformations are compatible, namely the following diagram commutes

$$\begin{tikzcd} \mathbf{D}^b(S) \arrow[r, "\Phi_{\mathcal{E}}"] \arrow[d, "v"] & \mathbf{D}^b(M) \arrow[d, "v"] \\ H^*(S,\mathbb{Q}) \arrow[r, "f_{\mathcal{E}}"] & H^*(M,\mathbb{Q}) \end{tikzcd}$$

Furthermore, [Huy06, Proposition 5.33] states that

Proposition 7.11. *Let $\mathcal{E} \in \mathrm{Coh}(X \times Y)$ for smooth projective varieties X and Y. If $\Phi_{\mathcal{E}}$ is an equivalence of categories then $f_{\mathcal{E}}$ is a bijection of rational vector spaces.*

As an immediate application we see that if M is derived equivalent to S, then $H^1(S,\mathbb{Q}) \simeq H^1(M,\mathbb{Q}) = 0$. In fact, in order to conclude that M is a K3 surface we do not really need the symplectic structure of M. The equivalence of categories also indicates that ω_M is trivial (see [Huy06, Prop. 4.1]).

Hence we need a criterion for derived equivalence. This is given by the following result of Bondal–Orlov and Bridgeland [Bri99, Theorem 1.1]

Theorem 7.12. *The functor $\Phi_{\mathcal{E}} \colon \mathbf{D}^b(M) \to \mathbf{D}^b(S)$ is fully faithful if and only if $\mathrm{Hom}(\mathcal{E}_t, \mathcal{E}_t) \simeq k$ and for any two distinct points $t \neq t' \in M$ satisfy*

$$\mathrm{Ext}^k(\mathcal{E}_t, \mathcal{E}_{t'}) = 0$$

for any integer k. It is an equivalence of categories if additionally $\mathcal{E}_t \simeq \mathcal{E}_t \otimes \omega_S$ for all $t \in M$.

Proof of Theorem 7.6. Assuming smoothness and irreducibility, we only need to show that $H^1(M,\mathbb{Q}) = 0$. The last condition in Theorem 7.12 is easily satisfied for $\mathcal{E}_t \in \mathrm{Coh}(S)$. For the first, note that since $\mathcal{E}_t$ are stable bundles $\mathrm{Hom}(\mathcal{E}_t, \mathcal{E}_{t'}) = 0$ for $t \neq t'$. Furthermore, $\mathrm{Ext}^2(\mathcal{E}_t, \mathcal{E}_{t'}) \simeq \mathrm{Hom}(\mathcal{E}_{t'}, \mathcal{E}_t)^\vee = 0$. Thus $\chi(\mathcal{E}_t, \mathcal{E}_{t'}) = -\langle v(\mathcal{E}_t), v(\mathcal{E}_{t'})\rangle = 0$. Hence $\mathrm{Ext}^1(\mathcal{E}_t, \mathcal{E}_{t'}) = 0$. So the first condition is also satisfied. Hence M is derived equivalent to S. Hence by Proposition 7.11 the cohomological Fourier–Mukai functor $f_{\mathcal{E}}$ is a bijection of $\mathbb{Q}$-vector spaces mapping $H^1(S,\mathbb{Q}) \to H^1(M,\mathbb{Q})$ and hence $H^1(M,\mathbb{Q}) = 0$.

Let M_1 denote the projective component of M^{s}. Since $M^s \subset M$ is open by [HL10, Prop. 2.3.1], M_1 is a also a connected component of M. Therefore we now show the connectedness of M. Smoothness of M then follows from smoothness of M^{s}.

To see connectedness, let $\mathcal{E} \in \mathrm{Coh}(S \times M_1)$ be a universal sheaf[23] restricted to the component M_1. Let $t \in M_1$ be a closed point. Since $\mathcal{E}_t$ is stable, for any $[F] \in M$ represented by a semistable sheaf F. Since $\mathcal{E}_t$ is stable and we have $p(F) = p(\mathcal{E}_t)$ with $\mathcal{E}_t$ and F having the same rank, we deduce that $\mathrm{Hom}(F, \mathcal{E}_t) = 0$ unless $[F] = t$ and by Serre duality $\mathrm{Ext}^2(F, \mathcal{E}_t) = 0$ unless $[F] = t$. Using $\chi(F, \mathcal{E}_t) = -v^2 = 0$, we obtain $\mathrm{Ext}^1(F, \mathcal{E}_t) = 0$ unless $[F] = t$.

