Categorical Resolutions of A_2 Singularities

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Abstract

Let X be a projective variety with an isolated A_2 singularity. We study its bounded derived category and prove that there exists a crepant categorical resolution $\pi_*\colon \widetilde{\mathcal{D}} \to D^b(X)$, which is a (Verdier) localization. Furthermore, we give an explicit description of a generating set for its kernel. In the case of a fourfold with a single A_2 singularity we also prove that this generating set is given by two 2-spherical objects. If X is a cubic fourfold with an isolated A_2 singularity, we show that this resolution restricts to a crepant categorical resolution $\widetilde{\mathcal{A}}_X$ of the Kuznetsov component $\mathcal{A}_X$ of X, which is equivalent to the bounded derived category of a smooth K3 surface S. Finally, we give an explicit description of a generating set for the kernel of $\pi_*\colon \widetilde{\mathcal{A}}_X \to \mathcal{A}_X$ as elements of the derived category $D^b(S)$. This work is based on recent results by A. Kuznetsov and E. Shinder about the derived category of projective varieties with isolated A_1 singularities.

Contents

Introduction

In recent decades, the study of the bounded derived category of coherent sheaves on a variety has become a powerful and versatile tool in algebraic geometry. For example, in the celebrated paper [Kuz10] by A. Kuznetsov, the study of the derived category of cubic fourfolds led to a new approach to the open problem of their rationality. Another example is [Bay+21], where the authors considered, among other things, the moduli space of Bridgeland semistable objects of the Kuznetsov component of a cubic fourfold, which is a K3 category, in order to construct an interesting family of hyperkähler manifolds of K3 type. While much is already known about derived categories of smooth projective varieties, the study of derived categories of singular varieties has become a very active topic of current research only in the past few years. When working classically with a singular variety, one often tries to understand its singularities by studying resolutions of singularities. This idea has a categorical manifestation, namely the notion of categorical resolutions of a triangulated category. Through this abstraction it is often possible to construct special kinds of resolutions on the categorical level that do not exist on the level of algebraic varieties, such as crepant resolutions, which, informally speaking, are certain particularly small resolutions.

Let X be a projective variety with rational singularities and consider a resolution of singularities $\pi\colon \widetilde{X} \to X$. On the level of bounded derived categories, there exist exact functors

$$\pi_*\colon D^b(\widetilde{X}) \to D^b(X) \quad \text{and} \quad \pi^*\colon D^{\mathrm{perf}}(X) \to D^b(\widetilde{X}), \tag{0.0.1}$$

where π^* is left adjoint to π_* on $D^{\mathrm{perf}}(X)$. Since X has rational singularities, the functor π^* is fully faithful. We recall that, more generally, a categorical resolution of a triangulated category $\mathcal{D}$ is defined as a triple $(\widetilde{\mathcal{D}}, \pi_*\colon \widetilde{\mathcal{D}} \to \mathcal{D}, \pi^*\colon \mathcal{D}^{\mathrm{perf}} \to \widetilde{\mathcal{D}})$, where the category $\widetilde{\mathcal{D}}$ is a full admissible subcategory of the bounded derived category of a smooth projective variety together with an adjunction $\pi^* \vdash \pi_*$ such that the functor π^* is fully faithful.[1]

A natural question one could ask is if there exists a minimal categorical resolution $\mathcal{D}$ of $D^b(X)$, i.e. a resolution $\mathcal{D}$, such that for every other resolution $\widetilde{\mathcal{D}}$ of $D^b(X)$ there exists a fully faithful embedding $\mathcal{D} \subset \widetilde{\mathcal{D}}$. To this end, let us recall the following conjecture by A. Bondal and D. Orlov.

Conjecture 0.1 ([BO02, Conjecture 10], [Abu16, Conjecture 1.0.1])**.** *Let X be a Gorenstein projective variety with canonical singularities*[2]*. Assume that there exists a crepant resolution of singularities $\widetilde{X} \to X$. Then for any other resolution of singularities $Y \to X$, there exists a fully faithful embedding*

$$D^b(\widetilde{X}) \hookrightarrow D^b(Y).$$

Thus, defining the "right" notion of crepancy for categorical resolutions could lead to a possible answer to the above question. In this thesis, we will use a notion of crepancy proposed by A. Kuznetsov in [Kuz08]. According to this definition, a categorical resolution $(\widetilde{\mathcal{D}}, \pi_*, \pi^*)$ is crepant if there exists an adjunction $\pi_* \vdash \pi^*$ restricted to $D^{\mathrm{perf}}(X)$. For an explanation why this is a very natural condition for crepant categorical resolutions, we refer to Remark 1.5. In the case of a variety X with an A_1 singularity, the existence of a crepant categorical resolution of $D^b(X)$ was proved in

[1] See [Kuz08, Definition 3.2].

[2] For the definition of canonical singularities we refer to [Rei85, Definition 1.1].

[KS23a, Theorem 5.8] and [Cat+22, Proposition 3.5]. It is done by using a general method, introduced by A. Kuznetsov in [Kuz08], which works as follows. Consider the blow-up $\pi\colon \widetilde{X} := \mathrm{Bl}_x(X) \to X$ of the singular point $x \in X$. It provides a resolution of singularities for X, where the exceptional divisor $Q \overset{j}{\hookrightarrow} \widetilde{X}$ is a smooth quadric, see Lemma 1.8. One can prove that the derived category $D^b(Q)$ admits a special kind of semiorthogonal decomposition, which, informally speaking, separates some simpler parts of the derived category from more interesting parts. By an application of [Kuz08, Proposition 4.1], we can descend this decomposition via the pushforward functor j_* to a semiorthogonal decomposition of the resolution $D^b(\widetilde{X})$, which also consists of "simple parts" and an interesting subcategory $\widetilde{\mathcal{D}}$. This subcategory, together with the restrictions of the pushforward and pullback functors π_* and π^* to $\widetilde{\mathcal{D}}$, will define a categorical resolution of $D^b(X)$. In the case X has an isolated A_2 singularity, we will use the same approach to construct a crepant categorical resolution of $D^b(X)$, see Theorem 2.1.

Since our goal is to understand the structure of the bounded derived category $D^b(X)$ of a variety with a single isolated A_2 singularity, we start by recalling what is already known in the case of A_1 singularities.

Theorem 0.1 ([Cat+22], [KS23a, Section 6])**.** *Let X be a projective variety with single isolated A_1 singularity and assume $\dim(X) \geq 2$. Then there exists a crepant categorical resolution $\pi_*\colon \widetilde{\mathcal{D}} \to D^b(X)$, which is a (Verdier) localization, whose kernel is generated by a single 2-spherical or 3-spherical object if X is even or odd dimensional, respectively. More precisely, we have*

$$\ker(\pi_*) = \begin{cases} \langle j_*\mathscr{S}\rangle, & \text{if } \dim(X) \text{ is odd} \\ \langle \mathrm{cone}(j_*\mathscr{S}_1 \to j_*\mathscr{S}_2[2])\rangle, & \text{if } \dim(X) \text{ is even,} \end{cases}$$

where $\mathscr{S}$ and $\mathscr{S}_1, \mathscr{S}_2$ denote the spinor bundles on the exceptional divisor $Q \overset{j}{\hookrightarrow} \widetilde{X} = \mathrm{Bl}_x(X)$, in the case Q is even or odd dimensional, respectively.

Furthermore, by an application of results of [KKS22], there already exists a statement similar to Theorem 0.1 in the case of surfaces with A_2 singularities. More precisely, let X be a surface with an isolated A_2 singularity. Then blowing up X at the singular point x yields a resolution of singularities and the exceptional divisor $Q \subset \mathbb{P}^2$ is a nodal quadric curve, which can be thought of as two curves C_1, C_2 isomorphic to $\mathbb{P}^1$ that intersect transversally, cf. Lemma 1.8. This can be summarized in a cartesian diagram

$$\begin{tikzcd} Q \arrow[r, hook, "j"] \arrow[d] & \widetilde{X} := \mathrm{Bl}_x(X) \arrow[d, "\pi"] \\ \{x\} \arrow[r, hook] & X. \end{tikzcd}$$

In this situation, we can apply [KKS22, Theorem 2.12] to the derived category $D^b(\widetilde{X})$[3] to see that the functor $\pi_*\colon D^b(\widetilde{X}) \to X$ induces an equivalence

$$\overline{\pi}_*\colon\ D^b(\widetilde{X})\Big/\langle j_*\mathcal{O}_{C_1}(-1), j_*\mathcal{O}_{C_2}(-1)\rangle \xrightarrow{\sim} D^b(X).$$

The left hand side denotes the Verdier quotient of $D^b(\widetilde{X})$ by the triangulated subcategory generated by the objects $j_*\mathcal{O}_{C_1}(-1), j_*\mathcal{O}_{C_2}(-1)$. Moreover, one easily verifies that these objects are in fact 2-spherical. This motivates the following theorem, which is the main result of this thesis and is proved in Section 3.3.

[3] To apply [KKS22, Theorem 2.12], we view this category as a trivial semiorthogonal decomposition of itself.

Theorem 0.2. *Let X be a fourfold with an isolated A_2 singularity at a point $x \in X$. Then the crepant categorical resolution $\pi_*\colon \widetilde{\mathcal{D}} \to D^b(X)$, constructed in Theorem 2.1, is a localization and its kernel is generated by two 2-spherical objects $j_*\mathcal{S}_1, j_*\mathcal{S}_2$. Here $\mathcal{S}_1, \mathcal{S}_2$ denote the spinor sheaves on the exceptional divisor $Q \overset{j}{\hookrightarrow} \mathrm{Bl}_x(X)$. In particular, the functor $\pi_*\colon \widetilde{\mathcal{D}} \to D^b(X)$ induces an equivalence of triangulated categories*

$$\overline{\pi}_*\colon\ \widetilde{\mathcal{D}}\Big/\langle j_*\mathcal{S}_1, j_*\mathcal{S}_2\rangle \xrightarrow{\ \sim\ } D^b(X). \tag{0.2.1}$$

At present it is not known for which kind of singularities on a variety X the kernel of a categorical resolution $\widetilde{\mathcal{D}} \to D^b(X)$ is generated by spherical objects, but if a resolution has this property, it is automatically crepant, cf. [KS23b, Lemma 5.8]. Moreover, observe that the spherical objects above induce autoequivalences of the triangulated category $\widetilde{\mathcal{D}}$, using the results of [AL17].

We now describe our proof strategy for Theorem 0.2. Its structure is similar to that of A_1 singularities. By an application of a powerful theorem by A. Efimov, see Theorem 3.11, we obtain that π_* is a (Verdier) localization. In Section 3.3, we explain in detail why we can apply it in our situation. This theorem also gives rise to an explicit set of generators of the kernel $\ker(\pi_*)$. In Section 3.2, we recall the definition of spinor sheaves on nodal quadrics and derive some basic facts about their cohomology. The key for demonstrating that the generators above are in fact 2-spherical is a calculation of the Ext-complexes of the spinor sheaves by Y. Kawamata, see Theorem 3.9. One other important result we use to explicitly determine the kernel $\ker(\pi_*)$ are the following semiorthogonal decompositions of the derived category of a nodal quadric, proved in Theorem 3.10, which can be thought of as a generalization of Kapranov's semiorthogonal decompositions of smooth quadrics, cf. Proposition 3.2.

Theorem 0.3. *Let $Y \subset \mathbb{P}^{n+1}$ be a nodal quadric hypersurface. In the case Y is odd dimensional, there exists a semiorthogonal decomposition*

$$D^b(Y) = \langle \mathcal{O}_Y(1-n), \mathcal{O}_Y(2-n), \ldots, \mathcal{O}_Y(-1), \langle \mathcal{S}_1, \mathcal{S}_2\rangle, \mathcal{O}_Y\rangle. \tag{0.3.1}$$

If Y is even dimensional, we have a semiorthogonal decomposition

$$D^b(Y) = \langle \mathcal{O}_Y(1-n), \mathcal{O}_Y(2-n), \ldots, \mathcal{O}_Y(-1), \mathcal{S}, \mathcal{O}_Y\rangle. \tag{0.3.2}$$

In Section 3.4, we refine Theorem 0.2 in the special case of cubic fourfolds. Recall that for a cubic fourfold with an A_1 singularity, A. Kuznetsov proved that there exists a crepant categorical resolution $\widetilde{\mathcal{A}}_X$ of the Kuznetsov component $\mathcal{A}_X$ of X that is equivalent to the bounded derived category of a (smooth) K3 surface S, which is a (2,3) complete intersection, see [Kuz10, Theorem 5.2]. In Section 2.2, we prove the analogous statement for a cubic fourfold with an A_2 singularity. The proof presented in [Kuz10, Theorem 5.2] generalizes to the case of an A_2 singularity without substantial changes, but the main difficulty lies in proving that the K3 surface S one can associate to X is in fact smooth, cf. Proposition 2.2. With this result at hand we can reformulate Theorem 0.2 in the following way.

Theorem 0.4. *Let X be a cubic fourfold with an isolated A_2 singularity and $t\colon S \to Q$ the inclusion map of the K3 surface S into the defining (nodal) quadric Q. Let $\mathcal{S}_1, \mathcal{S}_2$ denote the spinor sheaves on Q. Then the kernel of the crepant categorical resolution $D^b(S) \to \mathcal{A}_X$ constructed in Theorem 2.4 is generated by the spherical objects $t^*\mathcal{S}_1$ and $t^*\mathcal{S}_2$.*

The analogous statement for A_1 singularities was proved in [Cat+22, Section 4.1] and the proof can be extended to the A_2 case. Apart from the generalization of the results mentioned above, Proposition 2.2 also generalizes recent results of [BHS23], where the authors studied the Fano variety of lines $F(Y)$ of a special kind of cuspidal cubic fourfold Y. In this case the K3 surface S can be identified with a closed subscheme of $F(Y)$. Using the smoothness of the K3 surface in the A_2 case, proved in Theorem 2.2, their main result can be generalized to any cubic fourfold Y with an isolated A_2 singularity[4].

Outlook. We expect that Theorem 0.2 can be proven in any even dimension by using the results of [Add11] to perform necessary computations. Moreover, for higher A_n singularities we expect that there exists a sequence of n two spherical objects in a (crepant) categorical resolution $\widetilde{\mathcal{D}}$ of $D^b(X)$. By the results of [AL17] these objects would induce autoequivalences on $\widetilde{\mathcal{D}}$, which could be used to study the (crepant) categorical resolution $\widetilde{\mathcal{D}}$. These claims were already proved for surfaces. For example, if S is a K3 surface with a single isolated A_n singularity, then we can resolve it by successively blowing up the singular point $s \in S$. Let $\pi\colon \widetilde{S} \to S$ be this resolution. Then one can show that it is in fact crepant, cf. [Rei85], and the smooth variety $\widetilde{S}$ is again a K3 surface. In this case, the category $D^b(\widetilde{S})$ is indecomposable[5]. Moreover, the exceptional locus of π is a chain of intersections of (-2)-curves $C_1, \dots C_n$, with each of them being isomorphic to $\mathbb{P}^1$. By an application of [KKS22, Theorem 2.12] to the category $D^b(\widetilde{S})$, we obtain that the functor $\pi_*\colon D^b(\widetilde{S}) \to D^b(S)$ induces an exact equivalence

$$\overline{\pi}_*\colon\ D^b(\widetilde{S})\Big/\langle j_*\mathcal{O}_{C_1}(-1), \dots, j_*\mathcal{O}_{C_n}(-1)\rangle \xrightarrow{\sim} D^b(S).$$

Moreover, it is well-known that the 2-spherical objects $j_*\mathcal{O}_{C_1}(-1), \dots, j_*\mathcal{O}_{C_n}(-1)$ induce autoequivalences of $D^b(\widetilde{S})$ which furthermore generate an action of a braid group on this category, cf. [ST01, Proposition 3.19]. In the future it would be interesting to investigate similar results for higher dimensional varieties with A_n singularities.

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[4] The authors of [BHS23] remark at the beginning of Section 4.1 that their results can be generalized to any cuspidal cubic fourfold in the case where the associated K3 surface S is smooth.

[5] This is true for any smooth projective variety with trivial canonical bundle, see [Huy06, Exercise 8.8].

1 Preliminaries

Let X be an algebraic variety over a field k, i.e. X is an irreducible, noetherian, and separated scheme of finite type over k. If not stated otherwise, we assume k to be algebraically closed and of characteristic 0. By $D^b(X)$ we denote the bounded derived category of coherent sheaves on X, which is a k-linear triangulated category. For the definitions we refer to [Huy06].

1.1 (Geometric) resolutions of singularities

In this subsection $X, \widetilde{X}$ denote projective schemes over a field k of characteristic 0.

Definition 1.1. Let $d \in \mathbb{N}$ and $n = \dim(X)$. An isolated singularity $x \in X$ is an A_d *singularity*, if there exists an isomorphism

$$\widehat{\mathcal{O}_{X,x}} \cong k[[x_1, ..., x_{n+1}]]/(x_1^2 + \cdots + x_n^2 + x_{n+1}^{d+1}).$$

In the case $d = 1$ we say that X has a *node* at x, in the case $d = 2$ we say that X has a *cusp* at x.

Remark. In general, we say that a point $x \in X$ is a *hypersuface singularity*, if we have an isomorphism $\widehat{\mathcal{O}_{X,x}} \cong k[[x_1, ..., x_{n+1}]]/(f)$ for some $f \in k[x_1, \dots, x_{n+1}]$.

Definition 1.2. Let $(A, \mathfrak{m})$ be a noetherian local ring. Then A is called *Gorenstein* if it has finite injective dimension considered as an A-module, i.e. there exists an integer $n \in \mathbb{Z}$ such that $\operatorname{Ext}^N_A(-, A) = 0$ for all $N > n$. A locally noetherian scheme X is called *Gorenstein* if all the local rings $\mathcal{O}_{X,x}$ are Gorenstein.

Lemma 1.3 ([Eis95, Corollary 21.19])**.** *Let X be a variety with hypersurface singularities, which is smooth away from these points. Then X is Gorenstein.*

Definition 1.4. Let $\widetilde{X}$ be smooth. Then we call $\widetilde{X}$ together with a proper birational morphism $\pi\colon \widetilde{X} \to X$ a *resolution of singularities* for X.

Definition 1.5. Let X be normal and of finite type. Then X has *rational singularities* if we have

$$R^0\pi_*\mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X \quad \text{and} \quad R^i\pi_*\mathcal{O}_{\widetilde{X}} = 0$$

for any $i > 0$ and every resolution of singularities $\pi\colon \widetilde{X} \to X$.

Definition 1.6. Let $\pi\colon \widetilde{X} \to X$ be a resolution of singularities for X. Then we say that π is *crepant* if there exists an isomorphism $\omega_{\widetilde{X}} \cong \pi^*\omega_X$.

Let us recall the following basic result on the classification of quadrics over an algebraically closed field k with $\operatorname{char}(k) \neq 2$.

