Families of Divisors after János Kollár

Informal Preprint Seminal

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Nodes $\rightsquigarrow$ Nodes in codimension 1. genus>2 $\rightsquigarrow$ $K_X + D$ ample.

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$\mathscr{Y}$ =Weighted blow up of $(w^{25}(xyz^4 + y^6) + w^{21}z^{10} + t^{30}w^{31} + x^{31} + y^{31} + z^{31} = 0) \subset \mathbb{P}(x; y; z; w) \times \mathbb{C}[t]$ at $(0, 0, 0, 1) \times 0$.

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What fails: Chow semistablity $\overset{\text{Mumford}}{\Longrightarrow}$ $\operatorname{mult}_x \mathscr{Y}_0 < (\dim \mathscr{Y}_0 + 1)!$.

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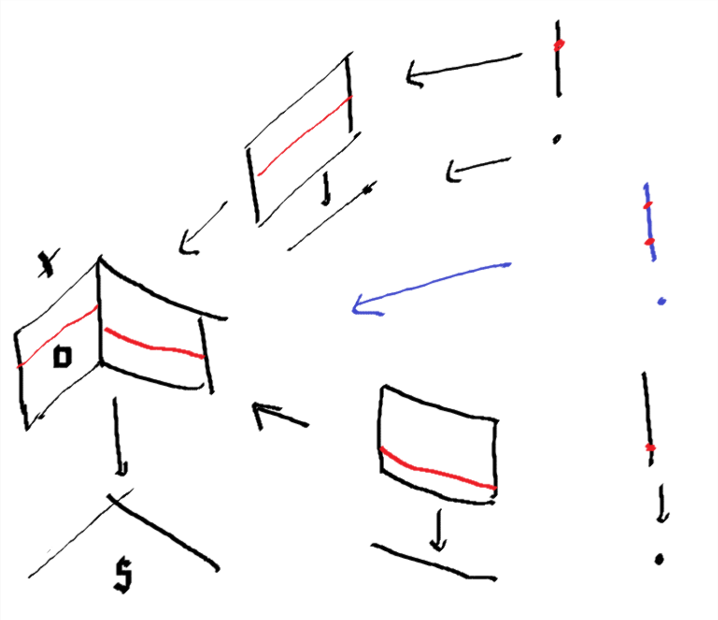
Why normality of S is necessary?



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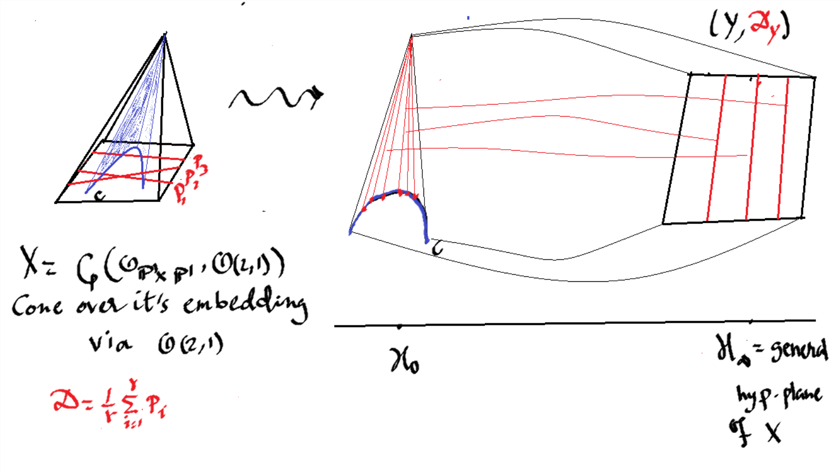