As an upshot we consider the transformation

$$\varphi \colon H^*(S) \to H^*(M)$$

given by $v(F) \mapsto v(Rp_* R\mathcal{H}om(q^*F, \mathcal{E}))$. By the previous discussion we obtain that on M_1, $Rp_* R\mathcal{H}om(q^*F, \mathcal{E})$ is supported at $[F] \in M_1$ and is 0 on all other irreducible components of M. By Grothendieck–Riemann–Roch, we know

$$\mathrm{ch}(Rp_* R\mathcal{H}om(q^*F, \mathcal{E})) \cdot \mathrm{td}(M) = p_* \,\mathrm{ch}(R\mathcal{H}om(q^*F, \mathcal{E}) \cdot \mathrm{td}(S \times M))$$

and hence $\mathrm{ch}(Rp_* R\mathcal{H}om(q^*F, \mathcal{E})) \cdot \mathrm{td}(M)$ should only depend on $\mathrm{ch}(q^*F)$ which does not vary with $[F] \in M$ and on $\mathrm{ch}(\mathcal{E})$. But if $[F] \in M_1$, an additional argument involvng homological algebra

[23] in general it is enough to assume $\mathcal{E}$ is a quasi-universal sheaf, which always exists by [HL10, §4.6].

shows that $Rp_*R\mathcal{H}om(q^*F, \mathcal{E}) \simeq k([F])[2] \in \mathbf{D}^b(M)$. Therefore, $\varphi(v(F)) \neq 0$. On the other hand $\varphi(v(F)) = 0$ if $[F] \notin M_1$ by the previous discussion, which is a contradiction. □

Exercise 22. Using the fact that the Chern character is multiplicative on tensor products and that $c_i(F^\vee) = (-1)^i c_i(F)$, show that if E and F are two locally free sheaves on S with isotropic Mukai vector v, write $v(\mathcal{H}om(E, F))$ in terms of $\mathrm{rk}(E) = \mathrm{rk}(F)$. [24].

7.4. **The symplectic form on M^{s}.**

Definition 7.13. A symplectic structure on a smooth variety X is a non-degenerate non-trivial closed two-form, i.e. a nonwhere vanishing global section $\sigma \in H^0(X, \Omega_X^2)$ such that $d\sigma = 0$.

A closed global 2-form is equivalent to giving a section $\mathcal{O}_X \to \Omega_X^2$, or equivalently a homomorphism $\mathcal{T}_X \simeq \Omega_X^\vee \to \Omega_X$ between the tangent and the cotangent bundle of X.

The symplectic structure σ is non-degenerate if $\sigma\colon \mathcal{T}_X \to \Omega_X$ induces a natural isomorphism, or equivalently $\sigma\colon \mathcal{T}_X \otimes \mathcal{T}_X \to \mathcal{O}_X$ is an everywhere non-degenerate alternating bilinear form.

When a smooth projective variety X admits a non-degenerate 2-form σ, we have an isomorphism $\wedge\sigma\colon \omega_X^{-1} \to \omega_X$. In particular $\omega_X^2 \simeq \mathcal{O}_X$.

A hyperkähler manifold a generalisation of complex K3 surfaces in higher dimension. Just like K3 surfaces over the complex numbers, a hyperkähler manifold X is compact Kähler together with an everywhere non-degenerate symplectic form $\sigma \in H^0(X, \Omega_X^2)$. Furthermore they are simply connected. In particular, $H^1(X, \mathbb{Z}) = 0$. The non-degeneracy of the 2-form then implies that $\omega_X \simeq \mathcal{O}_X$. Indeed, consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

and identify $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^*)$ and the first chern class with the boundary map $c_1\colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$. Since $H^1(S, \mathcal{O}_S) = 0$, c_1 is injective. Thus $\mathcal{O}_X$ is the only line bundle with trivial chern class on X.

When v is primitive, the moduli space of stable sheaves $M_v(S)$ on a K3 surface S is a hyperkähler manifold.