Proposition 1.7 ([GW10, Section 1.26])**.** *Let $Q \subset \mathbb{P}^n_{x_0,\dots,x_n}$ be a quadric hypersurface. Then the following statements hold.*

a) There exists an isomorphism $Q \xrightarrow{\sim} V_+(x_0^2 + \cdots + x_{r-1}^2)$ for some integer $r \geq 1$. We say that r is the rank of Q.

b) The quadric Q is smooth if and only if it has full rank, i.e. $r = n + 1$. It has one isolated A_1 singularity (and is smooth away from this point) if and only if it has corank 1, i.e. $r = n$.

c) *For $s \neq r$ the two quadrics $V_+(x_0^2 + \cdots + x_{r-1}^2)$ and $V_+(x_0^2 + \ldots x_{s-1}^2)$ are non-isomorphic.*

In particular, let Q_1 and Q_2 be quadrics (not necessarily in the same projective space), then they are isomorphic if and only if they have the same dimension and the same rank.

Lemma 1.8. *Let X be a variety with an isolated A_1 or A_2 singularity at a point $x \in X$. Then the blow-up $\widetilde{X} := \mathrm{Bl}_x(X)$ of X at the singular point x and the corresponding proper birational morphism $\pi\colon \mathrm{Bl}_x(X) \to X$ is a resolution of singularities for X. Let $Q \subset \widetilde{X}$ denote the exceptional divisor. Then Q is a smooth quadric if X has an A_1 singularity and Q is a nodal quadric if X has an A_2 singularity.*

Proof. Let us assume that X is of dimension $n \geq 2$. Then blowing up the singular point $x \in X$, gives us a cartesian diagram

$$\begin{tikzcd} Q \arrow[r, "j"] \arrow[d, "p"] & \widetilde{X} \arrow[d, "\pi"] \\ \{x\} \arrow[r, "i"] & X, \end{tikzcd}$$

where Q denotes the exceptional divisor and i and j denote the embeddings $\{x\} \subset X$ and $Q \subset \widetilde{X}$, respectively. Let us choose coordinates such that $x = [1 : 0 : \cdots : 0]$ and consider the affine chart $D(x) \cong \mathbb{A}^{n+1}$ with coordinates $x_1, \ldots, x_{n+1}$. We can compute the blow-up $\widetilde{X}$ in a formal local neighborhood of $x \in X$ as blowing up commutes with flat base change. Therefore, we can assume that $X = V(x_1^2 + \cdots + x_{n+1}^2)$, if X has an A_1 singularity and $X = V(x_1^2 + \cdots + x_n^2 + x_{n+1}^3)$ if it has an A_2 singularity. A computation of the embedded blow-up $\mathrm{Bl}_x(X) \subset \mathrm{Bl}_x(\mathbb{A}^{n+1})$ in local coordinates shows that the exceptional divisor Q is smooth in the first case and nodal in the latter case. In both cases this computation yields that the blow-up $\widetilde{X}$ is smooth. $\square$

Lemma 1.9. *Let X be Gorenstein variety with an isolated A_1 or A_2 singularity at a point $x \in X$ and let $\dim(X) = n + 1$. Denote by $\pi\colon \widetilde{X} := \mathrm{Bl}_x(X) \to X$ the blow-up that resolves the singularity of X and by $Q \subset \widetilde{X}$ the corresponding exceptional divisor, independent of the type of the singularity. Then there exists an isomorphism*

$$\omega_{\widetilde{X}} \cong \pi^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}((n-1)Q).$$

Proof. The proof is independent of the type of the singularity and stated in the A_1 case it was already done by [Cat+22, Proposition 3.5]. For the sake of completeness we recall the proof. Observe that $\pi\colon \widetilde{X} \to X$ is an isomorphism away from Q. Therefore we have an isomorphism $\omega_{\widetilde{X}} \cong \pi^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}(kQ)$ for some $k \in \mathbb{Z}$. By the adjunction formula, there exist isomorphisms

$$\mathcal{O}_Q(-n) \cong \omega_Q \cong \omega_{\widetilde{X}} \otimes \mathcal{O}_{\widetilde{X}}(Q)\big|_Q \cong \pi^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}((k+1)Q)\big|_Q \cong \mathcal{O}_Q(-k-1).$$

As the Picard group $\mathrm{Pic}(Q)$ is torsion free, see [Har77, Ex.II.6.5c], this implies $k = n - 1$. $\square$

Remark. The divisor $(n-1)Q$ on $\widetilde{X}$ is called the discrepancy of $\pi\colon \mathrm{Bl}_x(X) \to X$, see [Rei85, Section 1.1]. This also explains the neologism “crepant” of Definition 1.6, which was introduced by M. Reid.

1.2 Semiorthogonal decompositions of triangulated categories

Let $\mathcal{D}$ be a triangulated category and $\mathcal{A} \subset \mathcal{D}$ a full triangulated subcategory. The *left orthogonal* to $\mathcal{A}$ in $\mathcal{D}$ is defined as the full triangulated subcategory

$$\mathcal{A}^{\perp} = \{B \in \mathcal{D} \mid \operatorname{Hom}_{\mathcal{D}}(A, B) = 0 \ \forall A \in \mathcal{A}\}.$$

Analogously, the *right orthogonal* to $\mathcal{A}$ in $\mathcal{D}$ is defined as the full triangulated subcategory

$$^{\perp}\mathcal{A} = \{B \in \mathcal{D} \mid \operatorname{Hom}_{\mathcal{D}}(B, A) = 0 \ \forall A \in \mathcal{A}\}.$$

Definition 1.10 ([Bon90], [BK90])**.** Let $\mathcal{A} \subset \mathcal{D}$ be a full triangulated subcategory and denote the inclusion functor by $i_* \colon \mathcal{A} \to \mathcal{D}$. Then $\mathcal{A}$ is called *left admissible* (resp. *right admissible*), if i_* admits a left adjoint $i^* \colon \mathcal{D} \to \mathcal{A}$ (resp. a right adjoint $i^! \colon \mathcal{D} \to \mathcal{A}$). If i_* admits both left and a right adjoints, then $\mathcal{A}$ is said to be an *admissible* subcategory of $\mathcal{D}$.

Definition 1.11. ([BK90], [BO95]) A *semiorthogonal decomposition* of $\mathcal{D}$ consists of full triangulated subcategories $\mathcal{A}_1, \ldots, \mathcal{A}_n$, such that

1. the sequence $\mathcal{A}_1, \ldots, \mathcal{A}_n$ is *semiorthogonal*, i.e.

$$\operatorname{Hom}_{\mathcal{D}}(A_j, A_i) = 0$$

   for all $j > i$ and $A_j \in \mathcal{A}_j, A_i \in \mathcal{A}_i$.

2. The category $\mathcal{D}$ is the smallest triangulated subcategory of $\mathcal{D}$ containing the subcategories $\mathcal{A}_1, \ldots, \mathcal{A}_n$.

We denote a semiorthogonal decomposition by $\mathcal{D} = \langle \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \rangle$. We call it *admissible*, if all the subcategories $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are admissible.

Remark. Let X be a smooth projective variety. Then any semiorthgonal decomposition $D^b(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ is admissible, see [Orl16, Proposition 3.17].

Lemma 1.12 ([Bon90, Lemma 3.1])**.** *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be an semiorthogonal sequence in $\mathcal{D}$, such that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are left admissible and $\mathcal{A}_{k+1}, \ldots, \mathcal{A}_n$ are right admissible, then*

$$\mathcal{D} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_k, {}^{\perp}\langle \mathcal{A}_1, \ldots, \mathcal{A}_k \rangle \cap \langle \mathcal{A}_{k+1}, \ldots, \mathcal{A}_n \rangle^{\perp}, \mathcal{A}_{k+1}, \ldots, \mathcal{A}_n \rangle$$

is a semiorthogonal decomposition.

We recall the definition of an *exceptional collection* of objects in $\mathcal{D}$. Under certain conditions these collections give rise to semiorthogonal decompositions, cf. Remark 1.2.

Definition 1.13 ([Bon90, Section 2])**.** Let $\mathcal{D}$ be k-linear. We say that a collection of objects $E_1, \ldots, E_n \in \mathcal{D}$ is *exceptional*, if it satisfies the following properties.

1. For any $i \in \{1, \ldots, n\}$, the object E_i is *exceptional*, i.e. we have

$$\operatorname{Hom}(E_i, E_i[l]) = \begin{cases} k, & \text{if } l = 0, \\ 0, & \text{if } l \neq 0. \end{cases}$$

2. For $j > i$ and for all $l \in \mathbb{Z}$ it holds that $\operatorname{Hom}(E_j, E_i[l]) = 0$.

Definition 1.14. Let $\mathcal{D}$ be a k-linear triangulated category. We say that $\mathcal{D}$ is Hom-*finite* if $\dim_k \operatorname{Hom}_{\mathcal{D}}(F, G) < \infty$ for all $F, G \in \mathcal{D}$.

Observe that if X is a projective variety, the bounded derived category $D^b(X)$ is Hom-finite. We will see in Lemma 1.16 below that the subcategory generated by an exceptional object is always admissible if the category $\mathcal{D}$ is Hom-finite and if the exceptional object is *homologically finite*. Let us recall the latter definition.

Definition 1.15 ([Orl06, Definition 1.6])**.** Let $\mathcal{D}$ be a k-linear triangulated category. An object $F \in \mathcal{D}$ is said to be *homologically finite* if for any $G \in \mathcal{D}$ there is only a finite number of values $i \in \mathbb{Z}$, such that $\operatorname{Hom}_{\mathcal{D}}(F, G[i]) \neq 0$. We denote the full triangulated subcategory of homologically finite below objects by $\mathcal{D}^{\text{perf}}$.

Remark ([Orl06, Lemma 1.11]). Let X be a quasi-projective variety. For $\mathcal{D} = D^b(X)$ the subcategory of perfect complexes coincides with the subcategory of homologically finite objects, which justifies the notation $\mathcal{D}^{\text{perf}}$.

Lemma 1.16. *Let $\mathcal{D}$ be a* Hom*-finite k-linear triangulated category and $E \in \mathcal{D}$ an exceptional object which is homologically finite. Then the subcategory $\langle E \rangle \subset \mathcal{D}$ is admissible.*

Proof. For any $F \in \mathcal{D}$ the corresponding left- and right adjoint functors are given by

$$i^*F = \bigoplus_m \operatorname{Hom}(F, E[m])^\vee \otimes E[m] \quad \text{and} \quad i^!F = \bigoplus_m \operatorname{Hom}(E, F[m]) \otimes E[-m] \text{ for } F \in \mathcal{D},$$

respectively, see [Huy, Chapter 7.1.3]. □

Remark. Let $(E_1, \dots, E_n)$ be an exceptional collection in $\mathcal{D}$ and let us assume the setting of Lemma 1.16. Then there exist semiorthogonal decompositions

$$\mathcal{D} = \langle \langle E_1, \dots, E_n \rangle^\perp, E_1, \dots, E_n \rangle = \langle E_1, \dots, E_n, {}^\perp\langle E_1, \dots, E_n \rangle \rangle$$

by Lemma 1.12.

Proposition 1.17. *Let $X \subset \mathbb{P}^{n+1}$ be a Gorenstein hypersurface of degree d and assume $d \leq n+1$. Then $(\mathcal{O}_X, \dots, \mathcal{O}_X(n+1-d))$ is an exceptional collection in $D^b(X)$. Let*

$$\mathcal{A}_X := \langle \mathcal{O}_X, \dots, \mathcal{O}_X(n+1-d) \rangle^\perp$$

denote the left orthogonal of this collection. Then there exists a semiorthogonal decomposition

$$D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \dots, \mathcal{O}_X(n+1-d) \rangle.$$

Proof. One verifies that the collection $(\mathcal{O}_X, \dots, \mathcal{O}_X(n+1-d)) \subset D^b(X)$ is exceptional, using results on the cohomology of $\mathbb{P}^{n+1}$, see for example [Huy, Section 7.1.5]. Moreover, since line bundles on X are in particular homologically finite objects in $D^b(X)$, Lemma 1.16 implies that the subcategory $\langle \mathcal{O}_X, \dots, \mathcal{O}_X(n+1-d) \rangle \subset D^b(X)$ is admissible. Finally, we obtain the desired semiorthogonal decomposition by an application of Lemma 1.12. □

Remark. Many authors refer to the subcategory $\mathcal{A}_X$ as the *Kuznetsov Component* of $D^b(X)$. We will also use this terminology.

1.3 Mutations

Let $\mathcal{D}$ be a triangulated category. The group $\operatorname{Aut}(\mathcal{D})$ acts naturally on the set of semiorthogonal decompositions of $\mathcal{D}$. Apart from the automorphisms induced from elements of $\operatorname{Aut}(\mathcal{D})$, we are interested in the class of so called mutation functors (or mutations), which, informally speaking, permute the components of a semiorthogonal decomposition. In this section, we recall their definition and basic properties, which we use throughout the thesis.

Proposition 1.18 ([Bon90]). *Let $\mathcal{A} \subset \mathcal{D}$ be an admissible subcategory. By Lemma 1.12 we have semiorthogonal decompositions $\mathcal{D} = \langle \mathcal{A}^\perp, \mathcal{A}\rangle$ and $\mathcal{D} = \langle \mathcal{A}, {}^\perp\mathcal{A}\rangle$. Then there exist functors $\mathbb{L}_\mathcal{A}, \mathbb{R}_\mathcal{A} \colon \mathcal{D} \to \mathcal{D}$, vanishing on $\mathcal{A}$, that restrict to mutually inverse equivalences $\mathbb{L}_\mathcal{A} \colon {}^\perp\mathcal{A} \to \mathcal{A}^\perp$ and $\mathbb{R}_\mathcal{A} \colon \mathcal{A}^\perp \to {}^\perp\mathcal{A}$.*

Proof. Since $\mathcal{A}$ is admissible, we obtain the following exact triangles (induced by the unit and counit of the respective adjunction) for every $F \in \mathcal{D}$:

$$i_*i^!F \longrightarrow F \longrightarrow C_F \quad \text{and} \quad B_F[-1] \longrightarrow F \longrightarrow i_*i^*F, \tag{1.18.1}$$

where C_F and B_F denote the respective cones. We define

$$\mathbb{L}_\mathcal{A}(F) = C_F \qquad \mathbb{R}_\mathcal{A}(F) = B_F[-1].$$

Semiorthogonality implies that the cones are functorial, therefore the above formulas define functors. It is checked directly that they are mutually inverse equivalences. □

We call the functors $\mathbb{L}_\mathcal{A}$ and $\mathbb{R}_\mathcal{A}$ the *left-* and *right mutation functors* corresponding to $\mathcal{A}$. In the case the subcategory $\mathcal{A}$ is generated by a single exceptional object the left- and right mutations are of a particularly simple form, more precisely:

Remark ([Huy, Example 7.1.7]). Let $\mathcal{D}$ be a Hom-finite k-linear triangulated category and $E \in \mathcal{D}^{\mathrm{perf}}$ an exceptional object in $\mathcal{D}$. Then we have

$$\mathbb{L}_E(F) = \operatorname{cone}(\bigoplus_m \operatorname{Hom}(E, F[m]) \otimes E[-m] \to F),$$

$$\mathbb{R}_E(F) = \operatorname{cone}(F \to \bigoplus_m \operatorname{Hom}(F, E[m])^\vee \otimes E[m])$$

for any $F \in \mathcal{D}$. This can be deduced from Lemma 1.16 and the exact triangles (1.18.1).

Lemma 1.19 ([Bon90]). *Let $\mathcal{D} = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n\rangle$ be an admissible semiorthogonal decomposition. Then for each $1 \le k \le n-1$, there exists a semiorthogonal decomposition*

$$\mathcal{D} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{k-1}, \mathbb{L}_{\mathcal{A}_k}(\mathcal{A}_{k+1}), \mathcal{A}_k, \mathcal{A}_{k+2}, \dots, \mathcal{A}_n\rangle.$$

Furthermore, for each $2 \le k \le n$, there exists a semiorthogonal decomposition

$$\mathcal{D} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{k-2}, \mathcal{A}_k, \mathbb{R}_{\mathcal{A}_k}(\mathcal{A}_{k-1}), \mathcal{A}_{k+1}, \dots, \mathcal{A}_n\rangle.$$

We recall the following lemma describing the mutation functors in the situation where two consecutive components of the decomposition are orthogonal to each other.

Lemma 1.20 ([Bon90]). *Let $\mathcal{D} = \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n\rangle$ be an admissible semiorthogonal decomposition, such that the components $\mathcal{A}_k$ and $\mathcal{A}_{k+1}$ are orthogonal for some k, i.e.* $\operatorname{Hom}_\mathcal{D}(\mathcal{A}_k, \mathcal{A}_{k+1}) = 0$. *Then we have:*

$$\mathbb{L}_{\mathcal{A}_k}(\mathcal{A}_{k+1}) = \mathcal{A}_{k+1} \quad \text{and} \quad \mathbb{R}_{\mathcal{A}_{k+1}}(\mathcal{A}_k) = \mathcal{A}_k.$$

In particular, swapping the positions of the subcategories A_k and A_{k+1} yields a semiorthogonal decomposition

$$\mathcal{D} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{k-1}, \mathcal{A}_{k+1}, \mathcal{A}_k, \mathcal{A}_{k+2}, \dots, \mathcal{A}_n\rangle.$$

1.4 Serre functors and spherical objects

Definition 1.21. Let $\mathcal{D}$ be a k-linear triangulated category. Then an exact equivalence $\mathbb{S}_{\mathcal{D}} \colon \mathcal{D} \to \mathcal{D}$ is called *Serre functor* if there exist natural isomorphisms

$$\mathrm{Hom}(F, G) \cong \mathrm{Hom}(G, \mathbb{S}_{\mathcal{D}}(F))^{\vee}$$

of k-vector spaces for any $F, G \in \mathcal{D}$.

Example 1.22. Let X be a smooth projective variety. Then it follows by Grothendieck-Verdier duality, see [Huy06, Theorem 3.34], that $D^b(X)$ admits a Serre functor which is given by

$$\mathbb{S}_X \colon D^b(X) \xrightarrow{\sim} D^b(X), \quad F \mapsto F \otimes \omega_X[\dim(X)].$$

We now recall how Serre functors behave with respect to semiorthogonal decompositions.

Lemma 1.23 ([Huy, Exercise 7.1.13])**.** *Let X be a smooth projective variety and assume we have a semiorthogonal decomposition $D^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle$. Then exist semiorthogonal decompositions $D^b(X) = \langle \mathbb{S}_X(\mathcal{B}), \mathcal{A} \rangle = \langle \mathcal{B}, \mathbb{S}_X^{-1}(\mathcal{A}) \rangle$.*

Since we are generally working with varieties that are not necessarily smooth but at least Gorenstein, we recall the following generalization of the previous results.

Example 1.24 ([KSP21, Section 2.3])**.** Let X be a Gorenstein projective variety. Then we can define a Serre functor on $D^{\mathrm{perf}}(X)$ by setting $\mathbb{S}_X(-) := - \otimes \omega_X[\dim(X)]$. This does not define a Serre functor on $D^b(X)$, but for all $F \in D^{\mathrm{perf}}(X)$ and $G \in D^b(X)$ we obtain natural isomorphisms

$$\mathrm{RHom}(F, G) \cong \mathrm{RHom}(G, \mathbb{S}_X(F))^{\vee},$$

by Grothendieck-Verdier duality.

