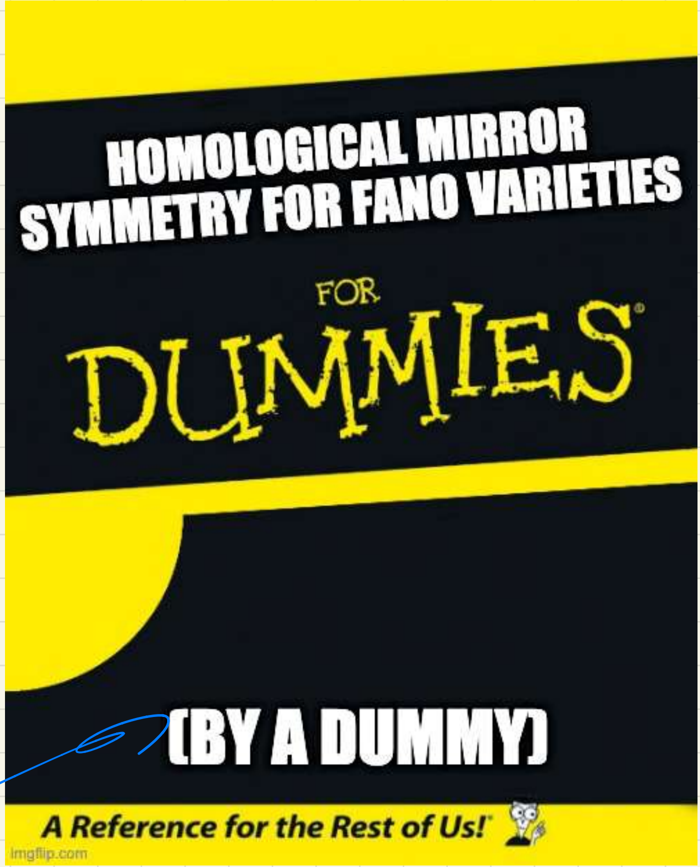
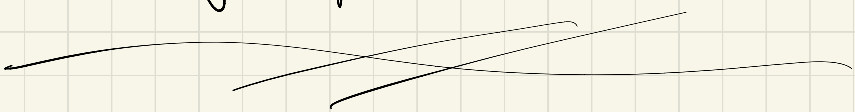


important disclaimer

work w/ a running example: $\mathbb{P}^2$



X Fano variety

$D \hookrightarrow X$ anticanonical divisor, defined by σ_D

then $\omega_{X \setminus D} \cong \mathcal{O}_{X \setminus D}$ induced by σ_D

$\Rightarrow U = X \setminus D$ Calabi-Yau

Pretend: we understand aspects of (H)MS for U

say that V is the mirror to U

Goal: put D back in to U

= choose a regular function on V

reference: Auroux, 2007, [MR 2386535]

What should HMS then look like?

Def (Y, w) Landau-Ginzburg model

Y smooth quasiprojective

$w : Y \longrightarrow \mathbb{A}^1$ superpotential

$$\boxed{\text{HMS picture}}$$

Fano: 4 types of categories, for 2 types objects

CY: 2 types of categories, for 1 type of objects — Fukaya / $D^b(X)$

A-side = work-in-progress   B-side = well-established

$$\begin{tikzcd} X & \text{④ } \mathrm{Fuk}(X) \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & D^b(X) \text{ ①} \arrow[dl, leftrightarrow] \\ (Y, w) & \mathrm{FS}(Y, w) \text{ ②} \arrow[r, leftrightarrow] & \mathrm{MF}(Y, w) \text{ ③} \end{tikzcd}$$

Fukaya-Seidel   matrix factorizations

[!] the categories are very different

& interaction

Goal: "define" + discuss for $\mathbb{P}^2$

+ discuss the interaction

$\boxed{1)\ D^b(X)}$ a) determines X (Bondal-Orlov)

b) always has semiorthogonal decompositions

e.g. $\mathcal{O}_X$ exc. object

we want to see them in the mirror

$\boxed{2)\ FS(Y, w)}$ I / we don't know how to define this in general

in some sense: people didn't care / didn't realise the interest in these objects

very instance-based: dependent on situation

today: Seidel's category of lagrangian vanishing cycles

need $w: Y \to \mathbb{A}^1$ be very nice: symplectic Lefschetz fibration

i.e. singularities of fibers are

reference: Auroux-Katzarkov-Orlov, 2006 [MR2257391]