7.4.1. *Hilbert scheme of points.* These were historically the first example of a higher dimensional hyperkähler manifolds [Bea83]. Let $v = (1, 0, 1-n)$, $n \in \mathbb{Z}_{\geq 1}$ be a Mukai vector on a K3 surface S. Sheaves with Mukai vector v, must be of rank 1. Therefore of the form $\mathcal{I}_Z \otimes L$ where $\mathcal{I}_Z$ is the ideal sheaf of a closed subscheme $Z \subset S$ of dimension 0 and L is a line bundle. Recall the general principle that any non-trivial rank 1 torsion free subsheaf $F \subset \mathcal{I}_Z \otimes L$ automatically satisfy $p(F) < p(I_Z \otimes L)$ since the quotient will be supported at zero dimensional subschemes. Therefore $\mathcal{I}_Z \otimes L$ is stable. Hence $M \coloneqq M_{(1,0,1-n)}(S)$ is smooth projective.

Furthermore, $c_1(I_Z \otimes L) = c_1(L)$. Since $c_1(I_Z \otimes L) = 0$, L must be a line bundle with zero Chern class. On K3 surfaces, only such line bundle is the structure sheaf $\mathcal{O}_S$.

One can identify $M_{(1,0,1-n)}(S)$, the moduli space of sheaves of the form $\mathcal{I}_Z$ where the length $\ell(Z) = n$ with the Hilbert scheme of points $S^{[n]} \coloneqq \mathrm{Quot}_{S/k}(\mathcal{O}_X, n)$, sometimes referred to as the punctual Hilbert scheme. Given $[q \coloneqq \mathcal{O}_{S_T} \twoheadrightarrow \mathcal{O}_{\mathcal{Z}}] \in S^{[n]}(T)$ we let $\mathcal{I}_{\mathcal{Z}} \coloneqq \ker(q)$ and note that $\mathcal{I}_{\mathcal{Z}} \in M(T)$. Thus we obtain a morphism of smooth projective schemes $S^{[n]} \to M$. Therefore, it is enough to show that the map is bijective on k-points. But that is obvious since from the discussion above we know that k-points of M are given by $\mathcal{I}_Z$ for some closed subscheme $Z \subset S$ of length n.

Beauville showed in [Bea83]

Theorem 7.14. *The Hilbert scheme of points $S^{[n]}$ on a K3 surface admits a unique symplectic 2-form $\sigma \in H^0(S^{[n]}, \Omega^2_{S^{[n]}})$.*

[24]Thanks Chenji and Moritz for catching a mistake in the earlier version of the exercise

Proof. To show this we use the geometric interpretation of the Hilbert scheme in terms of the n-fold product space $S^n \coloneqq S \times \cdots \times S$. The points on S^n are not necessarily distinct. The symmetric group $\mathfrak{S}_n$ acts on S^n by permuting the points and hence the quotient variety $S^{(n)} \coloneqq S^n/\mathfrak{S}_n$[25], the orbit space with respect to this action, known as the n-th symmetric product of S, does parametrize n unordered points on S.

Although the symmetric product $S^{(n)}$ parametrises the unordered set of n-points on S, along the image of the diagonal $\Delta_{ij} \coloneqq \{(x_i)_{1\le i\le n} \in S^n | x_i = x_j\}$, $i \neq j$ it will have singularities. We let $\Delta \coloneqq \bigcup_{i\neq j} \Delta_{ij}$ and its image $D \subset S^{(n)}$. The Hilbert scheme $S^{[n]}$ admits a morphism $h\colon S^{[n]} \to S^{(n)}$ by sending a 0-dimensional scheme to the points coresponding to it. It is an isomorphism on $V \coloneqq S^{(n)} \setminus D$. Letting $D' \subset D$ to be the closed subset given by union of image of all intersections $\Delta_{ij} \cap \Delta_{kl}$, i.e. on $D \setminus D'$, precisely two points could be the same and nothing else. Then the Hilbert scheme outside $h^{-1}D'$ can be identified with $Bl_\Delta S^n/\mathfrak{S}_n$.