Lemma 1.25 ([KSP21, Lemma 2.15])**.** *Let X be a Gorenstein projective variety and let $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition, where either $\mathcal{A}$ or $\mathcal{B}$ is contained in $D^{\mathrm{perf}}(X)$. Then the subcategories $\mathcal{A}, \mathcal{B} \subset D^b(X)$ are admissible and there exist semiorthogonal decompositions*

$$D^b(X) = \langle \mathcal{B} \otimes \omega_X, \mathcal{A} \rangle = \langle \mathcal{B}, \mathcal{A} \otimes \omega_X^{\vee} \rangle.$$

Let $\mathcal{D}$ be a full admissible subcategory of $D^b(X)$ for a smooth projective variety X. Since $D^b(X)$ admits a Serre functor, one can easily verify that also $\mathcal{D}$ admits a Serre functor, see [Huy, Lemma 7.1.14]. This enables us to define the notion of a *spherical object* in such a category $\mathcal{D}$.

Definition 1.26. Let $n \in \mathbb{Z}$. We say that an object $E \in \mathcal{D}$ is n-spherical, if it satisfies the following properties.

1. We have an isomorphism

$$\mathrm{Hom}(E, E[l]) \cong \begin{cases} k, & \text{if } l = 0, n \\ 0, & \text{else.} \end{cases}$$

2. The Serre functor applied to E is the shift by n, i.e. we have $\mathbb{S}_{\mathcal{D}}(E) \cong E[n]$.

1.5 (Crepant) categorical resolutions

In this subsection we recall the definition of a categorical resolution and a method for constructing (crepant) categorical resolutions of the bounded derived category $D^b(X)$ of a variety X with rational singularities, following [Kuz08].

Definition 1.27. Let $\mathcal{D}$ be a triangulated category. We say that $\mathcal{D}$ is *smooth* if there exists a smooth projective variety X such that $\mathcal{D}$ is equivalent to an admissible subcategory of the bounded derived category $D^b(X)$.

Remark. Nowadays, Definition 1.27 is considered outdated, but we will still use it since it suffices for our purposes. We refer to [Orl16, Definition 3.23] for the "right" definition of smoothness for any enhanced triangulated category. As one would expect, a smooth triangulated category in our sense will be smooth in the sense of [Orl16, Definition 3.23], see [Orl16, Proposition 3.31].

Definition 1.28 ([Kuz08, Lemma 3.2])**.** A *categorical resolution* of a triangulated category $\mathcal{D}$ consists of a smooth triangulated category $\widetilde{\mathcal{D}}$ and a pair of functors

$$\pi_*\colon \widetilde{\mathcal{D}} \to \mathcal{D} \quad \text{and} \quad \pi^*\colon \mathcal{D}^{\mathrm{perf}} \to \widetilde{\mathcal{D}}$$

satisfying the following properties:

1. The functor π^* is left adjoint to π_*. That is, there exist natural isomorphisms
$$\mathrm{Hom}_{\widetilde{\mathcal{D}}}(\pi^* F, G) \cong \mathrm{Hom}_{\mathcal{D}}(F, \pi_* G) \qquad \text{for any } F \in \mathcal{D}^{\mathrm{perf}},\ G \in \widetilde{\mathcal{D}}.$$

2. The natural transformation $\mathrm{id}_{\mathcal{D}^{\mathrm{perf}}} \to \pi_*\pi^*$ is an isomorphism[6].

Remark. Let X be a variety with rational singularities and $\pi\colon \widetilde{X} \to X$ a resolution of singularities. Then the derived category $D^b(\widetilde{X})$ together with the pushforward and pullback functors $\pi_*\colon D^b(\widetilde{X}) \to D^b(X)$ and $\pi^*\colon D^b(X)^{\mathrm{perf}} \to D^b(\widetilde{X})$ is a categorical resolution of $D^b(X)$. By imposing the second condition in the above definition, we restrict ourselves to the case where X has at most rational singularities.

Definition 1.29 ([Kuz08, Lemma 3.4])**.** A categorical resolution $(\widetilde{\mathcal{D}}, \pi_*, \pi^*)$ is called *crepant*[7] if the functor π^* is also right adjoint to π_* when restricted to $\mathcal{D}^{\mathrm{perf}}$, i.e. there exist natural isomorphisms

$$\mathrm{Hom}_{\widetilde{\mathcal{D}}}(G, \pi^* F) \cong \mathrm{Hom}_{\mathcal{D}}(\pi_* G, F) \qquad \text{for any } F \in \mathcal{D}^{\mathrm{perf}},\ G \in \widetilde{\mathcal{D}}.$$

Remark. Let X be a Gorenstein projective variety. A crepant (geometric) resolution $\pi\colon \widetilde{X} \to X$ induces a crepant categorical resolution $(D^b(\widetilde{X}), \pi_*, \pi^*)$ of $D^b(X)$ as follows. By Grothendieck-Verdier duality the right adjoint $\pi^!$ of $\pi_*\colon D^b(\widetilde{X}) \to D^b(X)$ can be given explicitly by $\pi^!(F) = \pi^* F \otimes \omega_\pi$ for all $F \in D^{\mathrm{perf}}(X)$. Therefore, if π is crepant, it immediately follows that the categorical resolution is crepant as well.

We now recall a method for constructing (crepant) categorical resolutions of the bounded derived category $D^b(X)$ of a variety X with rational singularities. The construction starts with a geometric resolution $\pi\colon \widetilde{X} \to X$, for which we assume that the

[6]We consider this natural transformation as a morphism in the functor category $\mathrm{Fun}(\mathcal{D}^{\mathrm{perf}}, \mathcal{D})$.

[7]In [Kuz08], they use a slightly different terminology and call crepant resolution "weakly crepant".

exceptional locus E of π is an irreducible divisor. Let Z denote the image of E under π. Then we have a cartesian diagram

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "p"'] & \widetilde{X} \arrow[d, "\pi"] \\ Z \arrow[r, "i"] & X, \end{tikzcd}$$

where the morphisms i and j denote the respective inclusions of the subvarieties $Z \subset X$ and $E \subset \widetilde{X}$. The construction is based on the existence of a specific semiorthogonal decomposition of the category $D^b(E)$, which is called a *dual Lefschetz decomposition*, see Definition 1.30 below. If we have such a decomposition, Theorem 1.31 explains how to define a subcategory $\widetilde{\mathcal{D}} \subset D^b(\widetilde{X})$ which, under certain conditions, gives rise to a categorical resolution of $D^b(X)$. In Proposition 1.32, we recall some additional assumptions which ensure that $\widetilde{\mathcal{D}}$ is in fact a crepant categorical resolution of $D^b(X)$.

Definition 1.30 ([Kuz08, Definintion 2.16])**.** Let X be a variety and let $\mathcal{O}(1)$ denote a line bundle on X. A *Lefschetz decomposition* of $D^b(X)$ is a semiorthogonal decomposition of the form

$$D^b(X) = \langle \mathcal{B}_0, \mathcal{B}_1(1), \dots, \mathcal{B}_{m-1}(m-1) \rangle,$$

where $\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_{m-1}$ are subcategories of $D^b(X)$, satisfying

$$0 \subset \mathcal{B}_{m-1} \subset \cdots \subset \mathcal{B}_1 \subset \mathcal{B}_0 \subset D^b(X).$$

Similarly, a *dual Lefschetz decomposition* of $D^b(X)$ is a semiorthogonal decomposition of the form

$$D^b(X) = \langle \mathcal{B}_{m-1}(1-m), \dots, \mathcal{B}_1(-1), \mathcal{B}_0 \rangle, \quad \text{where } 0 \subset \mathcal{B}_{m-1} \subset \cdots \subset \mathcal{B}_1 \subset \mathcal{B}_0 \subset D^b(X).$$

Remark. These two notions are equivalent: given a Lefschetz decomposition, one can construct a dual Lefschetz decomposition with the same $\mathcal{B}_0$ and vice versa, see [Kuz08, Lemma 2.15].

Theorem 1.31 ([Kuz08, Lemma 4.1, Theorem 4.4])**.** *We assume the setting of the paragraph preceding Definition 1.30 and additionally assume that there exists a dual Lefschetz decomposition*

$$D^b(E) = \langle \mathcal{B}_{m-1}(1-m), \mathcal{B}_{m-2}(2-m), \dots, \mathcal{B}_1(-1), \mathcal{B}_0 \rangle \tag{1.31.1}$$

with respect to the conormal bundle $\mathcal{O}_E(1)$ of the exceptional divisor $E \subset \widetilde{X}$. We define a full triangulated subcategory of $D^b(X)$ by

$$\widetilde{\mathcal{D}} = \{\mathcal{F} \in D^b(\widetilde{X}) \mid j^*\mathcal{F} \in \mathcal{B}_0\}.$$

Then the functor $j_* \colon D^b(E) \to D^b(\widetilde{X})$ is fully faithful when restricted to the subcategories $\langle \mathcal{B}_k(-k) \rangle$ for all $1 \le k \le m-1$ and there exists a semiorthogonal decomposition

$$D^b(\widetilde{X}) = \langle j_*\mathcal{B}_{m-1}(1-m), j_*\mathcal{B}_{m-2}(2-m), \dots, j_*\mathcal{B}_1(-1), \widetilde{\mathcal{D}} \rangle.$$

Moreover, assume the image of the pullback functor $\pi^* \colon \mathrm{D}^{\mathrm{perf}}(X) \to D^b(\widetilde{X})$ is contained in $\widetilde{\mathcal{D}}$. Then the triple $(\widetilde{\mathcal{D}}, \pi_*, \pi^*)$ is a categorical resolution of $D^b(X)$.

The next result gives us conditions under which the categorical resolution of the previous theorem is crepant.

Proposition 1.32 ([Kuz08, Proposition 4.5])**.** *Let X be Gorenstein and assume that we have an inclusion $p^*(D^{\mathrm{perf}}(Z)) \subset \mathcal{B}_{m-1}$. Furthermore, we assume that there exists an isomorphism $\omega_{\widetilde{X}} = \pi^*(\omega_X) \otimes \mathcal{O}_{\widetilde{X}}((m-1)E)$. Then the categorical resolution $(\widetilde{\mathcal{D}}, \pi_*, \pi^*)$ is crepant.*

Finally, let us recall the definition of a *localization*.

Definition 1.33. Let $F \colon \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories. We say that F is a *localization* if the induced functor $\overline{F} \colon \mathcal{D}/\ker(F) \to \mathcal{D}'$ is an equivalence of triangulated categories.

Later in Section 3.3 we will prove that the crepant categorical resolution of $D^b(X)$ for a variety X with an isolated A_2 singularity that we construct in Theorem 2.1 is in fact a localization. In the A_1 case this was already shown by [KS23a, Section 6] and [Cat+22, Corollary 2.22].

2 Crepant categorical resolutions of A_2 singularities

In Section 2.1 we prove that the bounded derived category $D^b(X)$ of a projective variety X with an isolated A_2 singularity admits a crepant categorical resolution $\widetilde{D}$, using the results stated in the previous subsection. For A_1 singularities this result is already known by [Cat+22, Proposition 3.5]. In Section 2.2 we take a closer look at cubic fourfolds with A_2 singularities: Recall that If X is a cubic fourfold with an isolated A_1 singularity, it is known that there exists a crepant categorical resolution $\widetilde{\mathcal{A}}_X$ of the Kuznetsov component $\mathcal{A}_X \subset D^b(X)$ and an equivalence $\widetilde{\mathcal{A}}_X \cong D^b(S)$, where S denotes a (smooth) K3 surface associated to X, cf. [Kuz10, section 5]. We will show that this result generalizes to the case of a cubic fourfold with an isolated A_2 singularity.

2.1 Construction of a crepant categorical resolution

Let X be a projective variety with a single isolated A_1 singularity. Then, under the application of the results we presented in Section 1.5, it was shown by [Cat+22, Proposition 3.5] and [KS23a, Section 5.2] that there exists a crepant categorical resolution $\widetilde{\mathcal{D}}$ of $D^b(X)$. The following theorem will generalize this result to A_2 singularities.

Theorem 2.1. *Let X be a projective variety with an isolated A_1 or A_2 singularity at a point $x \in X$. Then there exists a crepant categorical resolution $\widetilde{\mathcal{D}}$ of $D^b(X)$.*

Proof. Let $n = \dim(X) - 1$, for some integer $n \geq 1$. In Lemma 1.8 we saw that we can resolve the singularity of X by a single blow-up at the singular point x. Let $\pi \colon \widetilde{X} := \mathrm{Bl}_x(X) \to X$ be the corresponding proper birational morphism. We obtain a cartesian diagram

$$\begin{tikzcd} Q \arrow[r, hook, "j"] \arrow[d, "p"'] & \widetilde{X} \arrow[d, "\pi"] \\ \{x\} \arrow[r, hook, "i"] & X, \end{tikzcd}$$

where Q denotes the exceptional divisor of $\widetilde{X}$ and i and j the inclusions of the point $\{x\}$ into X and Q into $\widetilde{X}$, respectively. In the case $x \in X$ is an A_1 singularity, Q is smooth quadric and in the case $x \in X$ is an A_2 singularity, Q is a nodal quadric hypersurface in $\mathbb{P}^{n+1}$, cf. Lemma 1.8. Since Q is Gorenstein in both cases, Proposition 1.17 implies that we have a semiorthogonal decomposition

$$D^b(Q) = \langle \mathcal{A}_Q, \mathcal{O}_Q, \mathcal{O}_Q(1), \dots, \mathcal{O}_Q(n-1)\rangle. \tag{2.1.1}$$

Moreover, by Lemma 1.25 this decomposition is admissible and we can permute a single component by tensoring with the canonical sheaf ω_Q. By adjunction formula we have $\omega_Q = \mathcal{O}_Q(-n)$, for any quadric $Q \subset \mathbb{P}^{n+1}$. A successive application of the functor $- \otimes \mathcal{O}_Q(-n)$ to the line bundles $\mathcal{O}_Q(1), \dots, \mathcal{O}_Q(n-1)$ gives rise to a semiorthogonal decomposition

$$D^b(Q) = \langle \mathcal{O}_Q(1-n), \mathcal{O}_Q(2-n), \dots, \mathcal{O}_Q(-1), \mathcal{A}_Q, \mathcal{O}_Q\rangle. \tag{2.1.2}$$

This is a dual Lefschetz decomposition with respect to the conormal bundle $\mathcal{N}^{\vee}_{Q/\widetilde{X}} \cong \mathcal{O}_Q(1)$, by setting $\mathcal{B}_{n-1} = \mathcal{B}_1 = \langle \mathcal{O}_Q\rangle$ and $\mathcal{B}_0 = \langle \mathcal{A}_Q, \mathcal{O}_Q\rangle$. We now apply Theorem 1.31 and obtain a semiorthogonal decomposition

$$D^b(\widetilde{X}) = \langle j_*\mathcal{O}_Q(1-n), \dots, j_*\mathcal{O}_Q(-1), \widetilde{\mathcal{D}}\rangle, \tag{2.1.3}$$

where

$$\widetilde{\mathcal{D}} = \{F \in D^b(\widetilde{X}) |\ j^*F \in \mathcal{B}_0\}.$$

Let $\pi_*\colon \widetilde{\mathcal{D}} \to D^b(X)$ denote the restriction of the pushforward functor along π to $\widetilde{\mathcal{D}}$. All that is left to show is that the image of the pullback functor $\pi^*\colon D^{\mathrm{perf}}(X) \to D^b(\widetilde{X})$ is contained in $\widetilde{\mathcal{D}}$. For this it suffices to prove that we have an inclusion

$$p^*(D^{\mathrm{perf}}(x)) \subset \mathcal{B}_0. \tag{2.1.4}$$

Indeed, since

$$j^*\pi^*(F) = p^*i^*(F) \in p^*(D^{\mathrm{perf}}(x)),$$

for any $F \in D^{\mathrm{perf}}(X)$, the inclusion (2.1.4) would imply that $j^*\pi^*F \in \mathcal{B}_0$, i.e. $\pi^*F \in \widetilde{\mathcal{D}}$. The inclusion (2.1.4) holds, since all free $\mathcal{O}_Q$-modules are contained in $\mathcal{B}_0$ and for any finite dimensional k-vector space V we have $p^*(V) \cong \mathcal{O}_Q^{\oplus \dim(V)}$. By Theorem 1.31 we obtain that $(\widetilde{\mathcal{D}}, \pi_*, \pi^*)$ is a categorical resolution of $D^b(X)$. This resolution is in fact crepant, which follows by Proposition 1.32. Indeed, note that $p^*(D^{\mathrm{perf}}(x)) \subset \mathcal{B}_0 = \mathcal{B}_{n-1}$ and the canonical bundle of $\widetilde{X}$ is given by $\omega_{\widetilde{X}} = \pi^*\omega_X \otimes \mathcal{O}_{\widetilde{X}}((n-1)Q)$, which follows from Lemma 1.9. □

2.2 Special case of a cubic fourfold

In the previous subsection we proved the existence of a crepant categorical resolution of the bounded derived category $D^b(X)$ of a projective variety X with an isolated A_2 singularity. We now apply this result to cubic fourfolds and show that the Kuznetsov component $\mathcal{A}_X$ admits a crepant categorical resolution $\widetilde{\mathcal{A}}_X$ which is equivalent to the bounded derived category $D^b(S)$ of a (smooth) K3 surface. This generalizes the analogous result for A_1 singularities, which was proved in [Kuz10, Theorem 5.2]. The proof stated in [Kuz10, Theorem 5.2] will essentially work the same way for a cubic fourfold with an A_2 singularity. The main difficulty is to see that the K3 surface one can associate to a singular cubic fourfold is still smooth in the A_2 case. We prove this in Proposition 2.2 and recall in Lemma 2.3 how one associates a K3 surface to a singular cubic fourfold.

Proposition 2.2. *Let $X \subseteq \mathbb{P}^{n+1}$ be a cubic hypersurface with an isolated A_d singularity at $x = [1 : 0 : \cdots : 0]$. Then it is defined by an equation of the form*

$$F(x_0, \ldots, x_{n+1}) = x_0 Q(x_1, \ldots, x_{n+1}) + G(x_1, \ldots, x_{n+1}),$$

for a suitable quadric Q and cubic G in $V_+(x_0) \cong \mathbb{P}^n$. Furthermore:

- *$d = 1$ if and only if Q has maximal rank;*
- *$d = 2$ implies that Q has corank 1 and $V_+(G)$ does not pass through the node of $V_+(Q)$.*

Moreover, in both cases the intersection $V_+(Q, G)$ is smooth if X is smooth away from the singularity x.

Proof. We consider the affine neighborhood $D(x_0)$ of x, where the variety X is given by a defining equation of the form

$$\widetilde{F}(x_1, \ldots, x_{n+1}) = C + L(x_1, \ldots, x_{n+1}) + Q(x_1, \ldots, x_{n+1}) + G(x_1, \ldots, x_{n+1}), \quad (2.2.1)$$

for some homogeneous polynomials G, Q and L of degrees 3, 2 and 1 respectively and a scalar $C \in k$. Observe that $y = (0 : \cdots : 0)$ is a singularity of $\widetilde{F}$, so we have $L = 0$ and $C = 0$.

