Classifying covering spaces through the fundamental group

A fundamental concept in algebraic topology is that of a covering space. A covering space for a topological space X is a space $\tilde{X}$ and a map $p : \tilde{X} \to X$ such that every $x \in X$ has an open neighborhood U for which $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$ each of which are projected homeomorphically onto U by p. This formal definition can be understood intuitively by imagining that the covering space is 'wrapped' around X. An example to keep in mind is the covering space $p : \mathbb{R} \to S^1$ given by $x \mapsto e^{2\pi i x}$. This covering space is in fact used to calculate the fundamental group of the circle — one of the practical applications of covering spaces. On a more theoretical note, one can wonder how many different covering spaces a topological space X has. This is precisely the question we will be investigating.

In order to begin to do this, there has to be some way of telling when two covering spaces are 'the same'. We will call two covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ isomorphic if there exists a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ such that the following diagram is commutative:

$$\begin{tikzcd}
\tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\
& X. &
\end{tikzcd}$$

Let X be a topological space that is path-connected, locally path-connected, and semilocally simply-connected (this will be explained) and let $x_0 \in X$. Then there is a bijective correspondence between the path-connected covering spaces of X with a basepoint in $p^{-1}(x_0)$ (up to the isomorphism described above) and the subgroups of $\pi_1(X, x_0)$. During this 45 minute talk I will prove most of this correspondence.