To keep notations lighter, let us assume $n = 2$. In this case $D' = D = \Delta/\mathfrak{S}_2$ and $S^{[2]} \simeq Bl_\Delta S^2/\mathfrak{S}_2$. Let $\eta\colon Bl_\Delta S^2 \to S^2$ be the blow up map. We have the following fibre square

$$\begin{array}{ccc} Bl_\Delta S^2 & \xrightarrow{\widetilde{\rho}} & S^{[2]} \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle h} \\ S^2 & \xrightarrow[\rho]{} & S^{(2)} \end{array}$$

Let $\sigma \in H^0(S, \Omega^2_S)$ be the unique holomorphic symplectic form on the K3 surface and let $p_1\colon S^2 \to S$ be the natural projections. Then define $\widetilde{\sigma} \coloneqq p_1^*\sigma + p_2^*\sigma$ is a 2-form on S^2. Indeed, $\widetilde{\sigma} \in H^0(S^2, p_1^*\Omega^2_S \oplus p_2^*\Omega^2_S) \subseteq H^0(S^2, \Omega^2_{S^2})$. Note that it is invariant under the action of $\mathfrak{S}_2$. Hence its pullback $\eta^*\widetilde{\sigma}$ is also $\mathfrak{S}_2$ invariant, which implies that there exists $\tau \in H^0(S^{[2]}, \Omega^2_{S^{[2]}})$ such that $\widetilde{\rho}^*\tau = \eta^*\widetilde{\sigma}$. Since $\omega_{S^2} \simeq p_1^*\omega_S \otimes p_2^*\omega_S \simeq \mathcal{O}_{S^2}$, the divisor associated to the section $\wedge^2\widetilde{\sigma} \in H^0(S^2, \omega_{S^2})$ is zero. But since $\widetilde{\rho}$ is ramified along the exception E, we have the following equality

$$\widetilde{\rho}^*\mathrm{div}(\wedge^2\tau) + E = \mathrm{div}(\wedge^2\widetilde{\rho}^*\tau) = \mathrm{div}(\wedge^2\eta^*\widetilde{\sigma}) = \eta^*\mathrm{div}(\wedge^2\widetilde{\sigma}) + E = E.$$

All in all, we have $\widetilde{\rho}^*\mathrm{div}(\wedge^2\tau) = 0$ and hence $\mathrm{div}(\wedge^2\tau) = 0$. In general when $n > 2$, one needs to use Hartog's theorem to extend the 2-form from the locus of the Hilbert scheme where it looks like an $\mathfrak{S}_n$ quotient of a blow up to all of $S^{[n]}$.

We now argue that this is the only symplectic form $S^{[n]}$ could have. To see this, note that

$$h^0(S^{[2]}, \Omega^2_{S^{[2]}}) = h^0(Bl_\Delta S^2, \Omega^2_{Bl_\Delta S^2})^{\mathfrak{S}_2} = h^0(S^2, \Omega^2_{S^2})^{\mathfrak{S}_2} = 1.$$

The general case will again require Hartog's theorem. □

$S^{[n]}$ is in fact simply connected. For an argument see the second part of the proof of [Bea83, Lemme 1] or [HL10, Prop. 6.2.4]. Therefore $S^{[n]}$ is a Hyperkähler manifold.

7.4.2. *Other moduli of sheaves.* Let v be a primitive Mukai vector. In this case $M \coloneqq M_v(S)$ is a smooth projective moduli space of dimension $v^2 + 2$consisting of only stable sheaves. These are deformation equivalent to the Hilbert scheme of points $S^{[n]}$ for $n \coloneqq \frac{1}{2}(v^2) + 1$ (see [HL10, Theorem 6.2.16], i.e. there is family $f\colon \mathcal{X} \to \Delta$ over the unit disc Δ such that $\mathcal{X}_0 \simeq S^{[n]}$ and $\mathcal{X}_t \simeq M_v(S)$ for some $t \neq 0 \in \Delta$. In particular they have the same Betti numbers as $S^{[n]}$. Indeed, the flat family f of smooth projective varieties are smooth. Hence by the Ehresmann's lemma for any $t_0 \in \Delta$, there is an analytic open subset $U \ni t_0$ such that $f^{-1}(U) \simeq U \times \mathcal{X}_{t_0}$. Hence $\pi_1(M) = 0$. Also by semi-continuity of dimension of coherent cohomologies and the Hodge decomposition we have $h^0(M, \Omega^2_M) = 1$.

[25]Note that when a finite group G acts on a quasi-projective vareity X, the quotient X/G is constructed by taking quotients of G-invariant affine covers. Hence the quotient with quotient topology is again a variety.