We now assume $d = 1$. To check that Q has full rank, i.e., that Q is smooth, we consider the blow-up of X at the point $x = [1 : 0 : \cdots : 0]$. By Lemma 1.8, the corresponding exceptional divisor $E \subset V_+(x_0)$ is a smooth quadric. Note that E can be characterized as the projectivized tangent cone $\mathbb{P}\mathrm{TC}_x(X)$. At the same time, there exists an isomorphism between $\mathbb{P}\mathrm{TC}_x(X)$ and the quadric $V_+(Q)$, by the definition of the projectivised tangent cone and the defining equation (2.2.1) of X. Conversely, if we assume that Q has full rank we can argue by considering the completion of the local ring $\mathcal{O}_{X,x}$ to show that it is isomorphic to $k[[x_1, ..., x_{n+1}]]/(x_1^2 + \cdots + x_{n+1}^2)$, i.e., that d is necessarily equal to 1. To compute the completion of the local ring $\mathcal{O}_{X,x}$, we can restrict to the affine neighborhood $D(x_0) \cong \mathbb{A}_k^{n+1} = \mathrm{Spec}(k[x_1, \ldots, x_{n+1}])$ and since completion with respect to an ideal is an exact functor in our case, we obtain an isomorphism

$$\widehat{\mathcal{O}_{X,x}} \cong k[[x_1, ..., x_{n+1}]]/(x_1^2 + \cdots + x_{n+1}^2 + G(x_1, \ldots, x_{n+1})). \quad (2.2.2)$$

Here we assume w.l.o.g. that $Q = V_+(x_1^2 + \cdots + x_{n+1}^2)$, by the classification of quadric hypersufaces in $\mathbb{P}^{n+1}$. Now one can perform a series of coordinate transformations to conclude that $d = 1$. Note that in the case of $k = \mathbb{C}$ this is obtained by an application of the holomorphic Morse lemma, cf. [AGV12, Theorem 6.2]. The equation of (2.2.2) can be written as

$$\widetilde{F}(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2 + x_1 f_1 + x_1^2 f_2 + x_1^3 f_3 + f_4,$$

where $f_1, \ldots, f_4$ denote polynomials in the variables $x_2, \ldots, x_{n+1}$. Observe that the squares coming from the smooth quadric Q will "absorb" all higher terms in the same variable. More precisely, we have

$$x_1^2 + x_1^3 f_3 = x_1^2 (\underbrace{1 + x_1 f_3}_{=:g^2})$$

for some $g \in k[[x_1, \dots, x_{n+1}]]^\times$. Therefore, we can apply the coordinate transformation defined by $t_1 \mapsto gx_1$ and $t_i \mapsto x_i$ for all $i \neq 1$, which yields

$$\widetilde{F}(t_1, \dots, t_{n+1}) = (\frac{t_1}{g})^2 + \dots + t_{n+1}^2 + \frac{t_1}{g} f_1 + (\frac{t_1}{g})^2 f_2 + f_4.$$

We can proceed in the same way for the quadratic terms in t_1, that is

$$(\frac{t_1}{g})^2 + (\frac{t_1}{g})^2 f_2 = (\frac{t_1}{g})^2 (\underbrace{1 + f_2}_{=:\tilde{g}^2}),$$

for some $\tilde{g} \in k[[t_1, \dots, t_{n+1}]]^\times$. Again, replacing t_1 with $\frac{g}{\tilde{g}} t_1$ transforms the equation to

$$\widetilde{F}(t_1, \dots, t_{n+1}) = (\frac{t_1}{\tilde{g}})^2 + \dots + t_{n+1}^2 + \frac{t_1}{\tilde{g}} f_1(t_2, \dots, t_{n+1}) + f_4(t_2, \dots, t_{n+1}).$$

To absorb the linear terms, we complete the square:

$$(\frac{t_1}{\tilde{g}})^2 + \frac{t_1}{\tilde{g}} = (\frac{t_1}{\tilde{g}} + \frac{f_1}{2})^2 - \frac{f_1^2}{4}$$

and replace t_1 by the term $\tilde{g}t_1 - \frac{f_1}{2}$. This yields

$$\widetilde{F}(t_1, \dots, t_{n+1}) = t_1^2 + \dots + t_{n+1}^2 + f_4(t_2, \dots, t_{n+1}).$$

Since f_4 does not have terms containing the factor t_1, we can argue inductively over the number of variables and prove that X has a A_1 singularity at $x \in X$.

Now assume that X has an A_2-singularity at $x \in X$. Analogous to the $d = 1$ case we can use Lemma 1.8 to conclude that Q has corank 1. For the other part of the statement, we argue by contradiction. Let us assume that $V_+(G)$ passes through the node of $V_+(Q)$. For $Q = V_+(x_1^2 + \dots + x_n^2)$ this is equivalent to

$$G(0, 0, \dots, 0, 1) = 0. \tag{2.2.3}$$

Therefore, G cannot contain terms of the form cx_{n+1}^3, where $c \in k^\times$. The goal for the rest of the proof is to show that $d > 2$ by applying coordinate transformations. For this, we consider the completion of the local ring $\mathcal{O}_{X,x}$ in the affine neighborhood $D(x_0) \cong \mathbb{A}_k^{n+1} = \operatorname{Spec}(k[x_1, \dots, x_{n+1}])$. There we have an isomorphism

$$\widehat{\mathcal{O}_{X,x}} \cong k[[x_1, ..., x_{n+1}]]/(x_1^2 + \dots + x_n^2 + G(x_1, \dots, x_{n+1})).$$

Applying the same induction argument as before, the squares $x_1, \dots, x_n$ will "absorb" all the higher powers of the same variables. Therefore we obtain

$$\widehat{\mathcal{O}_{X,x}} \cong k[[x_1, ..., x_{n+1}]]/(x_1^2 + \dots + x_n^2 + \widetilde{G}(x_{n+1})),$$

where $\widetilde{G}(x_{n+1})$ is a polynomial over k in the variable x_{n+1}. The key observation now is that after applying the above algorithm, the polynomial $\widetilde{G}(x_{n+1})$ only contains terms of power 4 or higher. More precisely, the initial cubic $G(x_1, \dots, x_{n+1})$ does not contain elements of the form cx_{n+1}^3, where $c \in k^\times$. On the other hand, it must contain terms of the form $x_i x_{n+1}^2$, for some $i \in \{1, \dots n\}$, otherwise the squares in $x_1^2 + \dots + x_n^2$ would absorb all other terms linear in x_{n+1} and we would obtain that $d = 1$, which would be a contradiction. We saw above that "resolving" terms like $x_i x_{n+1}^2$ contributes a term

$(\frac{x_{n+1}^2}{2})^2$. Therefore, the smallest power of x_{n+1} in $\widetilde{G}(x_{n+1})$ has to be 4 or higher. Let $m \geq 4$ be the smallest non-trivial power of x_{n+1} contained in $\widetilde{G}$, then $\widetilde{G} = \sum_{i=m}^{n} a_i x_{n+1}^i$ for some $n \in \mathbb{N}$. Since $a_m \neq 0$, we have

$$\widetilde{G}(x_{n+1}) = x_{n+1}^m \underbrace{(a_m + f(x_{n+1}))}_{=:\bar{f}^m},$$

where $f(x_{n+1})$ denotes a suitable polynomial in x_{n+1} and $\bar{f} \in k[[x_1, \dots, x_{n+1}]]^\times$. Therefore, we can transform the coordinates by $t_{n+1} \mapsto \bar{f} x_{n+1}$ and $t_i \mapsto x_i$ for $i \neq n+1$, which yields an isomorphism

$$\widehat{\mathcal{O}_{X,x}} \cong k[[t_1, ..., t_{n+1}]]/(t_1^2 + \cdots + t_n^2 + t_{n+1}^m).$$

This implies that X has an A_d-singularity at x with $d > 2$, since $m \geq 4$, which is a contradiction. This shows that $V_+(G)$ does not pass through the node of $V_+(Q)$.

It is left to show that under the additional assumption of X being smooth everywhere away from x, the intersection $V_+(Q, G)$ is smooth in the cases $d = 1$ and $d = 2$. We will only verify this claim for $d = 2$, since the argument is analogous for $d = 1$ (it is even simpler, because Q is smooth in that case). Consider the partial derivatives of $V_+(Q, G)$ and X, respectively. We have

$$\frac{\partial(Q+G)}{\partial x_i} = \begin{cases} \frac{\partial Q}{\partial x_i} + \frac{\partial G}{\partial x_i}, & \text{if } i = 1, \dots n \\ \frac{\partial G}{\partial x_{n+1}}, & \text{else,} \end{cases} \quad \text{and} \quad \frac{\partial F}{\partial x_i} = \begin{cases} Q, & \text{if } i = 0 \\ x_0 \frac{\partial Q}{\partial x_i} + \frac{\partial G}{\partial x_i}, & \text{else.} \end{cases} \tag{2.2.4}$$

Observe that for all $y \in V_+(Q, G)$ there exists an $i \in \{1, \dots, n\}$ such that $\frac{\partial Q}{\partial x_i}(y) \neq 0$, since $V_+(G)$ does not pass through the node of Q. Assume that there exists a point $z \in V_+(Q, G)$ and a scalar $0 \neq \lambda$ such that $\lambda \frac{\partial Q}{\partial x_i}(z) = \frac{\partial G}{\partial x_i}(z)$ for some $i \in 1, \dots, n$. Considering the partial derivatives of F we see that $\tilde{z} = [-\lambda : z]$ would be a singular point of X. Since $\tilde{z} \neq x$, this is a contradiction to the smoothness of X outside the A_2 singularity x. $\square$

Corollary 2.2.1. *Let $X \subset \mathbb{P}^5$ be a cubic fourfold which is smooth away from an isolated A_2 singularity at a point $x \in X$. Then, with the notation of Proposition 2.2, the intersection $S = V_+(Q, G)$ is a smooth K3 surface.*

Proof. In Proposition 2.2 we proved the smoothness of $S = V_+(Q, G)$. Now it remains to show that S is indeed a K3 surface. By the adjunction formula, we immediately get an isomorphism $\omega_S \cong \mathcal{O}_S(2+3-5) = \mathcal{O}_S$. Since S is a complete intersection a standard calculation shows that $H^1(S, \mathcal{O}_S) = 0$, see for example [Har77, Ex.III.5.5c]. $\square$

Setting. From now on, let V be a 6-dimensional k-vector space and let $X \subset \mathbb{P}(V)$ denote a cubic fourfold with an isolated A_2 singularity at a point $x \in X$. We choose coordinates $x_0, \dots x_5$, such that $x = [1 : 0 \cdots : 0]$. Let $\pi \colon \widetilde{X} := \mathrm{Bl}_x(X) \to X$ denote the corresponding map of the blow-up of X at x, which is a resolution of singularities for X, see Lemma 1.8. Let $\sigma \colon \widetilde{X} \to \mathbb{P}^4$ be the extension of the projection away from the cuspidal point x to the blow-up $\widetilde{X}$. This projection is a rational map $X \dashrightarrow \mathbb{P}^4$. Recall that X has a defining equation of the form $x_0 Q + G$ for a nodal quadric $V_+(Q)$ and some cubic $V_+(G)$ in $V_+(x_0) \cong \mathbb{P}^4$, and the intersection $S = V_+(Q, G) \subset \mathbb{P}^4$ is a smooth K3 surface, see Corollary 2.2.1.

The following lemma was proved in [Kuz10, Lemma 5.1] for a cubic fourfold X with a single isolated A_1 singularity. Since we know that the K3 surface S is smooth in the A_2 case, the proof immediately generalizes to the A_2 case without further adjustments. We will recall this proof below for the sake of completeness.

Lemma 2.3. *The morphism σ is isomorphic to the blow-up of $\mathbb{P}^4$ along the K3 surface S. We denote by Q and D, the exceptional divisor of π and σ and the corresponding closed immersions by $j\colon Q \hookrightarrow \widetilde{X}$ and $\eta\colon D \hookrightarrow \widetilde{X}$, respectively. There exist two catesian diagrams*

$$\begin{tikzcd}
 & Q \arrow[r, hook, "j"] \arrow[dl, "p"'] & \widetilde{X} \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & D \arrow[l, hook', "\eta"'] \arrow[dr, "s"] & \\
x \arrow[r, hook, "i"] & X & & \mathbb{P}^4 & S. \arrow[l, hook']
\end{tikzcd}$$

The morphism $\sigma \circ j$ identifies Q with the quadric passing through S. Moreover, let H and h be pullbacks of classes of hyperplanes in $\mathbb{P}(V)$ and $\mathbb{P}^4$, respectively. Then we have the following relations in $Pic(\widetilde{X})$:

$$Q = 2h - D,\ H = 3h - D,\ h = H - Q,\ D = 2H - 3Q,\ K_{\widetilde{X}} = -5h + D = -3H + 2Q.$$

Proof. For the first two claims, we refer to [Huy, Section 1.5.4.] for a detailed proof. This reference provides a proof in the case of a nodal variety X, but it is proven analogously in the cuspidal case, essentially because both singularities can be resolved by one blow-up and they both have multiplicity 2. For the relations in the Picard group $\mathrm{Pic}(\widetilde{X})$, we first note that the right blow-up diagram immediately yields the relation $K_{\widetilde{X}} = -5h + D$. For the other relation, which contains the canonical divisor $K_{\widetilde{X}}$, it is easy to check that $K_{\widetilde{X}} = \pi^* K_X + rQ$ for some $r \in \mathbb{Z}$. Then, by adjunction formula, we have $\pi^* K_X = -3H$. Let $\tilde{h} \subset Q$ denote the pullbacks of classes of hyperplanes in $\mathbb{P}^4$. There exist equalities

$$-3\tilde{h} = K_Q = j^* K_{\widetilde{X}} + Q|_Q = \pi^* K_X + (r+1)Q|_Q = -(r+1)\tilde{h}.$$

Therefore r is equal to 2 and we obtain the relation $K_{\widetilde{X}} = -3H + 2Q$. Since σ is the extension of the projection away from the cuspidal point $x \in X$ on the blow-up $\widetilde{X}$, the relation $h = H - Q$ follows immediately. Finally, note that the proper transform (with respect to σ) of the quadric $\sigma \circ j(Q) \subset \mathbb{P}^4$ passing through S is contracted by π. Therefore we have an equation $Q = 2h - D$. The other relations follow by substitution from those we have proved. $\square$

Recall that Lemma 1.17 and Lemma 1.25 imply that the cuspidal cubic fourfold X has an admissible semiorthogonal decomposition

$$D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(H), \mathcal{O}_X(2H) \rangle, \tag{2.3.1}$$

where the subcategory $\mathcal{A}_X$ denotes the *Kuznetsov component* of X. The main theorem of this subsection is the following.

Theorem 2.4. *We assume Setting 2.2 above. Then there exists a smooth triangulated category $\widetilde{\mathcal{A}}_X \subset D^b(\widetilde{X})$ and functors*

$$\pi_* \colon \widetilde{\mathcal{A}}_X \to \mathcal{A}_X \quad \textit{and} \quad \pi^* \colon \mathcal{A}_X^{\mathrm{perf}} \to \widetilde{\mathcal{A}}_X,$$

where $\mathcal{A}_X^{\mathrm{perf}} = \mathcal{A}_X \cap D^{\mathrm{perf}}(X)$, such that $(\widetilde{\mathcal{A}}_X, \pi_*, \pi^*)$ is a crepant categorical resolution of $\mathcal{A}_X$. Moreover, there exists an equivalence of triangulated categories $D^b(S) \cong \widetilde{\mathcal{A}}_X$.

Proof. The relations in $\mathrm{Pic}(\widetilde{X})$ that we showed in Lemma 2.3 are central for computing the mutations of the semiorthogonal decomposition (2.4.4). Since they are the same as in the case of a nodal cubic fourfold X, the proof of the analogous statement in the nodal case, cf. [Kuz10, Theorem 5.2], generalizes to the case of a cuspidal cubic fourfold X without substantial changes.

By Theorem 2.1, we already know that there exists a crepant categorical resolution $(\widetilde{\mathcal{D}}, \pi_*, \pi^*)$ of $D^b(X)$. In the following proof we will first restrict this resolution to a crepant categorical resolution $\widetilde{\mathcal{A}}_X$ of $\mathcal{A}_X$ and then show that there exists an equivalence of triangulated categories $\widetilde{\mathcal{A}}_X \cong D^b(S)$.

Recall the semiorthogonal decomposition

$$D^b(\widetilde{X}) = \langle j_*\mathcal{O}_Q(-2h), j_*\mathcal{O}_Q(-h), \widetilde{\mathcal{D}}\rangle \tag{2.4.1}$$

which we considered in the proof of Theorem 2.1. Observe that $\pi^* \colon D^{\mathrm{perf}}(X) \to \widetilde{\mathcal{D}}$ is fully faithful, so we can use (2.3.1) to produce a semiorthogonal decomposition

$$\widetilde{\mathcal{D}} = \langle \widetilde{\mathcal{A}}_X, \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(H), \mathcal{O}_{\widetilde{X}}(2H)\rangle, \tag{2.4.2}$$

where the category $\widetilde{\mathcal{A}}_X$ is defined as the left orthogonal $\langle \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(H), \mathcal{O}_{\widetilde{X}}(2H)\rangle^{\perp}$. This refines (2.4.1) and we obtain

$$D^b(\widetilde{X}) = \langle j_*\mathcal{O}_Q(-2h), j_*\mathcal{O}_Q(-h), \widetilde{\mathcal{A}}_X, \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(H), \mathcal{O}_{\widetilde{X}}(2H)\rangle. \tag{2.4.3}$$

We now verify that the functors $\pi_* \colon \widetilde{\mathcal{D}} \to D^b(X)$ and $\pi^* \colon D^{\mathrm{perf}}(X) \to \widetilde{\mathcal{D}}$ restrict to a crepant categorical resolution of $\mathcal{A}_X$, i.e., we prove that $\pi_*(\widetilde{\mathcal{A}}_X) \subset \mathcal{A}_X$ and $\pi^*(\mathcal{A}_X^{\mathrm{perf}}) \subset \widetilde{\mathcal{A}}_X$. Using the definition of $\widetilde{\mathcal{A}}_X$ and the adjunction $\pi^* \dashv \pi_*$, we see that for any $\mathcal{F} \in \widetilde{\mathcal{A}}_X$ and any $k \in \{1, 2, 3\}$, there exist isomorphisms

$$\mathrm{Hom}(\mathcal{O}_X(k), \pi_*\mathcal{F}) \cong \mathrm{Hom}(\pi^*\mathcal{O}_X(k), \mathcal{F}) = \mathrm{Hom}(\mathcal{O}_{\widetilde{X}}(kH), \mathcal{F}) = 0.$$

Analogously, for any $\mathcal{G} \in \mathcal{A}_X^{\mathrm{perf}}$ we have

$$\mathrm{Hom}(\mathcal{O}_{\widetilde{X}}(kH), \pi^*\mathcal{G}) = \mathrm{Hom}(\pi^*\mathcal{O}_X(k), \pi^*\mathcal{G}) \cong \mathrm{Hom}(\pi_*\pi^*\mathcal{O}_X(k), \mathcal{G}) \cong \mathrm{Hom}(\mathcal{O}_X(k), \mathcal{G}) = 0.$$

Finally, we show that there exists an exact equivalence $\widetilde{\mathcal{A}}_X \cong D^b(S)$. We first apply Orlov's blow-up formula to $\pi \colon \widetilde{X} \to \mathbb{P}^4$, which yields a semiorthogonal decomposition

$$D^b(\widetilde{X}) = \langle \Psi(D^b(S)), \mathcal{O}_{\widetilde{X}}(-3h), \mathcal{O}_{\widetilde{X}}(-2h), \mathcal{O}_{\widetilde{X}}(-h), \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(h)\rangle, \tag{2.4.4}$$

where $\Psi(\mathcal{F}) = \eta_* s^*\mathcal{F} \otimes \mathcal{O}_{\widetilde{X}}(D)$ for $\mathcal{F} \in D^b(S)$. One now applies a series of mutations to the decomposition (2.4.4) to obtain a semiorthogonal decomposition

$$D^b(\widetilde{X}) = \langle j_*\mathcal{O}_Q(-2h), j_*\mathcal{O}_Q(-h), \Psi''(D^b(S)), \mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}(H), \mathcal{O}_{\widetilde{X}}(2H)\rangle, \tag{2.4.5}$$

where $\Psi'' = \mathbb{R}_{\mathcal{O}_{\widetilde{X}}(-h)} \circ \mathbb{R}_{\mathcal{O}_{\widetilde{X}}(-2h)} \circ \mathbb{T}_{\mathcal{O}_{\widetilde{X}}(D-2h)} \circ \eta_* \circ s^*$. Here, $\mathbb{T}_{\mathcal{O}_{\widetilde{X}}(D-2h)}$ denotes the functor defined by tensoring with the line bundle $\mathcal{O}_{\widetilde{X}(D-2h)}$. To avoid further repetition, we refer to [Kuz10, Thorem 5.2] for the detailed computation of the mutations. Finally, by comparing the semiorthogonal decomposition (2.4.5) with (2.4.3) it follows that the functor $\Psi'' \colon D^b(S) \to D^b(\widetilde{X})$ induces an equivalence of triangulated categories $D^b(S) \cong \widetilde{\mathcal{A}}$. □

3 Kernels of categorical resolutions of cuspidal singularities

In Theorem 2.1 we constructed a crepant categorical resolution $\widetilde{\mathcal{D}}$ of $D^b(X)$ for a projective variety X with an isolated A_2 singularity. The main goal of this section is a more explicit description of $D^b(X)$ as the Verdier quotient of $\widetilde{\mathcal{D}}$ by the kernel of the functor $\pi_*\colon \widetilde{\mathcal{D}} \to D^b(X)$, see Theorem 3.14. In the case of a fourfold with an isolated A_2 singularity, we prove that the kernel $\ker(\pi_*)$ is generated by two 2-spherical objects, see Theorem 3.15. In Subsection 3.4 we study the special case of a cuspidal cubic fourfold and explicitly describe generators of the kernel of the crepant resolution $\pi_*\colon \widetilde{\mathcal{A}}_X \to \mathcal{A}_X$ as elements of $D^b(S)$, using the equivalence $D^b(S) \cong \widetilde{\mathcal{A}}_X$ established in Theorem 2.4.