Construction

$$\begin{cases} X = \mathbb{P}^2 \\ D^b(\mathbb{P}^2) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle \end{cases}$$

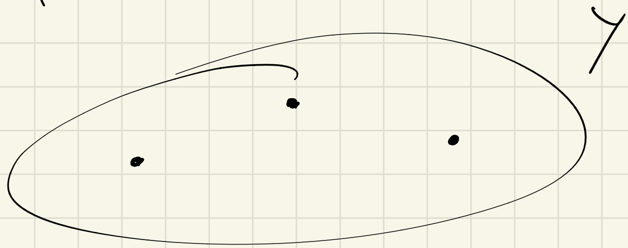
$$\begin{cases} Y = (\mathbb{G}_m)^2 = \operatorname{Spec} \mathbb{C}[x^{\pm}, y^{\pm}] \\ w = x + y + \dfrac{1}{xy} \end{cases}$$

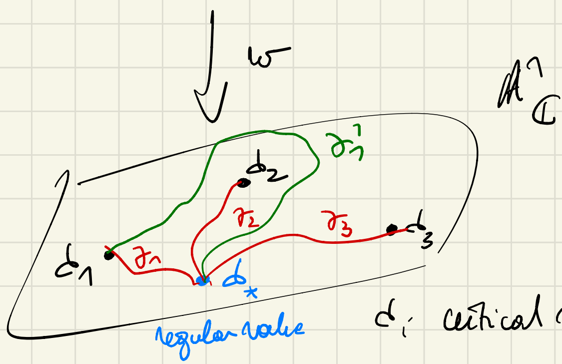
exceptional objects

$w : Y \to \mathbb{A}^1$ has isolated critical points

+ distinct critical values

Q: is Y big enough to see all critical values / critical points





1) pick regular value

2) choose paths γ_i from d_* to d_i ordered clockwise

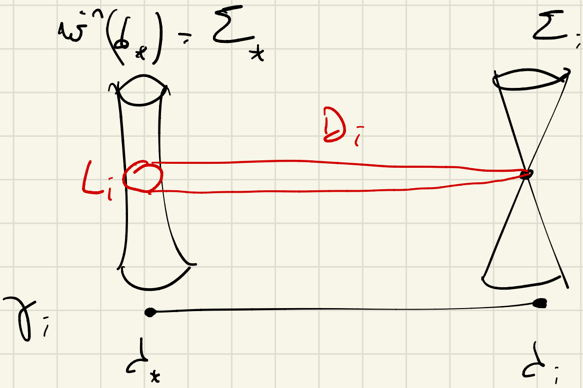
d_i critical values

$$d_1 = 3, \quad d_2 = 3\zeta, \quad d_3 = 3\zeta^2$$

$$\zeta = e^{\frac{2\pi i}{3}}$$

Remark: Seidel has shown independence of choices

ϑ_i:



D_i lagrangian thimble

$L_i := \partial D_i$ vanishing cycle $\subset \Sigma_*$

$\rightsquigarrow L_1, L_2, L_3$ in Σ_*

we can make them intersect transversely

<u>Def</u>: $Lag_{vc}(w, \{\gamma_i\})$ is A_∞-category

- objects: L_1, L_2, L_3
- morphisms: $\mathrm{Hom}(L_i, L_j) := \begin{cases} \mathbb{C}^{\# L_i \cap L_j} & i<j \\ \mathbb{C} \cdot \mathrm{id} & i=j \\ 0 & i>j \end{cases}$

composition law $=$ A_∞-category

$=$ compositions of all arities

$m_k : \mathrm{Hom}(L, L') \otimes \dots \otimes \mathrm{Hom}(L''', L'''') \to \mathrm{Hom}(L, L'''')$

these are given by Lagrangian Floer homology

cause of technical issues

but for $\mathbb{P}^2$: [AKO] show that only m_2 is nonzero

$\rightsquigarrow$ hence we can reinterpret the A_∞-category

as a quiver w/ relations

to understand the quiver: need to know $\# L_i \cap L_j$

[AKO]: this number is 3 $\forall i < j$.