The moduli spaces $M_v(S)$ admit a non-degenerate symplectic 2-form. When a universal sheaf $\mathcal{E}$ exists on $S \times M$, we let as before let $p\colon S \times M \to M$ and $q\colon S \times M \to S$ be the natural projections. Then roughly speaking the symplectic form is a global version of the following

$$\mathcal{T}_M \otimes \kappa(t) \simeq \operatorname{Ext}^1(\mathcal{E}_t, \mathcal{E}_t)$$

that we have seen in Remark 6.18. More precisely, let $\mathcal{E}xt^i_p(\mathcal{E}, \mathcal{E}) \coloneqq \mathcal{H}^i Rp_* R\mathcal{H}om(\mathcal{E}, \mathcal{E})$ be the relative ext-sheaves. Then

$$\mathcal{T}_M \simeq \mathcal{E}xt^i_p(\mathcal{E}, \mathcal{E})$$

(see [HL10, Theorem 10.2.1] for a thorough treatment in a more general set-up). The everywhere non-degenerate alternating 2-form is then given by the composition

$$\mathcal{T}_M \times \mathcal{T}_M \xrightarrow{\simeq} \mathcal{H}om_p(\mathcal{E}, \mathcal{E}[1]) \times \mathcal{H}om_p(\mathcal{E}[1], \mathcal{E}[2]) \xrightarrow{\alpha} \mathcal{E}xt^2_p(\mathcal{E}, \mathcal{E}) \simeq \mathcal{O}_M$$

where α is given by composition in the derived category and the last isomorphism follows from the fact that fibrewise $\operatorname{Ext}^2(\mathcal{E}_t, \mathcal{E}_t) \simeq k$.

7.5. **Example birational to the Hilbert scheme.** Let v be a primitive Mukai vector. While any moduli space of sheaves $M_v(S)$ is deformation equivalent to $S^{[n]}$ for $n = \frac{1}{2}(v^2 + 2)$, it is often the case that $M_v(S)$ is in fact birational to $S^{[n]}$, i.e. there exists an open subset in $M_v(S)$ that is isomorphic to an open subset of the Hilbert scheme. In this section we describe the example in [HL10, Theorem 11.3.1] which demonstrates this phenomenon.

Example 7.15. Let (S, H) be a polarised K3 surface. Define the numbers $k(n) \coloneqq (n^2 + n)H^2 + 2$. Then for $v = (2, (2n+1)H, -k(n))$ and $n \gg 0$ $M_v(S) \simeq S^{[\ell(n)]}$ where $\ell(n) = \frac{1}{2}H^2 + 2k(n) = \frac{(2n+1)^2 H^2}{2} + 3$.

Proof. The open set in $U \subset M_v(S)$ onto which we will construct an isomorphism from an open set in $S^{[\ell(n)]}$ is given by the set of μ-stable and locally free sheaves. [HL10, Theorem 9.4.3 and 9.4.2] ensure the openness of this locus.

Let Z be a general length $\ell(n)$ 0-dimensional subscheme of S. In what follows we construct a μ-stable locally free sheaf $F_Z \in U(\mathbb{C})$. This construction method, is known as *Serre construction* and is quite useful in constructing examples of μ-stable sheaves on surfaces. Note that since H is ample, by Kodaira vanishing and Hirzebruch–Riemann–Roch [26] we have

$$\chi(\mathcal{O}_S((2n+1)H)) = h^0(S, \mathcal{O}_S((2n+1)H)) = \frac{(2n+1)^2 H^2}{2} + 2 = \ell(n) - 1.$$

Therefore for Z general enough, there are no curve in the linear system $H^0(S, \mathcal{O}_S((2n+1)H))$ that vanishes along any subscheme Z' of length $\ell(n) - 1$ in Z, in other words the map $H^0(S, \mathcal{O}_S((2n+1)H)) \to H^0(Z', \mathcal{O}_{Z'})$ is surjective. Indeed, this is because H embeds $S \hookrightarrow \mathbb{P} \coloneqq \mathbb{P}^{\frac{1}{2}H^2+1}$ and general hypersurfaces of degree $(2n+1)$ on $\mathbb{P}$ intersects with S in $(2n+1)^2 H^2$ points. Therefore $(2n+1)^2 H^2 + 1$ points in general positions determine one such hypersurface and $(2n+1)^2 H^2 + 2$ points in general positions lie in no such hypersurfaces.

