DECOMPOSITION THEOREM FOR CURVES

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The main goal of this talk is to complete Step 1 from the proof of Statement 1 of Lecture 8. In other words, establish the $P(1,0)$ case of the decomposition theorem. We first recall the statement:

Theorem 1.1 [Sab05, Moc07]**.** *Let $\mathbb{X}$ be a smooth projective curve and let $a\colon \mathbb{X} \to \operatorname{Spec}\mathbb{C}$ denote the structure map for the curve. Let $(\mathcal{T}, \mathcal{S})$ be a polarized regular pure twistor D-module on $\mathbb{X}$ of weight 0. Then, the push-forward $(\oplus_{i=-1}^{1} a_+\mathcal{T}, \mathcal{L}, a_+\mathcal{S})$ is a polarized graded Lefschetz twistor structure.*

We refer to the previous lectures for the precise definitions of polarized regular pure twistor D-modules and their pushforwards. However we do briefly recall the following correspondence (see e.g. [Moc07, Theorem 20.1]). For simplicity, we restrict to the weight 0 case.

Theorem 1.2. *There is a one-to-one correspondence between the variation of polarized pure twistor structures of weight 0 which are generically defined over $\mathbb{X}$ and the regular pure twistor D-modules of weight 0 whose strict support is $\mathbb{X}$.*

The correspondence goes via harmonic bundles since a variation of polarized pure twistor structures of weight 0 underlies a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ on $X \coloneqq \mathbb{X} \setminus D$ for D a finite set of points (assume, only one point for simplicity). We let $\mathcal{X} \coloneqq X \times \mathbb{C}_\lambda$ and $\mathscr{X} \coloneqq \mathbb{X} \times \mathbb{C}_\lambda$ and p the respective projections to X (or $\mathbb{X}$). From this data, we obtain an $\mathcal{R}_{\mathcal{X}}$-triple $(\mathcal{E}, \mathcal{E}, C_0)$. On its naïve algebraic extension bundle $^\square\mathcal{E}$ (i.e. roughly speaking the twistor incarnation of the algebraic extension of E over $\mathbb{X}$) we have a V-filtration $U_\bullet^{(\lambda_0)\square}\mathcal{E}$ defined on $^\square\mathcal{E}|_{\mathscr{X}(\lambda_0,\epsilon_0)}$, where $\mathscr{X}(\lambda_0, \epsilon_0) \coloneqq \mathbb{X} \times \Delta(\lambda_0, \epsilon_0)$ namely a slice of the product space over the disc $\Delta(\lambda_0, \epsilon_0)$ around $\lambda_0 \in \mathbb{C}_\lambda$. Define $\mathfrak{E}(\mathscr{X}(\lambda_0, \epsilon_0)) \coloneqq$ the $\mathcal{R}_{\mathscr{X}}$- submodule of $^\square\mathcal{E}$ generated by $U_{<0}^{(\lambda_0)\square}\mathcal{E}$. Then the glued $\mathcal{R}_{\mathscr{X}}$-module gives rise to a polarized $\mathcal{R}_{\mathscr{X}}$-triple $((\mathfrak{C}, \mathfrak{E}, \mathfrak{C}), (\mathrm{Id}, \mathrm{Id}))$ underlying a pure regular twistor D-module.

Conversely, given such a triple, generically on $\mathbb{X}$, namely for a Zariski open subset X the $\mathcal{R}_{\mathcal{X}}$-triple $\mathcal{T}|_{\mathcal{X}}$ is a λ-deformed bundle of the harmonic bundle $(E, \overline{\partial}_E, \theta_E, h)$. The regularity implies tameness of this harmonic bundle.

Proof Sketch. For the proof of Theorem 1.1 we follow [Moc07, §20.2.2] and it relies on the Dolbeault lemma for a singular Hermitian line bundle due to Zucker. A different proof can be found in [Sab05, §6.2.b–6.2.f].

The proof goes via a series of quasi-isomorphisms leading up-to

$$\mathcal{H}^{i+\dim X}(\mathbb{R}a_*(\mathfrak{E} \otimes \Omega^\bullet_{\mathscr{X}}) \simeq \mathrm{Harm}^i \otimes \mathcal{O}_{\mathbb{C}_\lambda},$$

where Harm^i is a finite dimensional vector space. Thus by definition the pushforward is a twistor structure of weight i. The Lefschetz map in this case concerns only $i = 0$ and looks like

$$\mathcal{L} \coloneqq a_+^0\mathfrak{E} \to a_+^0\mathfrak{E} \otimes \mathbb{T}(0)$$

where $\mathbb{T}(0)$ is the Tate twistor structure of weight 0.