In the following two subsections, we recall the definition of spinor bundles (resp. spinor sheaves) on smooth (resp. nodal) quadrics and discuss some basic results on their cohomology. These sheaves play a central role in the proofs of the above results since they give rise to objects generating the kernel of the functor $\pi_*\colon \widetilde{\mathcal{D}} \to D^b(X)$, see Theorem 3.14.

3.1 Spinor sheaves on smooth quadrics

We first recall the definition and basic properties of spinor bundles on a smooth quadric hypersurface $Q \subset \mathbb{P}^{n+1}$, following [Ott88, Sections 1&2]. The definition depends on the parity of the dimension $n := \dim(Q)$.

First, we consider the odd dimensional case $n = 2k+1$. The linear subspaces in Q of maximal dimension are of dimension k and there exists a smooth projective variety T of dimension $(k+1)(k+2)/2$, parametrizing all the k-planes contained in Q. The Picard group $\mathrm{Pic}(T)$ is isomorphic to $\mathbb{Z}$ and for an ample generator $\mathcal{O}_T(1)$ we have $\dim H^0(T, \mathcal{O}_T(1)) = 2^{k+1}$. Pick a point $x \in Q$. Then it yields an embedding

$$T_x := \{[\mathbb{P}^k] \in T \mid x \in \mathbb{P}^k \subset Q\} \overset{i_x}{\hookrightarrow} \{[\mathbb{P}^k] \mid \mathbb{P}^k \subset Q\} = T.$$

In [Ott88, Corollary 1.2] it was shown that the restriction map $H^0(T, \mathcal{O}_T(1)) \to H^0(T_x, i_x^*\mathcal{O}_T(1))$ is surjective. This implies that we have an inclusion $H^0(T_x, i_x^*\mathcal{O}_T(1))^\vee \hookrightarrow H^0(T, \mathcal{O}_T(1))^\vee$ on the dual vector spaces for any $x \in Q$. Since $\dim H^0(T_x, i_x^*\mathcal{O}_T(1)) = 2^k$, we obtain an embedding into a Grassmannian:

$$s\colon Q \hookrightarrow \mathrm{Gr}(2^k, 2^{k+1}).$$

Let $\mathcal{U}$ denote the universal bundle of $\mathrm{Gr}(2^k, 2^{k+1})$. Then the spinor bundle $\mathscr{S}$ on Q is defined as the pullback $s^*\mathcal{U}$.

In the case of a quadric hypersurface $Q \subset \mathbb{P}^{n+1}$ of even dimension $n = 2k$, one can proceed in a similar way. The linear subspaces of maximal dimension are of dimension k and the k-planes in Q are parametrized by two smooth projective varieties T' and T'' of dimension $k(k+1)/2$. Let $\mathcal{O}_{T'}(1)$ and $\mathcal{O}_{T''}(1)$ denote the ample generators of $\mathrm{Pic}(T') \cong \mathbb{Z}$ and $\mathrm{Pic}(T'') \cong \mathbb{Z}$, respectively. It holds that $\dim H^0(T', \mathcal{O}_{T'}(1)) = \dim H^0(T'', \mathcal{O}_{T''}(1)) = 2^k$. We again pick a point $x \in Q$ and consider the embedding

$$T'_x \cup T''_x := \{[\mathbb{P}^k] \mid x \in \mathbb{P}^k \subset Q\} \overset{j_x}{\hookrightarrow} \{[\mathbb{P}^k] \mid \mathbb{P}^k \subset Q\} = T' \cup T'',$$

which restricts to two embeddings $j'_x\colon T'_x \hookrightarrow T'$ and $j''_x\colon T''_x \hookrightarrow T''$. In [Ott88, Corollary 1.2] it is shown that for any $x \in Q$ we have inclusions on the dual vector spaces

$$H^0(T'_x, j'^*_x\mathcal{O}_{T'}(1))^\vee \hookrightarrow H^0(T', \mathcal{O}_{T'}(1))^\vee \quad \text{and} \quad H^0(T''_x, j''^*_x\mathcal{O}_{T''}(1))^\vee \hookrightarrow H^0(T'', \mathcal{O}_{T''}(1))^\vee.$$

One can prove that $\dim H^0(T'_x, j'^*_x \mathcal{O}_{T'}(1)) = \dim H^0(T''_x, j''^*_x \mathcal{O}_{T''}(1)) = 2^{k-1}$, which gives rise to embeddings

$$s' \colon Q \hookrightarrow \mathrm{Gr}(2^{k-1}, 2^k) \quad \text{and} \quad s'' \colon Q \hookrightarrow \mathrm{Gr}(2^{k-1}, 2^k),$$

for every $x \in Q$. Now the spinor bundles on Q are defined as $\mathscr{S}_1 := s'^*\mathcal{U}$ and $\mathscr{S}_2 := s'^*\mathcal{U}$.

In the next proposition we will summarize some basic properties of spinor bundles, following [Ott88] and [Kap88].

Proposition 3.1. *Let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface. Let $\mathscr{S}_1, \mathscr{S}_2$ denote the spinor bundles for an even dimensional quadric Q and let $\mathscr{S}$ denote the spinor bundle in the odd dimensional case. Let $\widetilde{\mathscr{S}}$ denote any of the spinor bundles if the statement does not depend on the dimension of Q. Then:*

a)

$$\begin{aligned} &H^i(Q, \widetilde{\mathscr{S}}(k)) \cong 0 && \text{for } i \neq n-1, \text{ for all } k \in \mathbb{Z}, \\ &H^0(Q, \widetilde{\mathscr{S}}(k)) \cong 0 && \text{for } k \leq 0, \\ &H^{n-1}(Q, \widetilde{\mathscr{S}}(k)) \cong 0 && \text{for } k \geq 1-n. \end{aligned} \tag{3.1.1}$$

b) If Q is of odd dimension $2m+1$, then there exists a short exact sequence

$$0 \longrightarrow \mathscr{S} \longrightarrow \mathcal{O}_Q^{\oplus 2^{m+1}} \to \mathscr{S}(1) \longrightarrow 0, \tag{3.1.2}$$

and $\mathscr{S}^\vee \cong \mathscr{S}(1)$.

c) If Q is of even dimension $2m$, then there exist short exact sequences

$$\begin{aligned} 0 \longrightarrow \mathscr{S}_1 \longrightarrow \mathcal{O}_Q^{\oplus 2^m} \longrightarrow \mathscr{S}_2(1) \longrightarrow 0, \\ 0 \longrightarrow \mathscr{S}_2 \longrightarrow \mathcal{O}_Q^{\oplus 2^m} \longrightarrow \mathscr{S}_1(1) \longrightarrow 0. \end{aligned} \tag{3.1.3}$$

Moreover, we have isomorphisms

$$\begin{aligned} &\mathscr{S}_1^\vee \cong \mathscr{S}_1(1) \;\; \text{and} \;\; \mathscr{S}_2^\vee \cong \mathscr{S}_2(1), \;\; \text{for } m \text{ even}, \\ &\mathscr{S}_1^\vee \cong \mathscr{S}_2(1) \;\; \text{and} \;\; \mathscr{S}_2^\vee \cong \mathscr{S}_1(1), \;\; \text{for } m \text{ odd}. \end{aligned}$$

d) The spinor bundles of both even and odd dimensional quadrics are exceptional objects in $D^b(Q)$. If Q is even dimensional, the spinor bundles $\mathscr{S}_1$ and $\mathscr{S}_2$ are furthermore orthogonal to each other.

Proof. The first two isomorphisms in part a) are proven in [Ott88, Theorem 2.3]. The last one can be established using a simple Serre duality argument as follows. First, note that by the adjunction formula, the canonical bundle ω_Q is isomorphic to $\mathcal{O}_Q(1-n)$. Therefore we have an isomorphism

$$H^{n-1}(Q, \mathscr{S}(k)) \cong H^0(Q, \mathscr{S}(1-k-n))^\vee,$$

where the right hand side vanishes for $1-k-n \leq 0$, or, in other words, for $k \geq 1-n$. For part b) and c), we refer to [Ott88, Theorem 2.8]. The final statement d) is proved in [Kap88, Proposition 4.9]. □

We have a particularly nice decomposition of the derived category $D^b(Q)$ involving the spinor bundles introduced above.

Proposition 3.2. *Let $Q \subset \mathbb{P}^n$ be a smooth quadric hypersurface. Using the notation of Proposition 3.1, there exist the following full exceptional collections.*

$$D^b(Q) = \begin{cases} \langle \mathcal{O}_Q(2-n), \dots, \mathcal{O}_Q(-1), \mathscr{S}, \mathcal{O}_Q \rangle, & \textit{if } n \textit{ is odd,} \\ \langle \mathcal{O}_Q(2-n), \dots, \mathcal{O}_Q(-1), \mathscr{S}_1, \mathscr{S}_2, \mathcal{O}_Q \rangle, & \textit{if } n \textit{ is even.} \end{cases} \tag{3.2.1}$$

Proof. In [Kap88, Proposition 4.9, Theorem 4.10] it was shown that there exist semiorhtogonal decompositions

$$D^b(Q) = \begin{cases} \langle \mathscr{S}, \mathcal{O}_Q, \mathcal{O}_Q(1), \dots, \mathcal{O}_Q(n-2) \rangle, & \text{if } n \text{ is odd,} \\ \langle \mathscr{S}_1, \mathscr{S}_2, \mathcal{O}_Q, \mathcal{O}_Q(1), \dots, \mathcal{O}_Q(n-2) \rangle, & \text{if } n \text{ is even.} \end{cases}$$

Since Q is smooth, the category $D^b(Q)$ admits a Serre functor given by $\mathbb{S}_Q = - \otimes \omega_Q[n-1] \cong - \otimes \mathcal{O}_Q(1-n)[n-1]$. By an application of Lemma 1.25, we can permute the decompositions above by applying the Serre functor to the subcategories $\langle \mathcal{O}_Q(n-2) \rangle, \dots, \langle \mathcal{O}_Q(1) \rangle$ one by one. This gives rise to the decompositions (3.2.1). □

3.2 Spinor sheaves on nodal quadrics and their bounded derived categories

We now introduce the definition of spinor sheaves on a nodal quadric Y, following [KS23a, Proposition 6.4] and [Kaw22, Remark 6.6]. Informally, these sources define the spinor sheaves as pullbacks of the spinor bundles on a smooth quadric Q, along the rational map $\rho_0 \colon Y \dashrightarrow Q$, given by projecting away from the node $y \in Y$. Moreover, there also exists a definition of these sheaves via Clifford algebras, cf. [Add11, Section 2], but for our purposes the previous definition will suffice.

In this subsection we discuss basic results on the cohomology of the spinor sheaves on a nodal quadric and we provide a proof for the existence of a semiorthogonal decomposition of a nodal quadric, analogous to the one of Proposition 3.2 in the smooth case, using spinor sheaves, cf. Theorem 3.10. Later on in Section 3.3 we will use this result to explicitly determine generators of the kernel $\ker(\pi_* \colon \widetilde{\mathcal{D}} \to D^b(X))$ for a variety X with a single isolated A_2 singularity.

We fix the following notation throughout this subsection. Let $n \geq 2$ be an integer and consider the diagram

$$\begin{tikzcd} & E \arrow[r, hook, "\eta"] \arrow[dl] & \widetilde{Y} = \mathrm{Bl}_y(Y) \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\ \{y\} \arrow[r, hook] & Y \arrow[rr, dashed, "\rho_0"] & & Q, \end{tikzcd} \tag{3.2.2}$$

where $Q \subseteq \mathbb{P}^n$ is a smooth quadric hypersurface and Y denotes the cone over Q in $\mathbb{P}^{n+1}$ with an isolated nodal singularity at $y \in Y$. The lower dashed arrow ρ_0 is the projection away from the node y and ρ is the extension of the rational map ρ_0 to the blow-up $\widetilde{Y}$. The morphism ρ is a $\mathbb{P}^1$-bundle. More precisely, there exists an isomorphism $\widetilde{Y} \cong \mathbb{P}(\mathcal{O}_Q(1) \oplus \mathcal{O}_Q)$, for details we refer to [Huy, §1.5.1]. In the left part of the diagram, E denotes the exceptional divisor of the blow-up π and $\eta \colon E \hookrightarrow \widetilde{Y}$ the corresponding closed immersion. Note that $\rho \circ \eta \colon E \to Q$ is an isomorphism.

We start with an observation concerning the relationship between different hyperplane sections in the above diagram, which we will use later in the proof of Theorem 3.10.

Lemma 3.3. *Let H (resp. h) denote the pullback of classes of hyperplane sections in Y (resp. in Q) to $\widetilde{Y}$. Let L denote the class of hyperplanes on $\widetilde{Y}$ corresponding to the invertible sheaf $\mathcal{O}_\rho(1)$ of the projective bundle ρ. Then the following relations hold in the Picard group* $\mathrm{Pic}(\widetilde{Y})$*:*

$$h = H - E \quad \text{and} \quad L = H.$$

Proof. The first equation holds more generally for any projection away from a linear subspace and can be checked explicitly in local coordinates. For the other equation, note, as mentioned above, that there exists an isomorphism $\widetilde{Y} \cong \mathbb{P}(\mathcal{O}_Q(1) \oplus \mathcal{O}_Q)$, for details see [Huy, §1.5.1]. Let $\mathcal{E}$ denote the locally free sheaf $\mathcal{O}_Q(1) \oplus \mathcal{O}_Q$. Then the projective bundle formula yields the following equation in $\mathrm{Pic}(\widetilde{Y})$:

$$K_{\widetilde{Y}} = -2L + \rho^* K_Q + \rho^* \det(\mathcal{E}) = -2L + (1-n)h + h = -2L - (n-2)h.$$

In Lemma 1.9 we calculated the discrepancy of the blow-up $\pi\colon \widetilde{Y} \to Y$ and obtained that

$$K_{\widetilde{Y}} = \pi^* K_Y + (n-2)E = -nH + (n-2)E.$$

Together the equations imply

$$-2L = (n-2)h + (n-2)E - nH = (n-2)H - nH = -2H.$$

Finally, since ρ is a projective bundle, we have $\mathrm{Pic}(\widetilde{Y}) = \mathrm{Pic}(Q) \oplus \mathbb{Z}$ and since $\mathrm{Pic}(Q)$ is torsion-free, we obtain that $L = H$. $\square$

In Definition 3.7 we will introduce the spinor sheaves on Y as a pushforward of the pullback of the spinor bundles on Q along the right part of the diagram (3.2.2). For this construction to make sense, we need to ensure that the resulting complex of sheaves is indeed a sheaf in $D^b(Y)$. To this end, we recollect the following straightforward criterion.

Lemma 3.4 ([KS23a, Lemma 6.3])**.** *Let $\mathcal{F}$ be a sheaf on $\widetilde{Y}$ such that*

$$H^i(E, \eta^* \mathcal{F}(m)) = 0,$$

for all $m \geq 0$ and $i > 0$. Then the higher direct image sheaves $R^i \pi_* \mathcal{F}$ are trivial for all $i > 0$.

Next, we will provide a reminder of the property of a sheaf to be *maximal Cohen-Macaulay*. In Proposition 3.6 we will show that the spinor sheaves on a nodal quadric are maximal Cohen-Macaulay.

Definition 3.5. A coherent sheaf $\mathcal{F}$ on a Gorenstein scheme Y is called *maximal Cohen-Macaulay*, if $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_Y) = 0$ for all $i > 0$.

Remark. The definition immediately implies that locally free sheaves are maximal Cohen-Macaulay. Conversely, we have that maximal Cohen-Macaulay sheaves are reflexive, see [Buc86, Lemma 4.2.2(iii)]. Note that a reflexive sheaf on an integral normal scheme is already uniquely determined by its restriction to a subspace of codimension at least 2, cf. [Sch10, Corollary 2.11]. This fact will be frequently used in the following proofs.

Proposition 3.6. *Let $\widetilde{\mathscr{S}}$ denote any of the spinor bundles on $Q \subset \mathbb{P}^n$, regardless of the parity of its dimension. Then* *a) For any $k \geq 2-n$, the complex $\pi_*\rho^*\mathcal{O}_Q(k)$ is a sheaf on Y. There exist isomorphisms*

$$\mathcal{O}_Y(k) \cong \pi_*\rho^*(\mathcal{O}_Q(k)), \tag{3.6.1}$$

for any $0 \geq k \geq 2-n$.

b) For any $k \geq 1-n$, the complex $\pi_*\rho^*(\mathscr{S}(k))$ is a sheaf on Y. These sheaves are maximal Cohen-Macaulay for $2 \geq k \geq 1-n$.