In particular, $H^0(S, \mathcal{I}_Z \otimes \mathcal{O}_S((2n+1)H)) = 0$ where $\mathcal{I}_Z$ is the ideal sheaf defining Z, and hence we have the following short exact sequence in cohomologies

$$0 \to H^0(S, \mathcal{O}_S((2n+1)H)) \to H^0(Z, \mathcal{O}_Z) \to H^1(S, \mathcal{I}_Z \otimes \mathcal{O}_S((2n+1)H)) \to 0.$$

Since $h^0(Z, \mathcal{O}_Z) = \ell(Z) = \ell(n)$, we obtain $h^1(S, \mathcal{I}_Z \otimes \mathcal{O}_S((2n+1)H)) = \dim \operatorname{Ext}^1(\mathcal{O}_S, \mathcal{I}_Z \otimes \mathcal{O}_S((2n+1)H)) = 1$. By Serre duality we have $\dim \operatorname{Ext}^1(\mathcal{I}_Z \otimes \mathcal{O}_S((2n+1)H), \mathcal{O}_S) = 1$, and therefore there exists a unique extension

$$0 \to \mathcal{O}_S \to F_Z \to \mathcal{I}_Z \otimes \mathcal{O}_S((2n+1)H) \to 0$$

[26] or, Riemann–Roch on surfaces

Since $c_1(F_Z) = (2n+1)H$, if we show that F_Z is μ-stable and locally free, we have given a well-defined map from $S^{[n]} \to U$. To see why it is injective note that $H^0(S, F_Z) = 1$ and hence there can be a unique morphism $\mathcal{O}_S \to F_Z$.

First we argue why F_Z is locally free. By the choice of Z, any length $\ell(n)-1$ subscheme $Z' \subset Z$ satisfies $s|_{Z'} = 0 \Rightarrow s = 0$. Hence by the Cayley–Bacharach Theorem 7.16, we obtain the desired local freeness.

To see why it is μ-stable. Let $\mathcal{L} \subset F_Z$ be a rank 1 torsion free sheaf such that $\mu(\mathcal{L}) \geq \mu(F_Z)$. Since F_Z is locally free, we might as well replace $\mathcal{L}$ by its reflexive hull which is a line bundle [27]. Such a line bundle satisfies

$$c_1(\mathcal{L}) \cdot H \geq \frac{1}{2} c_1(F_Z) \cdot H = \frac{2n+1}{2} H^2,$$

and hence it cannot be a subsheaf of $\mathcal{O}_S$. Thus there is a non-trivial injection $\mathcal{L} \hookrightarrow \mathcal{I}_Z \otimes \mathcal{O}_S((2n+1)H)$ which defines a curve C as a section of $H^0(S, \mathcal{L}^{-1} \otimes \mathcal{O}_S((2n+1)H))$ vanishing along Z. We show that such curve cannot exist for Z in general position by arguing that $h^0(S, \mathcal{L}^{-1} \otimes \mathcal{O}_S((2n+1)H)) = h^0(C, \mathcal{O}_S(C)) \leq \ell(n) - 1$. To this end note that $h^0(C, \mathcal{O}_S(C)) = \frac{1}{2} C^2 + 2$ and that

$$C^2 = ((2n+1)H - c_1(\mathcal{L}))^2 = 2(2n+1)\left(\frac{2n+1}{2} H^2 - c_1(\mathcal{L}) \cdot H\right) + c_1(\mathcal{L})^2 \leq c_1(\mathcal{L})^2$$

where the last inequality follows from the fact that $\mathcal{L}$ destabilises F_Z. By the Hodge index theorem we know $c_1(\mathcal{L})^2 \leq \frac{(c_1(\mathcal{L}) \cdot H)^2}{H^2}$. And finally, observe that since $\mathcal{L} \hookrightarrow \mathcal{O}_S((2n+1)H)$, $c_1(\mathcal{L}) \cdot H \leq (2n+1)H^2$. Putting everything together, we obtain

$$h^0(S, \mathcal{O}_S(C)) \leq \frac{(2n+1)^2 H^2}{2} + 2 = \ell(n) - 1.$$

Hence the result.