Roughly speaking, the reason why such vector spaces Harm^i are independent of λ is they are generated by the kernel of the Laplace operator

$$D^{\lambda^*}D^\lambda + D^\lambda D^{\lambda^*} = (1 + |\lambda|^2)\left((\overline{\partial}_E + \theta)^*(\overline{\partial}_E + \theta) + (\overline{\partial}_E + \theta)(\overline{\partial}_E + \theta)^*\right)$$

acting on certain finite dimensional space of global L^2-sections. Here D^λ is the connection associated to E^λ.

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The crux of the proof lies in constructing this series of quasi-isomorphisms. It relies on the classical Dolbeault lemma for $\mathcal{C}^\infty$ rank 1 flat bundle (V, ∇) on the disc. This is due to Zucker [Zuc79] who established a quasi-isomorphism between the naïve algebraic extension ${}^\square V$ of V on $\mathbb{X}$ and the complex $\mathcal{L}^\bullet(V)_{(2)}$ of L^2-sections of $V \otimes A^p_X$ with L^2-derivatives [Zuc79, Theorem 6.2]. For all $\lambda \in \mathbb{C}_\lambda$, one can apply this construction to the bundles $\mathscr{E}^\lambda$ associated to $\mathfrak{E}|_{\mathcal{X}}$ and extend these ideas to construct a complex $S(\mathscr{E} \otimes \Omega^{\bullet,0}_{\mathscr{X}})$ on $\mathscr{X}$ whose fibres are certain λ-holomorphic sections of a sub-complex $\widetilde{\mathcal{L}}^\bullet(\mathcal{E}^\lambda)_{(2)} \subseteq \mathcal{L}^\bullet(\mathcal{E}^\lambda)_{(2)}$. This subcomplex is defined so that it is soft with respect to the global section functor and the i-th cohomology of the global section complex is a finite dimensional vector space Harm^i [Moc07, Lemma 20.23-24] and hence is independent of λ.

On the other hand, the relation between the complexes $a_*S(\mathcal{E} \otimes \Omega^{\bullet,0}_{\mathscr{X}})$ to $Ra_*(\mathfrak{E} \otimes \Omega^\bullet_{\mathscr{X}})$ is not so straightforward. To this end, one uses the V-filtration associated to ${}^\square\mathscr{E}$ defined over a neighbourhood $\Delta(\lambda_0, \epsilon_0)$ and the pieces of the weight filtrations whose sections are L^2 in order to construct $\mathcal{Q}^{(\lambda_0),\bullet}$ on $\mathscr{X}(\lambda_0, \epsilon_0)$ for each $\lambda_0 \in \mathbb{C}_\lambda$. Using Zucker's norm estimates one shows that that $\mathcal{Q}^{(\lambda),\bullet}|_{\mathbb{X}\times\{\lambda_0\}}$ is quasi-isomorphic to $S(\mathcal{E} \otimes \Omega^{\bullet,0}_{\mathscr{X}})|_{\mathbb{X}\times\{\lambda_0\}}$. For a discussion on how L^2-norm estimates on a harmonic bundle behaves with respect to the sections of parabolic filtrations, V-filtrations and the monodromy weight filtrations see Lecture 14. Since the cohomologies of the pushforwards of both complexes form coherent sheaves on the disc $\Delta(\lambda_0, \epsilon_0)$ and their fibres are already isomorphic, one can argue using the Nakayama lemma for graded rings to conclude

$$\mathcal{H}^i(Ra_*\mathcal{Q}^{(\lambda),\bullet}) \simeq \mathcal{H}^i(a_*S(\mathcal{E} \otimes \Omega^{\bullet,0}_{\mathscr{X}}))\big|_{\Delta(\lambda_0,\epsilon_0)} \simeq \mathrm{Harm}^i \otimes_{\mathbb{C}} \mathcal{O}_{\Delta(\lambda_0,\epsilon_0)}.$$

This is [Moc07, Lemma 20.38]. The quasi-isomorphism between the complexes $\mathcal{Q}^{(\lambda),\bullet}$ and $\mathfrak{E} \otimes \Omega^\bullet_{\mathscr{X}}$ follows from the properties of the V-filtration [Moc07, Lemma 20.35] completing the proof.

□

References

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