"FIBERS OVER INFINITY of LANDAU-GINZBURG MODELS"

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MIRROR SYMMETRY for CALABI-YAU VARIETIES

- phenomenon of duality b/w CY
- various approaches: HMS, SYZ MS, combinatorial/toric
- "topological MS": $h^{p,q}(X) = h^{n-q,p}(X^\vee)$

MIRROR SYMMETRY for FANO MANIFOLDS

- relation b/w Fano manifold X & (Y,w) Landau-Ginzburg model
- various approaches: HMS
- relate Hodge theoretic data of X to Hodge theoretic data of (Y,w)?
  geometry of X to invariants of (Y,w)?

defn: a Landau-Ginzburg (LG) model (Y, w) is

- Y smooth quasi-projective variety of dim n
- $w: Y \longrightarrow \mathbb{A}^1_{\mathbb{C}}$ regular function
- $K_Y \sim 0$
- fibres of w are cpt

} general fibre: CY of dim $n-1$

Interpretation of "X Fano manifold is mirror to (Y, w) LG model":

X Fano of $\dim(X) = n$ $(n=3)$

W smooth anticanonical divisor

$X \setminus W$ Calabi-Yau $\longleftrightarrow$ Y mirror CY

cpt $X \setminus W$ to X $\longleftrightarrow$ equip Y with $w: Y \longrightarrow \mathbb{A}^1_{\mathbb{C}}$

W cpt CY $\longleftrightarrow$ V fibre of w of dim $n-1$

K3 surface

$\bigoplus H^{p,p}(W)$ $\longleftrightarrow$ $H^{n-1}(V, \mathbb{C})$

cup product $\left((-) \cup c_1(X)|_W \right)$ $\longleftrightarrow$ $N_{n-1} = \log T_{n-1}$ monodromy action around ∞

ample

$N_{n-1}^{n-1} \neq 0$ maximally type III

__Setting:__ X Fano manifold $\dim X = n$ $\rightsquigarrow$

$$\begin{tikzcd} (\mathbb{C}^\times)^n \arrow[r, hook] \arrow[d, "p"] & Y \arrow[r, hook] \arrow[d, "w"] & Z \arrow[d, "f"] \\ \mathbb{C} \arrow[r, equal] & \mathbb{C} \arrow[r, hook] & \mathbb{P}^1_{\mathbb{C}} \end{tikzcd}$$

p Laurent polynomial

del Pezzo

Fano 3-folds; list

complete intersec in $\mathbb{P}^N$

Grass

c.i. in Grass

defn: (Z, f) log CY cpt of p

Z smooth proper variety

$f^{-1}(\infty)$ reduced snc divisor

$f^{-1}(\infty) \sim -K_Z$

X Fano, $\dim(X) = n$ $\rightsquigarrow$

$$\begin{tikzcd} (\mathbb{C}^\times)^n \arrow[r, hook] \arrow[d, "p"] & Y \arrow[r, hook] \arrow[d, "w"] & Z \arrow[d, "f"] \\ \mathbb{C} \arrow[r, equal] & \mathbb{C} \arrow[r, hook] & \mathbb{P}^1_{\mathbb{C}} \end{tikzcd}$$

number of reducible fibres in w (resp f) and number of irreducible components in reducible fibres of w (resp f) do not depend on choice of (Y, w), (Z, f)

Conj [Katzarkov - Kontsevich - Pantev]: $h^{p,q}(X) = f^{n-p,q}(Y, w) =$ dim graded pieces of MHS (Harder)

Conj: $h^{1,n-1}(X) = \begin{cases} \sum_{s \in \mathbb{C}} (\rho_s - 1) & \text{if } n > 2 \\ \sum_{s \in \mathbb{C}} (\rho_s - 1) + 1 & \text{if } n = 2 \end{cases}$

$\rho_s = \#$ components in $w^{-1}(s)$

Conj: $\chi(-K_X) = \rho_\infty + 1 \geq 2$ + conj on existence of toric LG (p)