Proof. To prove part a), we first verify that $\pi_*\rho^*\mathcal{O}_Q(k)$ is indeed a sheaf for $k \geq 2-n$. We have an isomorphism $\eta^*\rho^*\mathcal{O}_Q \cong \mathcal{O}_E$ and by Lemma 3.4, it suffices to show that the cohomology groups $H^i(E, \mathcal{O}_E(k+m))$ vanish for all $i > 0$ and $m \geq 0$. For this, we consider the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-2+k+m) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(k+m) \longrightarrow \mathcal{O}_E(k+m) \longrightarrow 0.$$

The higher cohomology groups of the left and middle terms vanish for any $m \geq 0$ and any twist $k \geq 2-n$. Indeed, note that in degree n we apply Serre duality and obtain the isomorphisms

$$\begin{aligned} H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-2+k+m)) &\cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2-k-m) \otimes \mathcal{O}_{\mathbb{P}^n}(-n-1))^\vee \\ &\cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1-k-m-n))^\vee, \end{aligned}$$

where the latter term vanishes for all $m \geq 0$, if $k \geq 2-n$. Next, we will verify that the sheaves $\pi_*\rho^*\mathcal{O}_Q(k)$ are maximal Cohen-Macaulay for any $0 \geq k \geq 2-n$. Using Grothendieck-Verdier duality, we obtain

$$\begin{aligned} R\mathcal{H}om(R\pi_*(\rho^*\mathcal{O}_Q(k)), \mathcal{O}_Y) &\cong R\pi_*(R\mathcal{H}om(\rho^*\mathcal{O}_Q(k), \omega_\pi)) \\ &\cong R\pi_*(R\mathcal{H}om(\rho^*\mathcal{O}_Q(k), \mathcal{O}_{\widetilde{Y}}((n-2)E))) \\ &\cong R\pi_*(\rho^*\mathcal{O}_Q(k)^\vee \otimes \mathcal{O}_{\widetilde{Y}}((n-2)E)). \end{aligned} \tag{3.6.2}$$

Recall that $\omega_\pi = \mathcal{O}_{\widetilde{Y}}((n-2)E)$, by the results of Lemma 1.9. We now apply Lemma 3.4 and remind that $\rho \circ \eta \colon E \to Q$ is an isomorphism. Therefore $\eta^*(\rho^*\mathcal{O}_Q(k)^\vee) \cong \mathcal{O}_E(-k)$, from which we obtain an isomorphism

$$\eta^*(\rho^*\mathcal{O}_Q(k)^\vee \otimes \mathcal{O}_{\widetilde{Y}}((n-2)E)) \cong \mathcal{O}_E(2-n-k).$$

The same calculation as before shows that the cohomology groups $H^i(E, \mathcal{O}_E(2+m-n-k))$ vanish for all $i > 0$ if $m \geq 0$ and $k \leq 0$. The isomorphisms (3.6.1) can be deduced from Remark 3.2, since the sheaves $\mathcal{O}_Y(k)$ and $\pi_*\rho^*\mathcal{O}_Q(k)$ are both reflexive and coincide on the smooth locus $Y \setminus \{x\}$.

For part b), we proceed in the same way as for a). Also note that the following calculation was already performed in [KS23a, Lemma 6.3] for the case $k = 0$. By Proposition 3.1a), the cohomology groups

$$H^i(E, \widetilde{\mathscr{S}}(k+m))$$

vanish for all $i > 0$ if $m \geq 0$ and $k \geq 1-n$. Therefore Lemma 3.4 implies that the complex $\pi_*\rho^*(\mathscr{S}(k))$ is a sheaf for all $k \geq 1-n$. Next, using Proposition 3.1 and Grothendieck-Verdier duality, there exist the isomorphisms

$$\begin{aligned} R\mathcal{H}om(R\pi_*(\rho^*\widetilde{\mathscr{S}}(k)), \mathcal{O}_Y) &\cong R\pi_*(R\mathcal{H}om(\rho^*(\widetilde{\mathscr{S}}(k)), \omega_\pi)) \\ &\cong R\pi_*(\rho^*\widetilde{\mathscr{S}}(k)^\vee \otimes \mathcal{O}_{\widetilde{Y}}((n-2)E)) \\ &\cong R\pi_*(\rho^*(\widetilde{\mathscr{S}}(1-k)) \otimes \mathcal{O}_{\widetilde{Y}}((n-2)E)), \end{aligned}$$

where the last one follows from Theorem 3.1b) and c). Moreover, we have

$$\eta^*(\rho^*(\widetilde{\mathscr{S}}(1-k))\otimes \mathcal{O}_{\widetilde{Y}}((n-2)E))\cong \mathscr{S}(3-n-k).$$

Finally, by Proposition 3.1a) the vector spaces $H^i(E,\mathscr{S}(3+m-n-k))$ vanish for all $m\geq 0$, $k\leq 2$ and $i>0$. Again, the claim follows from an application of Lemma 3.4. □

Definition 3.7 ([KS23a, cf. Proposition 6.4])**.** We define the *spinor sheaves* on an odd dimensional nodal quadric as

$$\mathcal{S}_1 := \pi_*\rho^*\mathscr{S}_1, \quad \mathcal{S}_2 := \pi_*\rho^*\mathscr{S}_2.$$

Analogously, we define the *spinor sheaf* on an even dimensional nodal quadric as

$$\mathcal{S} := \pi_*\rho^*\mathscr{S}.$$

These sheaves are maximal Cohen Macaulay by Proposition 3.6a) and therefore by Remark 3.2 in particular reflexive.

Remark. We have an isomorphism

$$\mathcal{S}_i(k) := \pi_*\rho^*(\mathscr{S}_i)\otimes \mathcal{O}_Y(k)\cong \pi_*\rho^*(\mathscr{S}_i\otimes \mathcal{O}_Q(k)),$$

for any $2\geq k\geq 1-n$, which follows from Proposition 3.6b) and Remark 3.2.

In the following proposition, we derive the analogues of the exact sequences of the spinor bundles in Proposition 3.1b) and c) for spinor sheaves on a nodal quadric.

Proposition 3.8. *Let Y be a nodal quadric of dimension n. If Y is even dimensional, we have a short exact sequence*

$$0\longrightarrow \mathcal{S}\longrightarrow \mathcal{O}_Y^{\oplus 2^{m+1}}\to \mathcal{S}(1)\longrightarrow 0, \qquad n=2m+2. \tag{3.8.1}$$

If Y is odd dimensional, we have the following short exact sequences, relating the two spinor sheaves $\mathcal{S}_1$ and $\mathcal{S}_2$:

$$\begin{aligned} 0\longrightarrow \mathcal{S}_1\longrightarrow \mathcal{O}_Y^{\oplus 2^m}\longrightarrow \mathcal{S}_2(1)\longrightarrow 0, &\qquad n=2m+1,\\ 0\longrightarrow \mathcal{S}_2\longrightarrow \mathcal{O}_Y^{\oplus 2^m}\longrightarrow \mathcal{S}_1(1)\longrightarrow 0, &\qquad n=2m+1. \end{aligned} \tag{3.8.2}$$

Proof. Observe that we already have the short exact sequences (3.1.2) and (3.1.3) on the smooth quadric. Applying the functor ρ^* to (3.1.2) we obtain a short exact sequence

$$0\longrightarrow \rho^*\mathscr{S}\longrightarrow \mathcal{O}_{\widetilde{Y}}^{\oplus 2^{m+1}}\to \rho^*\mathscr{S}(1)\longrightarrow 0,$$

since ρ is flat. Next, note that we have an isomorphism $\pi_*\mathcal{O}_{\widetilde{Y}}\cong \mathcal{O}_Y$, because Y has rational singularities. Moreover, the higher direct image sheaves $R^i\pi_*\rho^*\mathcal{S}(k)$ vanish for all $i>0$ if $k\geq 2-n$, by Proposition 3.6b). This implies that the following exact triangle is in fact a short exact sequence of sheaves:

$$0\longrightarrow \underbrace{\pi_*\rho^*\mathscr{S}}_{:=\mathcal{S}}\longrightarrow \mathcal{O}_Y^{\oplus 2^{m+1}}\longrightarrow \underbrace{\pi_*\rho^*\mathscr{S}(1)}_{\cong \mathcal{S}(1)}\longrightarrow 0.$$

The case of an odd dimensional nodal quadric Y works in a completely analogous way. □

In the next theorem we collect results on the cohomology of the spinor bundles on a nodal quadric Y. In particular, the Ext-groups below will play a central role in the proof that the generators of the kernel $\ker(\pi_*\colon \widetilde{\mathcal{D}} \to D^b(X))$ are spherical for a cuspidal fourfold X, cf. Theorem 3.15.

Theorem 3.9. *Let $\widetilde{\mathcal{S}}$ denote any of the spinor sheaves on a nodal quadric hypersurface $Y \subset \mathbb{P}^{n+1}$, independent of the parity of $\dim(Y)$. Then for $0 \geq k \geq 1-n$ and any $i \geq 0$ we have*

$$H^i(Y, \widetilde{\mathcal{S}}(k)) \cong 0. \tag{3.9.1}$$

If Y is 3-dimensional and $\mathcal{S}_1$, $\mathcal{S}_2$ denote the spinor sheaves on Y, the following statements hold:

$$\begin{aligned} &\operatorname{Hom}(\mathcal{S}_i, \mathcal{S}_i) \cong k \qquad \textit{for } i = 1, 2, \\ &\operatorname{Hom}(\mathcal{S}_i, \mathcal{S}_j) \cong 0 \qquad \textit{for } i \neq j,\ i, j \in \{1, 2\}; \end{aligned} \tag{3.9.2}$$

$$\operatorname{Ext}^p(\mathcal{S}_1, \mathcal{S}_1) \cong \operatorname{Ext}^p(\mathcal{S}_2, \mathcal{S}_2) \cong \begin{cases} k, & \textit{if } p \equiv 0 \pmod 2 \\ 0, & \textit{if } p \equiv 1 \pmod 2 \end{cases} \quad \textit{and} \quad p > 0; \tag{3.9.3}$$

$$\operatorname{Ext}^p(\mathcal{S}_1, \mathcal{S}_2) \cong \operatorname{Ext}^p(\mathcal{S}_2, \mathcal{S}_1) \cong \begin{cases} 0, & \textit{if } p \equiv 0 \pmod 2 \\ k, & \textit{if } p \equiv 1 \pmod 2 \end{cases} \quad \textit{and} \quad p > 0. \tag{3.9.4}$$

Proof. We begin with the proof of statement (3.9.1) and first recall that $\rho\colon \widetilde{Y} \to Q$ is a $\mathbb{P}^1$-bundle. Therefore the higher direct image sheaves $R^i\rho_*\mathcal{O}_{\widetilde{Y}}$ vanish for $i > 0$ and the projection formula implies that

$$H^i(Q, \mathcal{F}) \cong H^i(Q, \rho_*\rho^*\mathcal{F}) \cong H^i(\widetilde{Y}, \rho^*\mathcal{F}),$$

for any sheaf $\mathcal{F} \in D^b(Q)$. In Proposition 3.6b) we proved that all higher direct images $R^i\pi_*\rho^*(\mathscr{S}(k))$ vanish for $k \geq 1-n$ and $i > 0$. Therefore we obtain isomorphisms

$$H^p(Y, \mathcal{S}(k)) = H^p(Y, \pi_*\rho^*(\mathscr{S}(k))) \cong H^p(\widetilde{Y}, \rho^*(\mathscr{S}(k)))$$

for any $p \geq 0$ and $k \geq 1-n$. The claim now follows from the results of Proposition 3.1a) on the cohomology of spinor bundles on a smooth quadric.

To prove claim (3.9.2), note in the following that a smooth quadric Q of dimension 2 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and under this isomorphism the spinor bundles $\mathscr{S}_1$ and $\mathscr{S}_2$ correspond to the line bundles $\mathcal{O}_Q(0,-1)$ and $\mathcal{O}_Q(-1,0)$, respectively. The sheaf $\mathcal{H}om(\mathcal{S}_i, \mathcal{S}_j)$ is reflexive since the spinor sheaves are reflexive, cf. [Sch10, Corollary 2.9]. By restricting to the smooth locus $U := Y \setminus \{y\}$, there exist isomorphisms

$$\mathcal{H}om(\mathcal{S}_i, \mathcal{S}_i)|_U \cong \mathcal{H}om_{\mathcal{O}_U}(\mathcal{S}_i|_U, \mathcal{S}_i|_U) \cong \mathcal{O}_U,$$

for $i \in \{1, 2\}$, since $\mathcal{S}_i|_U$ are locally free of rank 1. Moreover, we have

$$\begin{aligned} \mathcal{H}om_{\mathcal{O}_U}(\mathcal{S}_i|_U, \mathcal{S}_j|_U) &\cong \mathcal{S}_j|_U \otimes \mathcal{S}_i|_U^\vee \cong \rho_0^*\mathscr{S}_j \otimes (\rho_0^*\mathscr{S}_i)^\vee \\ &\cong \begin{cases} \rho_0^*(\mathcal{O}_Q(1,-1)), & \text{if } i = 2,\ j = 1, \\ \rho_0^*(\mathcal{O}_Q(-1,1)), & \text{if } i = 1,\ j = 2. \end{cases} \end{aligned}$$

Here we abused the notation ρ_0 to denote the restriction of the projection away from the nodal point $y \in Y$ to the open subset U, which is a well-defined morphism $U \to Q$. By Proposition 3.6a) and under the application of the Künneth formula, it follows that $\pi_*\rho^*\mathcal{O}_Q(a,b)$ is a sheaf on Y for $a, b \geq -1$. Let us denote these sheaves by $\mathcal{O}_Y(a,b)$

for integers $a, b \geq -1$. One can straightforward verify that they are maximal Cohen Macaulay, if $a \leq 0$ or $b \leq 0$. This implies in particular that the sheaves $\mathcal{O}_Y(1,-1)$ and $\mathcal{O}_Y(-1,1)$ are reflexive. The fact that also $\mathcal{H}om(\mathcal{S}_i, \mathcal{S}_j)$ is reflexive and Remark 3.2 yield

$$\begin{aligned}
\mathrm{Hom}(\mathcal{S}_i, \mathcal{S}_j) \cong H^0(\mathcal{H}om(\mathcal{S}_i, \mathcal{S}_j)) &\cong \begin{cases} H^0(Y, \mathcal{O}_Y), & \text{if } i = j, \\ H^0(Y, \mathcal{O}_Y(1,-1)) \text{ or } H^0(Y, \mathcal{O}_Y(-1,1)), & \text{else.} \end{cases} \\
&\cong \begin{cases} k, & \text{if } i = j, \\ 0, & \text{else.} \end{cases}
\end{aligned}$$

For the last isomorphism we used that $H^0(Q, \mathcal{O}_Q(1,-1)) = H^0(Q, \mathcal{O}_Q(-1,1)) = 0$ and the fact that we can "pull back cohomology" of sheaves on Q to Y in this case, using the same arguments as in the beginning of the proof for the spinor bundles.

For the rest of the proof we sketch [Kaw22, Lemma 6.2], where the author calculates the Ext-groups (3.9.3) and (3.9.4). First, let us consider the local to global spectral sequences

$$E_2^{p,q} = H^p(Y, \mathcal{E}xt^q(\mathcal{S}_2, \mathcal{S}_1)) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{S}_2, \mathcal{S}_1) \tag{3.9.5}$$
$$E_2^{p,q} = H^p(Y, \mathcal{E}xt^q(\mathcal{S}_2(1), \mathcal{S}_1)) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{S}_2(1), \mathcal{S}_1) \tag{3.9.6}$$

Note that the E_2-page of both spectral sequences is trivial outside the coordinate axes p and q. Moreover, by the same arguments as before, we see that the cohomology groups $H^p(Y, \mathcal{H}om(\mathcal{S}_2, \mathcal{S}_1))$, $H^p(Y, \mathcal{H}om(\mathcal{S}_2(1), \mathcal{S}_1))$ vanish for all $p > 0$, which implies that the spectral sequences are in fact trivial. Then one obtains natural isomorphisms

$$\mathrm{Ext}^p(\mathcal{S}_2, \mathcal{S}_1) \cong H^0(\mathcal{E}xt^p(\mathcal{S}_2, \mathcal{S}_1)) \cong H^0(\mathcal{E}xt^p(\mathcal{S}_2(1), \mathcal{S}_1)) \cong \mathrm{Ext}^p(\mathcal{S}_2(1), \mathcal{S}_1). \tag{3.9.7}$$

The key of Kawamata's proof is to construct locally free extensions of $\mathcal{S}_2$ by $\mathcal{S}_1$ and $\mathcal{S}_2$ by $\mathcal{S}_1$, respectively, using the exact sequences (3.8.2). Let $\mathscr{S}_2 \hookrightarrow \mathscr{S}_2(1)$ be an injective morphism coming from a hyperplane section in Q. Applying the composition of functors $\pi_* \circ \rho^*$ to it we obtain an injective morphism $\psi \colon \mathcal{S}_2 \hookrightarrow \mathcal{S}_2(1)$ on Y, by the results of Proposition 3.6 and Remark 3.2. Pulling pack the extension defined by the first short exact sequence of (3.8.2) along ψ we obtain an extension $\mathcal{G}_1$ of $\mathcal{S}_2$ by $\mathcal{S}_1$, fitting into a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}_1 \arrow[r] \arrow[d, equal] & \mathcal{G}_1 \arrow[r] \arrow[d, hook, "\eta"] & \mathcal{S}_2 \arrow[r] \arrow[d, hook, "\psi"] & 0 \\
0 \arrow[r] & \mathcal{S}_1 \arrow[r] \arrow[d] & \mathcal{O}_Y^2 \arrow[r] \arrow[d, two heads] & \mathcal{S}_2(1) \arrow[r] \arrow[d, two heads] & 0 \\
& 0 \arrow[r] & C \arrow[r, "\cong"] & \mathcal{S}_2(1)|_H \arrow[r] & 0.
\end{tikzcd} \tag{3.9.8}$$

Here we denote by H a hyperplane section coming from $\mathbb{P}^4$ corresponding to the line bundle $\mathcal{O}_Y(1)$ and by C we denote the cokernel of the induced morphism η. An application of the snake lemma yields an isomorphism $C \cong \mathcal{S}_2(1)|_H$. We now explain why the coherent sheaf $\mathcal{G}_1$ is locally free. Note that H is a general hyperplane, so we can choose it in a way that it does not pass through the node of Y. By the middle vertical sequence this implies that we have an isomorphism $\mathcal{G}_y \cong \mathcal{O}_{Y,y}^2$ for all $y \in Y \setminus H$. Moreover, the reflexive sheaves $\mathcal{S}_1$ and $\mathcal{S}_2$ are locally free restricted to the smooth locus of

Y. This implies the vanishing of $\mathrm{Ext}^i(\mathcal{S}_{1y}, M)$ and $\mathrm{Ext}^i(\mathcal{S}_{2y}, M)$ for all $\mathcal{O}_{Y,y}$-modules M and any $y \in Y^{\mathrm{sm}}$. With the upper horizontal exact sequence we obtain the same vanishing for G_{1_y} for all $y \in Y^{\mathrm{sm}}$, which implies its local freeness along Y^{sm}. Together this implies that $\mathcal{G}_1$ is a locally free sheaf on Y. Above we showed that the cohomology groups of the spinor sheaves vanish. By applying the functor $\mathrm{RHom}(-, \mathcal{S}_1)$ to the middle vertical sequence this implies that we have isomorphisms

$$\mathrm{Ext}^p(\mathcal{G}_1, \mathcal{S}_1) \cong \mathrm{Ext}^{p+1}(\mathcal{S}_2(1)|_H, \mathcal{S}_1) = 0.$$

The latter Ext-groups vanish by considering the right vertical exact sequence of the diagram (3.9.8) and the isomorphisms (3.9.7). Analogously, we can deduce

$$\mathrm{Ext}^p(\mathcal{G}_1, \mathcal{S}_2) \cong \mathrm{Ext}^{p+1}(\mathcal{S}_2(1)|_H, \mathcal{S}_2) = 0.$$

One can perform the same calculation in the case the roles of $\mathcal{S}_1$ and $\mathcal{S}_2$ are reversed in the diagram (3.9.8). This defines a non-trivial extension $\mathcal{G}_2$ of $\mathcal{S}_1$ by $\mathcal{S}_2$. For the rest of the proof Kawamata uses the diagram and all the cohomological results gathered above to calculate the desired Ext-groups. To avoid repetition, we refer to [Kaw22, Lemma 6.2] for the details of these calculations. □

Remark. The calculation of the Ext-groups above does not immediately generalize to any dimension. If Y is of dimension 4 or higher, we cannot deduce that the spectral sequences (3.9.5) and (3.9.6) are trivial in the same way as above, since do not have control the cohomology groups $H^*(Y, \mathcal{H}om(\mathcal{S}_1, \mathcal{S}_2))$. However, A. Kuznetsov and E. Shinder calculated $\mathrm{Ext}^\bullet(\widetilde{\mathcal{S}}, \widetilde{\mathcal{S}})$ using a different approach, see [KS23a, Proposition 6.4].