$\square$

Theorem 7.16 (Cayley–Bacharach property)**.** *Let $Z \subset X$ be a local complete intersection of codimension two*[28]*. Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles on X. Let E be any extension of $\mathcal{M} \otimes \mathcal{I}_Z$ by $\mathcal{L}$, i.e.*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \otimes \mathcal{I}_Z \to 0.$$

Then $\mathcal{E}$ is locally free if and only if the pair $(\mathcal{L}^{-1} \otimes \mathcal{M} \otimes \omega_X, Z)$ has the Cayley–Bacharach property which is described as follows

Let $Z' \subsetneq Z$ is a subscheme with $\ell(Z') = \ell(Z) - 1$. Then for any $s \in H^0(X, \mathcal{L}^{-1} \otimes \mathcal{M} \otimes \omega_X)$ with $s|_{Z'} = 0$ satisfy $s|_Z = 0$.

7.6. **The singular moduli space.** When the Mukai vector v is not primitive the moduli space $M_v(S)$ is still symplectic in the sense that it admits a global 2-form that is non-degenerate on the regular locus. The stable part $M_v^{\mathrm{s}}(S)$ is non-compact, and $M_v(S) \setminus M_v^{\mathrm{s}}(S)$ consists of the S-equivalence classes of strictly semistable sheaves. It is natural to wonder whether this singular projective symplectic variety admits a resolution to a hyperkähler manifold. For a primitive Mukai vector v_0, we describe the answer on a case-by-case basis.

1. When $v = mv_0$ and $v_0^2 = -2$. We have seen in Theorem 7.5 that $M_{v_0}(S) = \{\mathrm{pt}\}$. Let E denote the stable sheaf corresponding to this point. For $m \geq 2$, $v^2 < -2$ and hence $M_v^{\mathrm{s}} = \emptyset$. Therefore $M_v(S) = \{[E^{\oplus m}]\}$.
2. When $v = mv_0$ and $v_0^2 = 0$. We have seen in Theorem 7.6 that $Y = M_{v_0}(S)$ is a K3 surface such that there exists a derived equivalence $\Phi\colon D^b(S) \to D^b(Y)$ sending $\Phi([F]) = \kappa([F])$ for $[F] \in Y(\mathbb{C})$. In the level of cohomology we can write $\varphi(v_0) = (0,0,1)$. Since $\varphi\colon H^*(S,\mathbb{Z}) \to$

[27] recall that $\mu(\mathcal{L}) \leq \mu(\mathcal{L}^\vee)$

[28] e.g. reduced points in a smooth surface

$H^*(Y,\mathbb{Z})$ is a Hodge isometry we have $\varphi(mv_0) = (0,0,m)$. Hence $M_v(S) \simeq M_{(0,0,m)}Y \simeq Y^{(m)}$ the symmetric product of m-copies of Y. Hence $M_v(S)$ admits a symplectic resolution given by the Hilbert–Chow map $Y^{[m]} \to Y^{(m)} \simeq M_v(S)$.

(3) When $v = mv_0$ and $v_0^2 = 2$: In this case we have seen that $M_{v_0}(S)$ is a hyperkhler manifold deformation equivalent to $S^{[n]}$ where $n = \frac{1}{2}v_0^2 + 1$.

When $m = 2$, O'grady in [O'G99] shows that the moduli space $X \coloneqq M_{(2,0,-2)}(S)$ is singular. Moreover, the singular locus $X_{\text{sing}} \simeq M_{v_0}(S)^{(2)}$ and hence it is of codimension 2 in 10-dimensional X. He showed that it can be desingularised to a hyperkhler manifold of dimension 10, now famously known as OG10. This means the symplectic form on the regular locus $X \setminus X_{\text{sing}}$ extends to a symplectic form on OG10. This hyperkähler's second Betti number is 24 and hence is topologically different from the Hilbert scheme of points. Its complete list of Betti numbers and the Hodge decompositions were calculated only recently in [dCRS21].

When $m > 2$ it is shown in [KLS06] that the singular locus of $M_v(S)$ does not admit a symplectic resolution.

(4) when $v_0^2 > 2$. In this case the results of [KLS06] shows that the singular locus of $M_v(S)$ does not admit a symplectic resolution.

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