$\boxed{?}$ $D^b(\mathbb{P}^2) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$

$$\begin{tikzcd} \circ \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \circ \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \circ \end{tikzcd} + \text{relations};$$

3 quadratic relations

$\Rightarrow$ $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(2)) = 6$-dim.

$\boxed{?}$ with $\# L_i \cap L_j = 3$

i.e. $\# L_1 \cap L_3 = 3$

$$= \langle \mathcal{O}, T_{\mathbb{P}^2}(-1), \mathcal{O}(1) \rangle$$

will have correct Hom. spaces

<u>Conjecture</u> (HMS) (if $Y \xrightarrow{w} \mathbb{A}^1$ is nice)

$$D^b(X) \cong D^b \mathrm{Lag}_{vc}(w, \{\gamma_i\})$$

A_∞-cat $\mathcal{A}$ $\rightsquigarrow \exists D^b(\mathcal{A})$ triangulated, or dg category

<u>Theorem</u> [AKO] Yes for $\mathbb{P}^2$, for del Pezzo surfaces

very ad hoc: just checking End-algebras of full exc. collections

quasi-equivalence of dg categories

<u>Example for Thomas</u> $\mathbb{P}^3$:

$$\begin{array}{c} Y = (\mathbb{C}^\times)^3 \\ \big\downarrow w \\ \mathbb{A}^1 \end{array} \qquad x + y + z + \frac{1}{xyz}$$

for the other equivalence: $\nexists$(?) proof of HMS

$\boxed{\text{3) } MF(Y, w)}$ $\mathbb{Z}/2\mathbb{Z}$ - graded dg category

$\exists$ explicit definition via objects + morphisms + differential

(± 2001) <u>Orlov</u>: + Buchweitz (1980)

$$\underbrace{H^0(MF(Y,w))}_{\text{triangulated cat.}} \cong D^b_{sg}(w^{-1}(0)) := D^b(w^{-1}(0)) / \mathrm{Perf}\, w^{-1}(0)$$

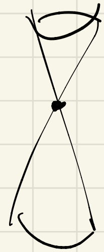
$w + d,\ d \in \mathbb{A}^1$; d

coh of the fiber

$\delta \in \mathbb{A}^n$ not a critical point, $w^{-1}(\delta)$ smooth

$$D^b_{sg} = 0$$

$d \in \mathbb{A}^n$ critical: singularities of $w^{-1}(\delta)$ will determine the geometry

e.g. for $\mathbb{P}^2$:  , $D^b_{sg}(w^{-1}(d_i)) \simeq D^b(k)$

i.e. $\exists!$ exceptional object

now we combine all them:

$$\boxed{\bigoplus_{d \in \mathbb{A}^n} D^b_{sg}(w^{-1}(d)) \;=\; MF(Y, w)}$$

completely orthogonal exc. collection of length 3

$\boxed{4)\ \mathrm{Fuk}(X)}$ X as symplectic manifold

run away

$\rightsquigarrow$ $\mathbb{Z}/2\mathbb{Z}$ A_∞-category (+ curvature)

reference: Sheridan, 2016, MR 3578916, §2 quasi-eq. of A_∞-cat.