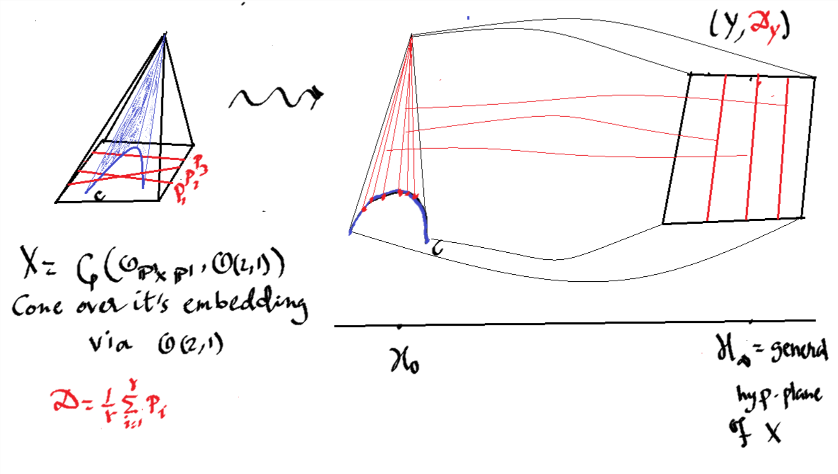

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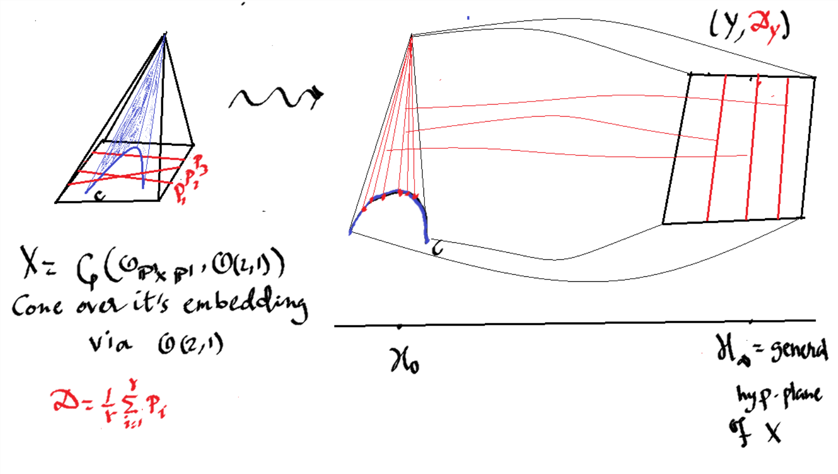

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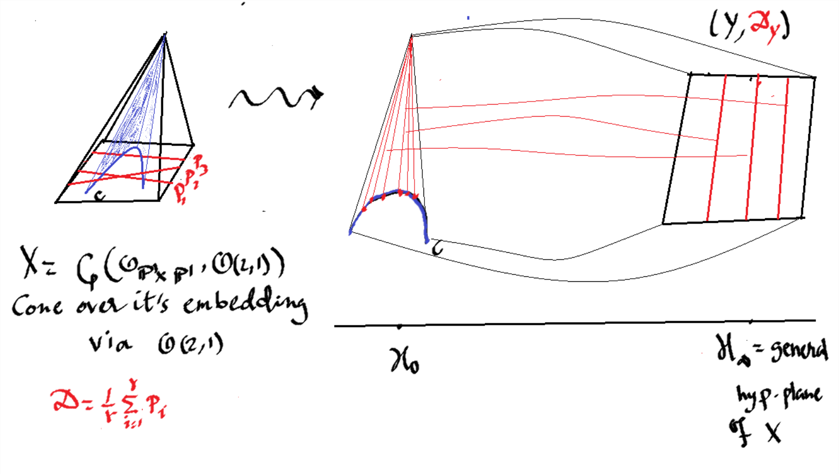

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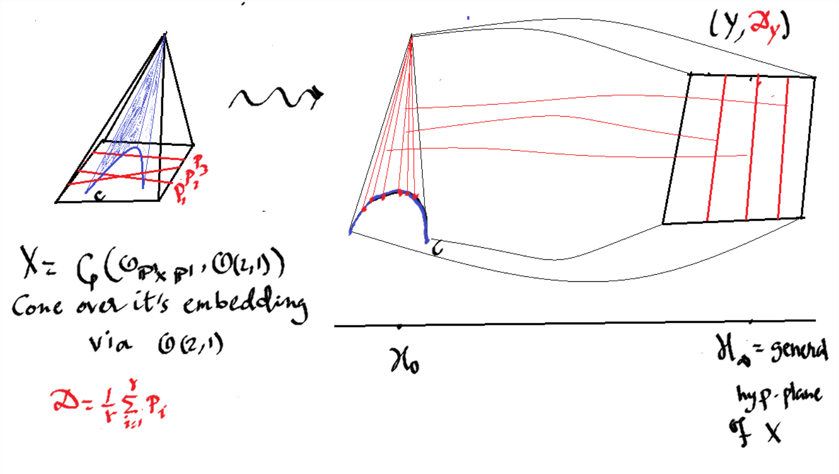

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- $g\colon D \to S$ is flat at generic points of D_s for every $s \in S$.