Corollary 3.9.1. *Let $Y \subset \mathbb{P}^4$ be a nodal quadric hypersurface and denote the spinor sheaves by $\mathcal{S}_1$ and $\mathcal{S}_2$. Then there exist isomorphisms*

$$\mathrm{Ext}^p(\mathcal{S}_1(1), \mathcal{S}_1) \cong \mathrm{Ext}^p(\mathcal{S}_2(1), \mathcal{S}_2) \cong \begin{cases} 0, & \text{if } p = 0, 1, \\ k, & \text{if } p \equiv 0 \pmod 2 \quad \text{and } p \geq 2 \\ 0, & \text{if } p \equiv 1 \pmod 2 \quad \text{and } p \geq 2 \end{cases}$$

$$\mathrm{Ext}^p(\mathcal{S}_1(1), \mathcal{S}_2) \cong \mathrm{Ext}^p(\mathcal{S}_2(1), \mathcal{S}_1) \cong \begin{cases} 0, & \text{if } p \equiv 0 \pmod 2 \\ k, & \text{if } p \equiv 1 \pmod 2 \end{cases} \quad \text{and } p \geq 0.$$

Proof. In Proposition 3.8, we showed that there exists a short exact sequence

$$0 \longrightarrow \mathcal{S}_2 \longrightarrow \mathcal{O}_Y^{\oplus 2} \longrightarrow S_1(1) \longrightarrow 0. \tag{3.9.9}$$

Applying the functor $\mathrm{RHom}(-, \mathcal{S}_1)$ to this exact sequence gives rise to an exact triangle

$$\mathrm{RHom}(\mathcal{S}_1(1), \mathcal{S}_1) \to \mathrm{RHom}(\mathcal{O}_Y^{\oplus 2}, \mathcal{S}_1) \to \mathrm{RHom}(\mathcal{S}_2, \mathcal{S}_1).$$

The complex in the middle vanishes by (3.9.1), therefore we obtain isomorphisms

$$\mathrm{Ext}^p(\mathcal{S}_2, \mathcal{S}_1) \cong \mathrm{Ext}^{p+1}(\mathcal{S}_1(1), \mathcal{S}_1).$$

Using the result (3.9.4) of the previous theorem, we obtain the desired claim. For the other claim we proceed in the same way. In Proposition 3.8 we showed the existence of a short exact sequence

$$0 \longrightarrow \mathcal{S}_1 \longrightarrow \mathcal{O}_Y^{\oplus 2} \longrightarrow \mathcal{S}_2(1) \longrightarrow 0, \tag{3.9.10}$$

i.e. the roles of the spinor sheaves are reversed. Applying $\mathrm{RHom}(-,\mathcal{S}_2)$ to this sequence and using (3.9.4), we obtain isomorphisms

$$\mathrm{Ext}^p(\mathcal{S}_2(1),\mathcal{S}_2) \cong \mathrm{Ext}^{p-1}(\mathcal{S}_1,\mathcal{S}_2) \cong \mathrm{Ext}^{p-1}(\mathcal{S}_2,\mathcal{S}_1) \cong \mathrm{Ext}^p(\mathcal{S}_1(1),\mathcal{S}_1).$$

The second statement is proven analogously. □

Remark. Let $Y \subset \mathbb{P}^{n+1}$ be a nodal quadric hypersurface of even dimension. Then it was shown in [KS23a, Proposition 6.4] that the Ext-complexes $\mathrm{Ext}^\bullet(\mathcal{S}_i,\mathcal{S}_i)$ admit a structure of a graded k-algebra for $i \in \{1,2\}$. More precisely, let $k[\theta_i]$ be the differential graded k-algebra with $\deg(\theta_i)=2$ and differential $d=0$. Then there exists a morphism $0 \neq \phi \in \mathrm{Hom}(\mathcal{S}_i,\mathcal{S}_i[2]) \cong \mathrm{Ext}^2(\mathcal{S}_i,\mathcal{S}_i)$ and an isomorphism of graded k-algebras

$$k[\theta_i] \xrightarrow{\sim} \mathrm{Ext}^\bullet(\mathcal{S}_i,\mathcal{S}_i), \quad \theta_i \mapsto [\phi\colon \mathcal{S}_i \to \mathcal{S}_i[2]]. \tag{3.9.11}$$

In the next theorem we prove that we have semiorthogonal decompositions of the derived categories of nodal quadrics, analogous to those of Proposition 3.2. Note, however, that in the nodal case they do not form full exceptional collections, since the spinor sheaves are not exceptional by the results of Theorem 3.9. In the case of a 3-dimensional nodal quadric, we already know that a decomposition like the one below exists by [Kaw22, Example 7.1].

Theorem 3.10. *Let $Y \subset \mathbb{P}^{n+1}$ be a nodal quadric hypersurface. In the case Y is odd dimensional, there exists a semiorthogonal decomposition*

$$D^b(Y) = \langle \mathcal{O}_Y(1-n), \mathcal{O}_Y(2-n), \ldots, \mathcal{O}_Y(-1), \langle \mathcal{S}_1, \mathcal{S}_2\rangle, \mathcal{O}_Y\rangle. \tag{3.10.1}$$

If Y is even dimensional, we have a semiorthogonal decomposition

$$D^b(Y) = \langle \mathcal{O}_Y(1-n), \mathcal{O}_Y(2-n), \ldots, \mathcal{O}_Y(-1), \mathcal{S}, \mathcal{O}_Y\rangle. \tag{3.10.2}$$

Proof. We only prove the statement in the case where Y is odd dimensional, as the even dimensional case is proved analogously. By Proposition 3.2 we have the following semiorthogonal decomposition for a smooth quadric Q of even dimension $n-1$:

$$D^b(Q) = \langle \mathcal{O}_Q(2-n), \ldots, \mathcal{O}_Q(-1), \mathscr{S}_1, \mathscr{S}_2, \mathcal{O}_Q\rangle.$$

Recall that the morphism $\rho\colon \widetilde{Y} \to Q$ is a $\mathbb{P}^1$-bundle. Therefore we can use Orlov's projective bundle formula [Orl93, Theorem 2.6] to induce a semiorthogonal decomposition of $D^b(\widetilde{Y})$. In Lemma 3.3 we verified that

$$\mathcal{O}_\rho(1) =: \mathcal{O}_{\widetilde{Y}}(L) \cong \mathcal{O}_{\widetilde{Y}}(H) = \pi^*\mathcal{O}_Y(1). \tag{3.10.3}$$

Hence we obtain the following semiorthogonal decomposition:

$$\begin{aligned} D^b(\widetilde{Y}) &= \langle \rho^*(D^b(Q))(-1), \rho^*(D^b(Q))\rangle \\ &= \langle \rho^*(\mathcal{O}_Q(2-n)) \otimes \mathcal{O}_{\widetilde{Y}}(-1), \ldots, \rho^*(\mathcal{O}_Q(-1)) \otimes \mathcal{O}_{\widetilde{Y}}(-1), \rho^*\mathscr{S}_1 \otimes \mathcal{O}_{\widetilde{Y}}(-1), \\ &\qquad \rho^*\mathscr{S}_2 \otimes \mathcal{O}_{\widetilde{Y}}(-1), \mathcal{O}_{\widetilde{Y}}(-1), \\ &\qquad \rho^*(\mathcal{O}_Q(2-n)), \ldots, \rho^*(\mathcal{O}_Q(-1)), \rho^*\mathscr{S}_1, \rho^*\mathscr{S}_2, \mathcal{O}_{\widetilde{Y}}\rangle. \end{aligned} \tag{3.10.4}$$

It was shown in [KS23a, Theorem 5.2] that the pushforward functor $\pi_*\colon D^b(\widetilde{Y}) \to D^b(Y)$ is a (Verdier) localization, i.e. in particular essentially surjective. Therefore we can apply π_* to the semiorthogonal decomposition of $\widetilde{Y}$ and obtain a collection of objects that generate the triangulated category $D^b(Y)$. Applying isomorphism (3.10.3), the objects in the semiorthogonal decomposition (3.10.4) are the following:

$$\pi_*(\rho^*(\mathcal{O}_Q(2-n))\otimes\pi^*\mathcal{O}_Y(-1)),\dots,\pi_*(\rho^*(\mathcal{O}_Q(-1)))\otimes\pi^*\mathcal{O}_Y(-1),\pi_*(\rho^*\mathscr{S}_1\otimes\pi^*\mathcal{O}_Y(-1)),$$
$$\pi_*(\rho^*\mathscr{S}_2\otimes\pi^*\mathcal{O}_Y(-1)),\pi_*\pi^*\mathcal{O}_Y(-1),\pi_*\rho^*(\mathcal{O}_Q(2-n)),\dots,\pi_*\rho^*(\mathcal{O}_Q(-1)),\mathcal{S}_1,\mathcal{S}_2,\mathcal{O}_Y.$$

Using projection formula together with Proposition 3.6 and Remark 3.2, we obtain the following collection of sheaves generating $D^b(Y)$:

$$(\mathcal{O}_Y(1-n),\mathcal{S}_1(-1),\mathcal{S}_2(-1),\mathcal{O}_Y(2-n),\dots,\mathcal{O}_Y(-1),\langle\mathcal{S}_1,\mathcal{S}_2\rangle,\mathcal{O}_Y).$$

Twisting the sequences of Proposition 3.8, we get short exact sequences

$$0\longrightarrow\mathcal{S}_1(-1)\longrightarrow\mathcal{O}_Y^{\oplus N}(-1)\longrightarrow\mathcal{S}_2\longrightarrow 0,$$
$$0\longrightarrow\mathcal{S}_2(-1)\longrightarrow\mathcal{O}_Y^{\oplus N}(-1)\longrightarrow\mathcal{S}_1\longrightarrow 0,$$

for suitable $N\geq 0$ depending on the dimension of Y. Therefore we can further simplify the collection generating $D^b(Y)$ to

$$(\mathcal{O}_Y(1-n),\mathcal{O}_Y(2-n),\dots,\mathcal{O}_Y(-1),\mathcal{S}_1,\mathcal{S}_2,\mathcal{O}_Y).$$

Using the results on the cohomology of the spinor sheaves, cf. Theorem 3.9, we obtain a semiorthogonal decomposition

$$D^b(Y)=\langle\mathcal{O}_Y(1-n),\mathcal{O}_Y(2-n),\dots,\mathcal{O}_Y(-1),\langle\mathcal{S}_1,\mathcal{S}_2\rangle,\mathcal{O}_Y\rangle. \tag{3.10.5}$$

□

Remark. This result shows that the Kuznetsov component $\mathcal{A}_Y$, cf. Proposition 1.17, is generated by the spinor sheaves $\mathcal{S}_1,\mathcal{S}_2$ in the case Y is odd dimensional and by $\mathcal{S}$ if Y is even dimensional.

3.3 Explicit description of the kernel

Let X be a projective variety with a single isolated A_1 singularity. Then the bounded derived category $D^b(X)$ admits a crepant categorical resolution $(\widetilde{\mathcal{D}},\pi_*,\pi^*)$ and the functor $\pi_*\colon\widetilde{\mathcal{D}}\to D^b(X)$ is a localization, see Definition 1.33. Moreover, the kernel $\ker(\pi_*)$ is generated by a single 2- or 3-spherical object in the case X is even or odd dimensional, respectively. These results were simultaneously proved by [KS23a, Theorem 5.8] and [Cat+22, Theorem 1.1]. The goal of this subsection is to show a similar result for a variety X with a single isolated A_2 singularity. Precisely, we will prove that the crepant categorical resolution $\pi_*\colon\widetilde{\mathcal{D}}\to D^b(X)$, constructed in Theorem 2.1, is a localization and if X is a fourfold, the kernel is generated by two 2-spherical objects in $\widetilde{\mathcal{D}}$. In fact, we will show that these two objects are the pushforwards of the spinor sheaves $\mathcal{S}_1,\mathcal{S}_2$ along the inclusion $j\colon Q\to\widetilde{X}$ of the exceptional divisor.

To prove this result we proceed as [KS23a] and [Cat+22]. We show that we can apply [Efi20, Theorem 8.22] to derive that $\pi_*\colon\widetilde{\mathcal{D}}\to D^b(X)$ is a localization. After that we use the decompositions of Theorem 3.10 to explicitly determine generators of the kernel $\ker(\pi_*)$, see Theorem 3.14. Finally, all the cohomological information about the spinor sheaves that we collected in Section 3.2 is used to verify that these generators are indeed 2-spherical in the case of a cuspidal fourfold X.

The next theorem is key to showing that the crepant resolution $\pi_*\colon\widetilde{\mathcal{D}}\to D^b(X)$ constructed in subsection 2.1 is in fact a localization.

Theorem 3.11 ([Efi20, Theorem 8.22], [KS23a, Theorem 5.2]). *Let $\pi\colon \widetilde{X} \to X$ be a proper birational morphism and $i\colon Z \to X$ a closed subscheme such that the schematic preimage $E := \pi^{-1}(Z)$ is a Cartier divisor. Assume that the restriction $\pi\colon \widetilde{X} \setminus E \to X \setminus Z$ is an isomorphism and we have*

$$\pi_* \mathcal{O}_{\widetilde{X}}(-mE) \cong \mathcal{J}_Z^m, \quad \textit{for all } m \geq 0. \tag{3.11.1}$$

We consider the cartesian diagram

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "p"'] & \widetilde{X} \arrow[d, "\pi"] \\ Z \arrow[r, "i"] & X. \end{tikzcd}$$

If the functor $p_*\colon D^b(E) \to D^b(Z)$ is a localization, then the functor $\pi_*\colon D^b(\widetilde{X}) \to D^b(X)$ is also a localization. The category $\ker(\pi_*)$ is generated by $j_*(\ker(p_*))$.

We apply this theorem in the case of a variety X with a single isolated A_2 singularity at a point $x \in X$. For this, we fix the following notation for the rest of the subsection. Let $n = \dim(X) - 1$ and denote the blow-up of X at x by $\widetilde{X}$. We denote by $Q \subset \mathbb{P}^{n+1}$ the exceptional divisor of $\widetilde{X}$, which is a nodal quadric. Moreover, we have a cartesian diagram

$$\begin{tikzcd} Q \arrow[r, "j"] \arrow[d, "p"'] & \widetilde{X} \arrow[d, "\pi"] \\ \{x\} \arrow[r, "i"] & X, \end{tikzcd}$$

where i and j denote the corresponding closed immersions of Q and $\{x\}$, respectively.

It is not obvious for what kind of singularities of X the requirements of Theorem 3.11 are met. In [KS23a, Corollary 5.6], the authors investigate this question in the case where the variety X has a single isolated singularity and the morphism π is a blow-up of that singularity. This leads to the definition of *acyclic normal singularities*. They show that this class of singularities satisfies the requirements of Theorem 3.11. We recall their definition and results below to avoid repeating standard calculations, especially when verifying the assumption (3.11.1).

Definition 3.12. Let $x \in X$ be a normal isolated singularity and let $\pi\colon \widetilde{X} := \mathrm{Bl}_x(X) \to X$ be the blow-up of X at x with exceptional divisor $Q \subset \widetilde{X}$. Then (X, x) is called *acyclic projectively normal*, if the following conditions hold for $m \geq 0$:

1. the canonical map $\mathfrak{m}_{X,x}^m / \mathfrak{m}_{X,x}^{m+1} \to H^0(Q, \mathcal{O}_Q(m))$ is an isomorphism;
2. $H^i(Q, \mathcal{O}_Q(m)) = 0$ for all $i > 0$.

Lemma 3.13. *Let X be a variety with an isolated A_2 singularity at $x \in X$. Then (X, x) is an acyclic projectively normal singularity.*

Proof. The proof is analogous to the case of a nodal variety X, cf. [KS23a, Lemma 5.7]. Observe that the statement can be checked in a formal neighborhood of the A_2 singularity x. In such a neighborhood, X is defined by the (affine) equation $F = x_1^2 + \cdots + x_{n+1}^2 + x_{n+2}^3 =: q + x_{n+2}^3$. We know that the exceptional divisor $Q = V_+(q) \subset \mathbb{P}^{n+1}$ is a (nodal) quadric. Therefore, we have an exact sequence

$$0 \to H^0(\mathbb{P}^{n+1}, \mathcal{O}(m-2)) \to H^0(\mathbb{P}^{n+1}, \mathcal{O}(m)) \to H^0(Q, \mathcal{O}_Q(m)) \to H^1(\mathbb{P}^{n+1}, \mathcal{O}(m-2)) = 0$$

for all $m \geq 0$. The last term of the sequence vanishes for all $m \geq 0$, since $n \geq 2$. Moreover, there exists an isomorphism $H^0(\mathbb{P}^{n+1}, \mathcal{O}(m)) \cong k[x_0, \dots, x_{n+1}]_m$, so we obtain

$$H^0(Q, \mathcal{O}_Q(m)) \cong k[x_1, \dots, x_{n+1}]_m / q \cdot k[x_1, \dots, x_{n+1}]_{m-2}.$$

A straightforward calculation shows that $\mathfrak{m}_{X,x}^m / \mathfrak{m}_{X,x}^{m+1}$ is canonically isomorphic to the latter expression. The second condition follows immediately, since $Q \subset \mathbb{P}^{n+1}$ is a quadric hypersurface. $\square$

We now apply Theorem 3.11 and describe generators of the kernel $\ker(\pi_*)$.

