# Construction of Galois representations in cohomology of Shimura curves

Theo van den Bogaart

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These notes sketch a construction of the Galois representations associated to Hilbert modular forms, using the cohomology of Shimura curves. For the most part, this text is based on a preprint of Saito, see [Sa].

The notes belong to a talk I gave at the Dutch intercity seminar on arithmetic geometry, which in the fall of 2005 had as aim to give an overview of the work of Khare on Serre's conjecture. Before me others already talked about Hilbert modular forms, a part of the local Langlands program and the Jacquet-Langlands correspondence; so I will assume some familiarity with these things. See [www] for more information about the seminar and for the texts of the other speakers.

Please inform me if there are any errors, misunderstandings, clumsy constructions, etc.

## 1 Introduction, notation

Start with a Hilbert cusp form $f$ over a totally real field $F$, and suppose that $f$ is an eigenform for all Hecke operators. A statement which already has come up in this seminar a few times is that associated to $f$ there is a representation of the Galois group $\mathrm{Gal}(\overline{F}/F)$. The most general form of this statement is still conjectural, but a lot has been proved already. See the introduction of [Ta] for a short overview.

Our goal is to deal with a particular, but already very general, situation where one can give an explicit geometric construction of the Galois representation.

Fix the following:

- A totally real number field $F$ of degree $n > 1$ over $\mathbf{Q}$.

- An ordering of the set $I := \mathrm{Hom}(F, \mathbf{R}) = \{\tau_1, \ldots, \tau_n\}$ of real embeddings of $F$.

- A prime $v$ of $F$ (this prime is only relevant if $n$ is even).

- A sufficiently large number field $L$. To be precise: it must contain a splitting field $F^{\mathrm{spl}}$ of $F$ and it must split the quaternion algebra that is ramified at $\tau_2, \ldots, \tau_n$ if $n$ is odd, and at $\tau_2, \ldots, \tau_n, v$ if $n$ is even. (See section 3 for an explanation of this.)

- A multi-weight $k = (k_1, \ldots, k_n, w) \in \mathbf{Z}^{n+1}$, such that $w \geq k_i \geq 2$ and $w \equiv k_i \pmod{2}$ for all $i$.

Let us recall some things from the talks by Van der Geer and Edixhoven (see [www]). Let $S_{\mathbf{C}}^k$ be the $\mathbf{C}$-vector space of all Hilbert cusp forms over $F$ of weight $k$ and arbitrary level. It is a representation of $\mathrm{GL}_2(\mathbf{A}_F^f)$ and as such it decomposes into a sum $\bigoplus_f \pi_f$ of irreducible subrepresentations, called cuspidal automorphic representations. This sum is indexed by the normalized newforms and is multiplicity free, which means $\pi_f \simeq \pi_{f'} \iff f = f'$. Each factor decomposes as a restricted tensor product $\pi_f = \bigotimes'_{\mathfrak{p}} \pi_{f,\mathfrak{p}}$ indexed by the primes $\mathfrak{p}$ of $F$, where each $\pi_{f,\mathfrak{p}}$ is an irreducible smooth representation of $\mathrm{GL}_2(F_{\mathfrak{p}})$. The representation $S_{\mathbf{C}}^k$ has a natural model over $F^{\mathrm{spl}}$. For each $f$, there is a finite extension $F^{\mathrm{spl}}(f)$ of $F^{\mathrm{spl}}$ over which $\pi_f$ and each $\pi_{f,\mathfrak{p}}$ are defined; it is obtained by adjoining to $F^{\mathrm{spl}}$ the eigenvalues of all the Hecke operators acting on $f$. We write $L(f)$ for the compositum of $F^{\mathrm{spl}}(f)$ and $L$.

Let $\lambda$ be a prime of $L(f)$ and let

$$\mathrm{Gal}(\overline{F}/F) \quad \xrightarrow{\ \rho\ } \quad \mathrm{GL}_2(L(f)_\lambda)$$

be a representation that is continuous for the $\lambda$-adic topology on the right and the Krull topology on the left. Then $\rho$ is the representation associated to $f$ if "$\rho$ is compatible with the representations obtained from $\pi_f$ via the local Langlands correspondence." For this talk, the precise meaning of this phrase is not so important. In fact, we can think of this as a criterium for the restrictions of $\rho$ to the decomposition groups at primes $\mathfrak{p}$ that do not divide the level of $f$ and do not divide $\ell$, where $\ell \in \mathbf{Z}$ is the prime that $\lambda$ divides. For such $\mathfrak{p}$ the compatibility condition is that $\rho$ is unramified $\mathfrak{p}$ and that the characteristic polynomial of a Frobenius element has a prescribed form, in terms of the $\mathfrak{p}$-Hecke eigenvalues of $f$. (See Van der Geer's talk ([www]) for the precise statement.) Alternatively, one can use $L$-functions. The full local Langlands correspondence is explained in the talks by Edixhoven and J. de Jong. To give a precise definition: $\rho$ is compatible with the representation obtained from $\pi_f$ via the local Langlands correspondence if for every prime $\mathfrak{p}$ of $F$