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The deformation space is infinite dimensional.

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Kollár's solution: K-flatness

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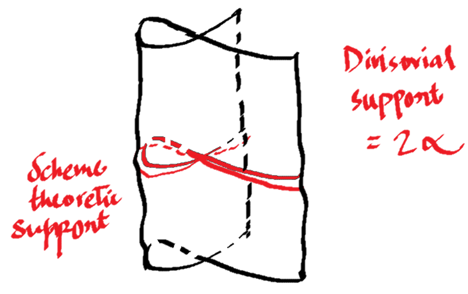
Key Theorem [Kollár, Moduli Book, Theorem 2.93]

D is relatively Cartier iff $\tau^{[*]}(D)$ is relatively Cartier for all Artin subscheme $\tau\colon A \hookrightarrow S$.

Divisorial support of a coherent sheaf



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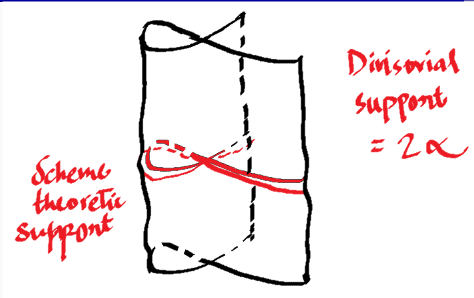

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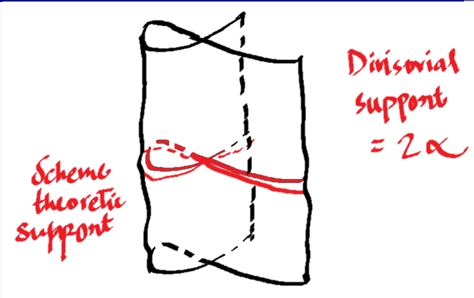

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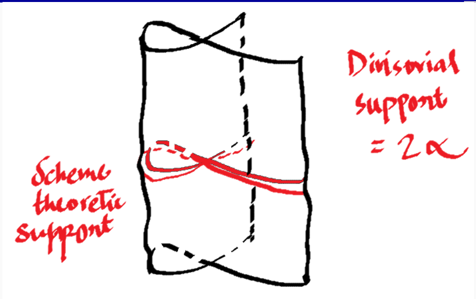

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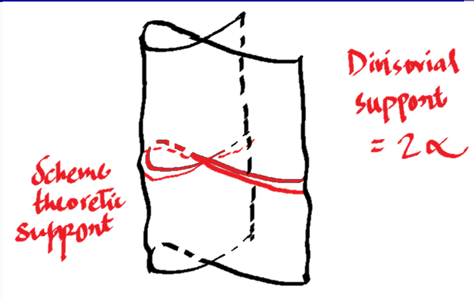

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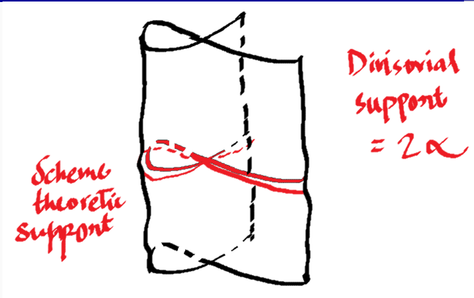

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Compatibility

Let $h\colon T \to S$ then $\operatorname{DSupp}_S(h_X^*\mathscr{F}) \simeq h^{[*]} \operatorname{DSupp}_S(\mathscr{F})$.

Where is the definition coming from?

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Theorem 61, Kollár'19.

$Ch(\mathcal{F})$ is relatively Cartier iff $p_*(\mathrm{Supp}(\mathcal{F}))$ is relatively Cartier for all $\mathcal{O}_S$-linear projection $p : \mathbb{P}^n_S \dashrightarrow \mathbb{P}^{d+1}_S$ that is finite on $\mathrm{DSupp}(\mathcal{F})$.

Main Theorem (Kollár'19)

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Theorem 4

Let $f\colon X \to S$ be a flat, projective morphism with S_2 fibres of pure dimension n. Then the functor $KDiv_d(X/S)$ of K-flat relative Mumford divisors of degree d is representable by a separated scheme of finite type $\mathrm{KDiv}_d(X/S)$.

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5. The obstruction theory of K-flat is not known.

New definition of the moduli functor

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Questions?

Thank You!