Theorem 3.14. *Let X be a variety with an isolated A_2 singularity at a point $x \in X$. Then the crepant categorical resolution $\pi_* \colon \widetilde{\mathcal{D}} \to D^b(X)$ constructed in Theorem 2.1 is a localization and its kernel is generated by the following objects:*

$$\ker(\pi_*) = \begin{cases} \langle j_* \mathcal{S} \rangle, & \text{if } \dim(X) \text{ is odd,} \\ \langle j_* \mathcal{S}_1, j_* \mathcal{S}_2 \rangle, & \text{if } \dim(X) \text{ is even.} \end{cases} \tag{3.14.1}$$

Proof. In Lemma 3.13 we proved that an isolated A_2 singularity (X, x) is acyclic projectively normal. It is shown in [KS23a, Corollary 5.6] that we can apply Theorem 3.11 to such singularities (X, x) and the resolution $\pi \colon \widetilde{X} \to X$ that is given by a single blow-up of the point x. This implies that the functor $\pi_* \colon D^b(\widetilde{X}) \to D^b(X)$ is a localization and the kernel is generated by $j_*(\langle \mathcal{O}_Q \rangle^\perp)$. Observe that the restriction $\pi_* \colon \widetilde{\mathcal{D}} \to D^b(X)$ to the crepant resolution $\widetilde{\mathcal{D}}$ of $D^b(X)$ is also a localization. This immediately follows from the existence of the semiorthogonal decomposition $D^b(\widetilde{X}) = \langle \widetilde{\mathcal{D}}^\perp, \widetilde{\mathcal{D}} \rangle$ and the inclusion $\widetilde{\mathcal{D}}^\perp \subseteq \ker(\pi_*)$ (3.14.3) that we prove below. We now compute the kernel of the restriction $\ker(\pi_*) \cap \widetilde{\mathcal{D}}$. We only do this in the case where X is even dimensional, as the odd dimensional case is analogous. We consider the semiorthogonal decomposition

$$D^b(Q) = \langle \mathcal{O}_Y(1-n), \mathcal{O}_Y(2-n), \dots, \mathcal{O}_Y(-1), \langle \mathcal{S}_1, \mathcal{S}_2 \rangle, \mathcal{O}_Y \rangle.$$

that we introduced in Theorem 3.10, where $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the spinor sheaves on the nodal quadric Q. We see that the left orthogonal has a decomposition

$$\langle \mathcal{O}_Q \rangle^\perp = \langle \mathcal{O}_Y(1-n), \dots, \mathcal{O}_Q(-1), \langle \mathcal{S}_1, \mathcal{S}_2 \rangle \rangle,$$

which implies

$$\ker(\pi_*) = \langle j_* \mathcal{O}_Q(1-n), \dots, j_* \mathcal{O}_Q(-1), \langle j_* \mathcal{S}_1, j_* \mathcal{S}_2 \rangle \rangle. \tag{3.14.2}$$

Recall the following semiorthogonal decomposition that we constructed in Theorem 2.1:

$$D^b(\widetilde{X}) = \langle j_* \mathcal{O}_Q(1-n), \dots, j_* \mathcal{O}_Q(-1), \widetilde{\mathcal{D}} \rangle.$$

It implies $\widetilde{\mathcal{D}}^\perp = \langle j_* \mathcal{O}_Q(1-n), \dots, j_* \mathcal{O}(-1) \rangle$ and together with (3.14.2) yields an inclusion

$$\widetilde{\mathcal{D}}^\perp \subseteq \ker(\pi_*). \tag{3.14.3}$$

Now it is clear that the semiorthogonal decomposition $D^b(\widetilde{X}) = \langle \widetilde{\mathcal{D}}^\perp, \widetilde{\mathcal{D}} \rangle$ is compatible with the kernel, i.e., we have

$$\ker(\pi_*) = \langle \widetilde{\mathcal{D}}^\perp, \ker(\pi_*) \cap \widetilde{\mathcal{D}} \rangle.$$

We compute the kernel $\ker(\pi_*) \cap \widetilde{\mathcal{D}}$ as the right mutation of $\ker(\pi_*)$ through the left orthogonal $\widetilde{\mathcal{D}}^{\perp}$:

$$\ker(\pi_*) \cap \widetilde{\mathcal{D}} = \mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}(\ker(\pi_*)) = \langle \mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}(j_*\mathcal{S}_1), \mathbb{R}_{\widetilde{\mathcal{D}}^{\perp}}(j_*\mathcal{S}_2)\rangle = \langle j_*\mathcal{S}_1, j_*\mathcal{S}_2\rangle.$$

For the last equality, observe that the objects $j_*\mathcal{S}_1, j_*\mathcal{S}_2$ are contained in $\widetilde{\mathcal{D}}$. This can be deduced in the following way. By the definition of $\widetilde{\mathcal{D}}$ it suffices to show that $j^*j_*\mathcal{S}_i \in \langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{O}_Q\rangle$ for $i = 1, 2$. Since $j\colon Q \to \widetilde{X}$ is a divisorial embedding, there exists an exact triangle

$$j^*j_*\mathcal{S}_1 \longrightarrow \mathcal{S}_1 \xrightarrow{\epsilon} \mathcal{S}_1 \otimes \mathcal{N}^{\vee}_{Q/\widetilde{X}}[2] = \mathcal{S}_1(1)[2].^{8} \tag{3.14.4}$$

This reduces the claim to showing that $\mathcal{S}_i(1) \in \langle \mathcal{S}_1, \mathcal{S}_2, \mathcal{O}_Q\rangle$, but this immediately follows from the exact sequences (3.8.2) relating the spinor sheaves $\mathcal{S}_1$ and $\mathcal{S}_2$. □

In the next theorem we prove that in the case of a fourfold the generators of $\ker(\pi_*)$ from the previous theorem are in fact 2-spherical.

Theorem 3.15. *In the setting of Theorem 3.14, assume that X is a fourfold. Then the sheaves $j_*\mathcal{S}_1$ and $j_*\mathcal{S}_2$ generating the kernel $\ker(\pi_*)$ are 2-spherical.*

Proof. Let us consider the exact triangle (3.14.4). Applying the functor $\mathrm{RHom}(-, \mathcal{S}_1)$ gives rise to an exact triangle

$$\mathrm{Ext}^{\bullet}(\mathcal{S}_1(1), \mathcal{S}_1)[-2] \xrightarrow{\epsilon^*} \mathrm{Ext}^{\bullet}(\mathcal{S}_1, \mathcal{S}_1) \longrightarrow \mathrm{Ext}^{\bullet}(j^*j_*\mathcal{S}_1, \mathcal{S}_1). \tag{3.15.1}$$

We already determined the complexes on the left and in the middle, cf. Theorem 3.9 and Corollary 3.9.1. We showed that

$$\mathrm{Ext}^p(\mathcal{S}_1, \mathcal{S}_1) \cong \begin{cases} k, & \text{if } p \equiv 0 \pmod 2 \\ 0, & \text{if } p \equiv 1 \pmod 2 \end{cases} \quad \text{and} \quad p \geq 0, \tag{3.15.2}$$

$$\mathrm{Ext}^p(\mathcal{S}_1(1)[2], \mathcal{S}_1) \cong \begin{cases} 0, & \text{if } p \in \{1, \dots, 3\}, \\ k, & \text{if } p \equiv 0 \pmod 2 \text{ and } p \geq 4 \\ 0, & \text{if } p \equiv 1 \pmod 2 \text{ and } p \geq 4. \end{cases} \tag{3.15.3}$$

To deduce the desired isomorphism $\mathrm{RHom}(j_*\mathcal{S}_1, j_*\mathcal{S}_1) \cong k \oplus k[-2]$ it is left to show that the morphism ϵ^* is an isomorphism in degrees greater or equal to 4. To this end, we use the structure of a graded k-algebra on $\mathrm{Ext}^{\bullet}(\mathcal{S}_i, \mathcal{S}_i)$, see Remark 3.2. Recall that the graded k-vector space $\mathrm{Ext}^{\bullet}(\mathcal{S}_i, \mathcal{S}_i)$ admits a structure of a polynomial algebra with a generator of degree 2. We use the notation of Remark 3.2 and denote this generator by θ_1. Consider the induced morphism of $\mathrm{Ext}^{\bullet}(\mathcal{S}_1, \mathcal{S}_1) \cong k[\theta_1]$-modules

$$\mathrm{Ext}^{\bullet}(\mathcal{S}_1(1)[2], \mathcal{S}_1) \xrightarrow{\epsilon^*} \mathrm{Ext}^{\bullet}(\mathcal{S}_1, \mathcal{S}_1), \quad f \mapsto f \circ \epsilon. \tag{3.15.4}$$

We first show that $\mathrm{Ext}^{\bullet}(\mathcal{S}_1(1)[2], \mathcal{S}_1)$ is a free $k[\theta_1]$-module and then prove that ϵ^* is an isomorphism in degrees ≥ 4. By [Kaw22, Lemma 6.2] (see also Theorem 3.9), there exists a locally free extension

$$0 \longrightarrow \mathcal{S}_1 \longrightarrow \mathcal{G}_1 \longrightarrow \mathcal{S}_2 \longrightarrow 0. \tag{3.15.5}$$

[8] For details we refer to [KM07, §3]

Let $0 \neq \phi \in \mathrm{Hom}(\mathcal{S}_2, \mathcal{S}_1[1]) \cong \mathrm{Ext}^1(\mathcal{S}_2, \mathcal{S}_1)$ be the morphism corresponding to $\mathcal{G}_1$. In the proof of Theorem 3.9, it was shown that $\mathrm{Ext}^\bullet(\mathcal{G}_1, \mathcal{S}_1) = 0$, which yields an isomorphism of $k[\theta_1]$-modules

$$\phi^*\colon \mathrm{Ext}^\bullet(\mathcal{S}_1, \mathcal{S}_1)[-1] \xrightarrow{\sim} \mathrm{Ext}^\bullet(\mathcal{S}_2, \mathcal{S}_1), \quad f \mapsto f \circ \phi. \tag{3.15.6}$$

Similarly, we consider the exact sequence from Proposition 3.8:

$$0 \longrightarrow \mathcal{S}_2 \longrightarrow \mathcal{O}_Y^{\oplus 2} \longrightarrow \mathcal{S}_1(1) \longrightarrow 0. \tag{3.15.7}$$

Let $0 \neq \alpha \in \mathrm{Hom}(\mathcal{S}_1(1), \mathcal{S}_2[1]) \cong \mathrm{Ext}^1(\mathcal{S}_1(1), \mathcal{S}_2)$ be the morphism corresponding to the non-trivial extension defined by (3.15.7). In Proposition 3.9 we proved that the cohomology groups $H^p(Y, \mathcal{S}_1)$ vanish for all $p \geq 0$. Therefore the extension (3.15.7) gives rise to an isomorphism of $k[\theta_1]$-modules

$$\alpha^*\colon \mathrm{Ext}^\bullet(\mathcal{S}_2, \mathcal{S}_1)[-1] \xrightarrow{\sim} \mathrm{Ext}^\bullet(\mathcal{S}_1(1), \mathcal{S}_1), \quad f \mapsto f \circ \alpha. \tag{3.15.8}$$

This implies that we can view $\mathrm{Ext}^\bullet(\mathcal{S}_1(1), \mathcal{S}_1)$ as a free $k[\theta_1]$-module under the isomorphism $\alpha^* \circ \phi^*$. Shifting by -2, we obtain an isomorphism

$$\alpha^* \circ \phi^*[-2]\colon \mathrm{Ext}^\bullet(\mathcal{S}_1, \mathcal{S}_1)[-4] \xrightarrow{\sim} \mathrm{Ext}^\bullet(\mathcal{S}_1(1)[2], \mathcal{S}_1), \tag{3.15.9}$$

i.e., $\mathrm{Ext}^\bullet(\mathcal{S}_1(1)[2], \mathcal{S}_1)$ is a free $k[\theta_1]$-module as well. We now consider the composition

$$\begin{tikzcd}
\mathcal{S}_1 \arrow[r, "\epsilon"] \arrow[ddr] & \mathcal{S}_1(1)[2] \arrow[d, "\alpha"] \\
& \mathcal{S}_2[3] \arrow[d, "\phi"] \\
& \mathcal{S}_1[4]
\end{tikzcd}$$

as an element of $\mathrm{Ext}^4(\mathcal{S}_1, \mathcal{S}_1)$. Then there exists a scalar $t \in k$, such that $\phi \circ \alpha \circ \epsilon = t\theta_1^2$. Consider the induced morphism of k-vector spaces

$$\mathrm{Ext}^\bullet(\mathcal{S}_1, \mathcal{S}_1)[-4] \xrightarrow{\alpha^* \circ \phi^*} \mathrm{Ext}^\bullet(\mathcal{S}_1(1)[2], \mathcal{S}_1) \xrightarrow{\epsilon^*} \mathrm{Ext}^\bullet(\mathcal{S}_1, \mathcal{S}_1), \quad \theta_1^i \mapsto \theta_1^i \circ \phi \circ \alpha \circ \epsilon = t_i \theta^{i+2},$$

where $t_i \in k$. In fact, this is a $k[\theta_1]$-linear map, therefore we obtain $\epsilon^* = c\theta_1$, for $c \in k$. Assume that $c = 0$. Then the exact triangle (3.15.1) and the results of Proposition 3.9 imply that the complex $\mathrm{Ext}^\bullet(j^* j_* \mathcal{S}_1, \mathcal{S}_1)$ is unbounded. Using the adjunction $j^* \vdash j_*$ we have an isomorphism

$$\mathrm{Ext}^\bullet(j^* j_* \mathcal{S}_1, \mathcal{S}_1) \cong \mathrm{Ext}^\bullet(j_* \mathcal{S}_1, j_* \mathcal{S}_1).$$

This implies that the complex $\mathrm{Ext}^\bullet(j_* \mathcal{S}_1, j_* \mathcal{S}_1)$ in $D^b(\widetilde{X})$ is unbounded, which is a contradiction to the smoothness of $\widetilde{X}$. Therefore we conclude that $c \neq 0$, i.e. the morphism (3.15.4) is an isomorphism in all degrees greater or equal to 4. Together with the exact triangle (3.15.1) and (3.15.2) this implies that we have an isomorphism $\mathrm{RHom}(j_* \mathcal{S}_1, j_* \mathcal{S}_1) \cong k \oplus k[-2]$. This argument works analogous for the sheaf $\mathcal{S}_2$.

We now verify the part of Definition 1.26 concerning the Serre functors. For this we apply the same arguments as in [KS23a, Lemma 5.10(iii)], where the authors assume

X has an A_1 singularity. Let us consider the semiorthognal decomposition (2.1.3) and compute

$$\begin{aligned}\mathbb{S}_{\widetilde{\mathcal{D}}}(j_*\mathcal{S}_1) = \mathbb{R}_{\widetilde{\mathcal{D}}^\perp}(\mathbb{S}_{\widetilde{X}}(j_*\mathcal{S}_1)) &= \mathbb{R}_{\widetilde{\mathcal{D}}^\perp}(j_*\mathcal{S}_1 \otimes \omega_{\widetilde{X}}[4]) \\ &= \mathbb{R}_{\widetilde{\mathcal{D}}^\perp}(j_*(\mathcal{S}_1 \otimes j^*\omega_{\widetilde{X}}))[4] \\ &= \mathbb{R}_{\widetilde{\mathcal{D}}^\perp}(j_*\mathcal{S}_1(-2))[4],\end{aligned} \tag{3.15.10}$$

where we applied the adjunction formula

$$j^*\omega_{\widetilde{X}} = \omega_Q \otimes j^*\mathcal{O}_{\widetilde{X}}(-Q) = \mathcal{O}_Q(-3) \otimes \mathcal{O}_Q(1) = \mathcal{O}(-2)$$

in the last step. By twisting the short exact sequences (3.8.2) by $\mathcal{O}_Q(k)$ for suitable $k \in \mathbb{Z}$ and pushing them forward along j_*, we obtain morphisms

$$j_*\mathcal{S}_1[2] \longrightarrow j_*\mathcal{S}_2(-1)[3] \longrightarrow j_*\mathcal{S}_1(-2)[4] \tag{3.15.11}$$

with cones $j_*\mathcal{O}_Q^2(-1)[3]$ and $j_*\mathcal{O}_Q^2(-2)[4]$, respectively. We now consider the cone

$$\operatorname{cone}(j_*\mathcal{S}_1[2] \longrightarrow j_*\mathcal{S}_1(-2)[4]) \in \langle j_*\mathcal{O}_Q(-2), j_*\mathcal{O}(-1)\rangle = \widetilde{\mathcal{D}}^\perp$$

of the composition (3.15.11) above. In the proof of Theorem 3.14, we showed that the sheaf $j_*\mathcal{S}_1$ is an object of the resolution category $\widetilde{\mathcal{D}}$. Therefore, we can apply the right mutation functor $\mathbb{R}_{\widetilde{\mathcal{D}}^\perp}$ to the composition (3.15.11), to obtain an isomorphism

$$j_*\mathcal{S}_1[2] \xrightarrow{\sim} \mathbb{R}_{\widetilde{\mathcal{D}}^\perp}(j_*\mathcal{S}_1(-2))[4].$$

Finally, this isomorphism together with calculation (3.15.10) yields $\mathbb{S}_{\widetilde{\mathcal{D}}}(j_*\mathcal{S}_1) = j_*\mathcal{S}_1[2]$. By the symmetry of the sequences (3.8.2) in $\mathcal{S}_1$ and $\mathcal{S}_2$, the same argument holds for the sheaf $j_*\mathcal{S}_2$. □

Remark. In order to generalize Theorem 3.15 to all dimensions, one needs to calculate the respective Ext-groups (3.9.3) and (3.9.4) for a nodal quadric Y of any dimension, the rest of the proof works analogously.

3.4 Special case of a cubic fourfold

Let X be a cubic fourfold with an isolated A_2 singularity. In Theorem 2.1 we proved that the singular category $D^b(X)$ admits a crepant categorical resolution $(\widetilde{\mathcal{D}}, \pi_*, \pi^*)$. By Theorem 3.15 we know that the kernel of the functor $\pi_*\colon \widetilde{\mathcal{D}} \to D^b(X)$ is generated by two 2-spherical objects. In this subsection we connect these results with the ones of Section 2.2. Precisely, we consider the crepant cateogrical resolution $\pi_*\colon D^b(S) \cong \widetilde{\mathcal{A}}_X \to \mathcal{A}_X$ of the Kuznetsov component $\mathcal{A}_X$ which exists by Theorem 2.4 and explicitly describe the spherical generators of the subcategory $\ker(\pi_*) \subset \widetilde{\mathcal{A}}_X$ as elements of the bounded derived category $D^b(S)$ of the (smooth) K3 surface we associated to X in Section 2.2.

Theorem 3.16. *Let X be a cubic fourfold with an isolated A_2 singularity and assume we are in the Setting 2.2. Let $t\colon S \hookrightarrow Q$ be the inclusion map of the K3 surface S into the defining (nodal) quadric Q and let $\mathcal{S}_1, \mathcal{S}_2$ denote the spinor sheaves on Q. Then the kernel of the crepant categorical resolution $D^b(S) \to \mathcal{A}_X$ constructed in Theorem 2.4 is generated by the spherical objects $t^*\mathcal{S}_1$ and $t^*\mathcal{S}_2$.*

In the case of a cubic fourfold X with an isolated A_1 singularity this was done in [Cat+22, Section 4], the proof of Theorem 3.16 follows verbatim.

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