$$'(\rho|_{\mathrm{Gal}(\overline{F}_{\mathfrak{p}}/F_{\mathfrak{p}})})^{F\text{-ss}} = \check{\sigma}_h(\pi_{f,\mathfrak{p}}).$$

On the left we take $\rho_{f,\lambda}$ which we subsequently restrict to a decomposition group in $\mathfrak{p}$, replace by the associated representation of the Weil-Deligne group and $F$-semi-simplify. On the right we take the representation of the Weil-Deligne group associated to $\pi_{f,\mathfrak{p}}$ by the local Langlands correspondence using the Hecke normalisation. (As said, the meaning of these words is not so important in this text.)

The main theorem reads:

THEOREM.　　*Let $f$ be a Hilbert cusp form over $F$ of weight $k$, which is a normalized newform. If $n$ is even, assume that $\pi_{f,v}$ is not a principal series representation. Then for every prime $\lambda$ of $L(f)$ there exists a unique representation*

$$\mathrm{Gal}(\overline{F}/F) \xrightarrow{\ \rho_{f,\lambda}\ } \mathrm{GL}_2(L(f)_\lambda)$$

*which is continuous for the $\lambda$-adic topology on $L(f)_\lambda$ and which is compatible with the representations obtained from $\pi_f$ via the local Langlands correspondence.*

This theorem was proved, in increasing generality, by Eichler, Shimura, Deligne, Langlands, Ohta, Carayol and Saito. See [Sa] for further details. In the seminar we will also need a version of this theorem where we lift the assumption on $\pi_{f,v}$ if $n$ is even. This has been obtained in [Ta], starting with the above theorem and using congruences between modular forms of different level.

In this text, we will construct $\rho_{f,\lambda}$ that appears in the theorem, but there will not be given a proof that it is compatible with the representations obtained from the local Langlands correspondence. We will however see that $\rho_{f,\lambda}$ is non-zero. The cases $F = \mathbf{Q}$ and $F \neq \mathbf{Q}$ turn out to be somewhat different. In the next section there is a very rough account of the first case. Starting from section 3, we will assume $F \neq \mathbf{Q}$.

## 2　Classical modular forms

As a motivation, there now follows a sketch of the construction of the Galois representation if $F = \mathbf{Q}$. In most respects — but not all — this is the easier case. We take $L = \mathbf{Q}$ in this section.

Let $X = \mathbf{P}^1\mathbf{C} - \mathbf{P}^1\mathbf{R}$ be the union of the upper and lower complex half plane. Let $K \subset \mathrm{GL}_2(\mathbf{A}^f)$ be a compact open subgroup. If $K$ is small enough, the double quotient space

$$Y_K(\mathbf{C}) = \mathrm{GL}_2(\mathbf{Q})\backslash(X \times \mathrm{GL}_2(\mathbf{A}^f)/K)$$

has the structure of a, not necessarily connected, smooth variety of dimension 1. It is the moduli space of elliptic curves with a certain level structure (depending on $K$). Using this moduli description, we obtain a model $Y_K$ of $Y_K(\mathbf{C})$ defined over $\mathbf{Q}$. The variety $Y_K$ is a smooth (non-connected) curve which is not proper. There is a standard way to compactify it to a proper scheme $X_K$. The non-compactness of $Y_K$ is a feature that does not appear for the curves we need for the other real fields. Therefore, we will not elaborate on the subtleties that turn up because of this. The space of all cusp forms of weight $k$ can be described as the space

$$S_{\mathbf{Q}}^k = \varinjlim_K \mathrm{H}^0(X_K, \mathcal{L}_K),$$

where $\mathcal{L}_K$ is a line bundle on $X_K$ that depends on $k$, and where we take the limits over smaller and smaller compact open subgroups. As we are only interested in

these kinds of limits we can conveniently assume $K$ to be "sufficiently small" in the things that follow.