<u>Conjecture</u> $\mathrm{Fuk}(X) \cong^{?}_{\text{mod}} MF(Y, w)$

$\leftarrow$? natural ^orthogonal decomposition

<u>Remarks</u> o At Fukaya side the decomposition is given by eigenvalues of $c_1(X) * -$ on $QH^\bullet(X)$

is / should be $HH^\bullet(\mathrm{Fuk}(X))$

- orthogonal decomposition
- $\mathrm{Fuk}(X)$ is always a Calabi-Yau

$\exists$ relation between the 2 mirror pictures

= Dubrovin's conjecture + various amplifications

<u>Conjecture</u> if $QH^\bullet(X)$ is semisimple then $D^b(X)$ has full exc. collection

$\overset{\text{12}}{FS(Y,w)}$

eigenvalues

critical values

Fun coda to the talk: = down-to-earth way of understanding MS for Fanos

1) X Fano

GW theory gives us the quantum period

$$\boxed{G_X(t) := \sum_{n\geq 0} p_n t^n}$$

$$p_n := \int_{[\mathcal{M}_{0,1}(X,n)]^{vir}} \psi^{n-2}\, ev^*(pt) \in \mathbb{Q}$$

here $\psi := c_1(\omega_\pi)$

$$\pi : \mathcal{M}_{0,1}(X,n) \longrightarrow \mathcal{M}_{0,0}(X,n)$$

$$\boxed{\hat{G}_X(t) := \sum_{n\geq 0} n!\, p_n t^n}$$

regularised quantum period

2) (Y, w) LG model

f is Laurent polynomial in $\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$

$$\begin{tikzcd}
(\mathbb{C}^\times)^n \arrow[r, phantom, "\subseteq"] \arrow[d, "f"'] & Y \arrow[d, "w"] \\
\mathbb{A}^1 \arrow[r, phantom, "="] & \mathbb{A}^1
\end{tikzcd}$$

classical period of f

$$\pi_f(t) := \left(\frac{1}{2\pi i}\right)^n \int_{|x_1| = \dots = |x_n| = 1} \frac{1}{1 - tf} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

$$= \sum_{m \geq 0} q_m t^m$$

q_m is the constant coefficient of the mth power of f

Conjecture: $$\boxed{\hat{G}_X(t) = \pi_f(t)}$$

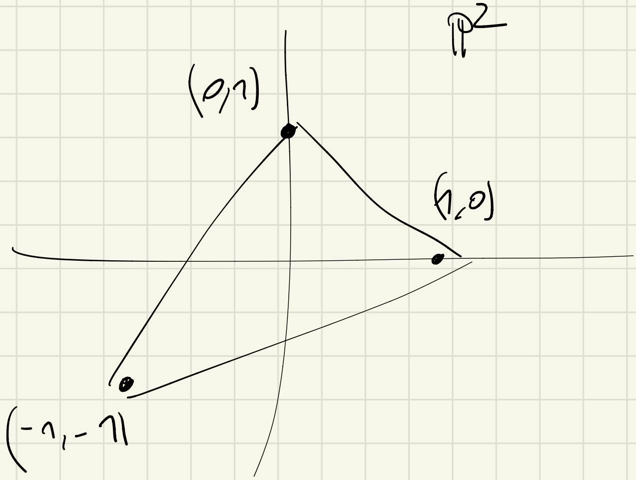
use RHS as fingerprint of a Fano

Fanosearch: find all possible $\pi_f(t)$ via combinatorial methods (toric geometry enters)

then match these to the desired classification of Fano varieties

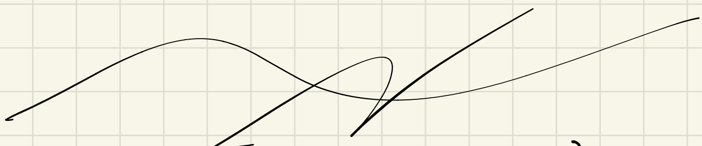
Mandel, ... Frobenius ...

1901.06155, Petracci



$$x + y + \frac{1}{xy} = \sum_{v \in F} x^v$$

P. Smith, 2016?, ... quadrics ...



$$\begin{tikzcd}
\overset{K3}{D^b(D)} \arrow[r, "\text{spherical functor}"'] \arrow[d, equal] & \overset{\mathbb{P}^3}{D^b(X)} \arrow[d, equal] \\
FS(Y, w, \dots) \arrow[r] & FS(Y, w)
\end{tikzcd}$$

$X \setminus D = U$

wrapped FS