$\parallel$

$h^0(-K_X)$ $\dim X \leq 5$

variety X	KKP conj: $h^{p,q} = f^{q,n-p}$	$h^{1,n-1}(X) = \sum(\rho_s - 1)$ $(+1)$	$\chi(-K_X) = \rho_\infty + 1$
del Pezzo surf	✓ Lunts - Prz.	✓	✓ Auroux - Katzarkov - Orlov
Fano threefold (105 families)	Harder's work: $f^{p,q}$ can be computed using geometry of log CY cpt Z ✓ Cheltsov - Prz [18] family by family comp.	Prz: rk $\mathrm{Pic}(X) = 1$ [13] (17 families) ✓	Prz: $-K_X$ very ample [17] (98 families) complete proof for all Fano threefold
complete int. in $\mathbb{P}^N$		✓ Prz - Shramov - combinatorial comp of Hodge numbers - resolution procedure	Prz: description [18] fibre over infinity complete proof for complete intersection
toric variety w/ dual toric variety admitting crep res			✓

EXAMPLE: CUBIC SURFACE and CUBIC THREEFOLD
(particular case of complete intersection in $\mathbb{P}^N$)

$X_d \subseteq \mathbb{P}^N$, $I = N+1-d$

Laurent polynomial: [Givental]

$$P_X = \frac{(x_1 + x_2 + \ldots + x_{d-1}+1)^d}{x_1 \cdots x_{d-1}\, y_1 \cdots y_{I-1}} + y_1 + \ldots + y_{I-1} \in \mathbb{C}[x_i^{\pm}, y_i^{\pm}]$$

Singular LG model:

$$\{ y_0^I (x_1 + \ldots + x_d)^d = (\lambda y_0 + y_1 + \ldots + y_{I-1}) y_1 \cdots y_{I-1}\, x_1 \cdots x_d \}$$

$$\subseteq \mathbb{P}^{d-1}_{x_1,\ldots,x_d} \times \mathbb{P}^{I-1}_{y_0,\ldots,y_{I-1}} \times \mathbb{A}^1_\lambda \qquad (d, I)$$

in general:

Let X be a Fano complete intersection in $\mathbb{P}^N$ of hypersurfaces of degrees $d_1, \ldots, d_k$, let i_X be its Fano index, and let p be the Laurent polynomial

$$\frac{\prod_{i=1}^k (x_{i,1} + \ldots + x_{i,d_i-1} + 1)^{d_i}}{\prod_{i=1}^k \prod_{j=1}^{d_i-1} x_{i,j} \prod_{j=1}^{i_X-1} y_j} + y_1 + \ldots + y_{i_X-1} \in \mathbb{C}[x_{i,j}^{\pm 1}, y_s^{\pm 1}],$$

which we consider as a regular function on the torus $(\mathbb{C}^*)^n$, where $n = \dim(X)$. Let Δ be

Strategy: crepant resolution of $LG_5(X)$

$X_3 \subseteq \mathbb{P}^3$

$$p = \frac{(x_1+x_2+1)^3}{x_1x_2}$$

$$LG_5(X) = \{\mu(x_1+x_2+x_3)^3 = \lambda x_1x_2x_3\} \subseteq \mathbb{P}^2_x \times \cancel{\mathbb{A}^1_\lambda}\ \mathbb{P}^1_{[\lambda:\mu]}$$

state: $\{a := x_1+x_2+x_3 = x_J = 0\}$

locally: $a^3 = \lambda x_J$ sing of type A_2

$$\boxed{\rho_0 = 1 + 3\cdot 2 = 7 = h^{1,1}(X_3)}$$

$\rho_\infty = 3$, $\boxed{\rho_\infty + 1 = \chi(-K_X) = 1 + (-K_X)^2 = 4}$

$X_3 \subseteq \mathbb{P}^4$

$$p = \frac{(x_1+x_2+1)^3}{x_1x_2y_1} + y_1$$

$$LG_5(X) = \{y_0^2(x_1+x_2+x_3)^3 = (\lambda y_0 + y_1)y_1x_1x_2x_3\}$$
$$\subseteq \mathbb{P}^2_x \times \mathbb{P}^1_y \times \mathbb{A}^1_\lambda$$

state: $\{\underset{x_1+x_2+x_3}{\overset{}{a}} = y_1 = x_J = 0\}$

locally: $a^3 = \lambda y_1 x_J$

$$\boxed{\rho_0 = 1 + 5 = 1 + h^{1,2}(X_3)}$$