There is the universal elliptic curve $f \colon \mathbf{E} \to Y_K$, The local system $\mathcal{F}_K = \operatorname{Sym}^{k-2} \operatorname{R}^1 f_* \mathbf{Q}$ on $Y_K(\mathbf{C})$ encodes information about the fundamental groups of the elliptic curves in the family. In the same way, there is a constructible $\ell$-adic sheaf $\mathcal{F}_{\ell,K} = \operatorname{Sym}^{k-2} \operatorname{R}^1 f_* \mathbf{Q}_\ell$ on the étale site of $Y_{K,\mathrm{et}}$, which encodes the $\ell$-torsion of the elliptic curves. This last sheaf allows us to make a continuous Galois representation $\operatorname{H}^1_{\mathrm{par}}(Y_{K,\overline{\mathbf{Q}},\mathrm{et}}, \mathcal{F}_{\ell,K})$. Here "par" means that we use parabolic cohomology, which is the image of compactly supported cohomology in ordinary cohomology.

Taking the direct limit we obtain a space

$$ H_\ell = \varinjlim_K \operatorname{H}^1_{\mathrm{par}}(Y_{K,\overline{\mathbf{Q}},\mathrm{et}}, \mathcal{F}_{\ell,K}) $$

with commuting actions of $\operatorname{GL}_2(\mathbf{A}^f)$ and $\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on it. The representation $\rho_{f,\lambda}$ that we want can now be "cut out" as

$$ \rho_{f,\lambda}^\vee = \operatorname{Hom}_{\mathbf{Q}(f)_\lambda[\operatorname{GL}_2(\mathbf{A}^f)]}\big(\pi_f \otimes_{\mathbf{Q}(f)} \mathbf{Q}(f)_\lambda, H_\ell \otimes_{\mathbf{Q}_\ell} \mathbf{Q}(f)_\lambda\big). $$

To show that this representation is non-zero (and, in fact, 2-dimensional), we need Hodge theory. This identifies $H_{\mathbf{C}} := \varinjlim_K \operatorname{H}^1(X_K(\mathbf{C}), \mathcal{F}_K)$ with two copies of $\mathbf{S}_{\mathbf{C}}^k$.

In principle, the above recipe can also be used if $F \neq \mathbf{Q}$, using Hilbert modular varieties. But it gives the wrong representation. For one thing, as the Hilbert modular variety has a model over $\mathbf{Q}$ one obtains a representation of the absolute Galois group of $\mathbf{Q}$ instead of $F$. Also, as the dimension of the Hilbert modular variety is greater then 1, the degree of the cohomology will not be right. In fact: this representation can be obtained from the right one by forgetting information, i.e., by using induction and tensor constructions.

To overcome this problem we will use the Jacquet-Langlands correspondence to switch to Shimura curves defined by quaternion algebras. These have the advantage that they are defined over the right field and that they have the right dimension. Unfortunately, they have not as nice a moduli interpretation as Hilbert moduli varieties.

## 3 The Shimura curve

From now on we will assume $n = \dim_{\mathbf{Q}} F > 1$.

Recall that a quaternion algebra over $F$ is uniquely determined by the set of places where it ramifies. The subsets of places of $F$ which turn up in this way are exactly the finite sets which have an even number of elements. Now if $n$ is odd, define $B$ to be the quaternion algebra that ramifies at the infinite places $\tau_2, \ldots, \tau_n$. If on the other hand $n$ is even, we need in addition to fix a finite place $v$ to make the number of places even: in this case, $B$ is the quaternion algebra ramifying at $\tau_2, \ldots, \tau_n$ and $v$.

Let $G$ be $B^\times$ considered as an algebraic group over $\mathbf{Q}$. So $G(\mathbf{Q}) = B^\times$, $G(\mathbf{A}^f) = (B \otimes_F \mathbf{A}^f_F)^\times$ and

$$G(\mathbf{R}) \simeq \mathrm{GL}_2(\mathbf{R}) \times \mathbf{H}^\times \times \cdots \times \mathbf{H}^\times,$$

where on the right there appear $n-1$ copies of the multiplicative group of non-zero elements of the classical Hamiltonian quaternions $\mathbf{H}$.

Let $\mathbf{S}$ be $\mathbf{C}^\times$ considered as algebraic group over $\mathbf{R}$. Consider the space of homomorphisms $\mathbf{S} \to G_{\mathbf{R}}$. On it, $G(\mathbf{R})$ acts via conjugation. Let $X$ be the orbit of the element that on real points is given by

$$
\begin{array}{ccccccccc}
h\colon & \mathbf{C}^* & \longrightarrow & G(\mathbf{R}) = & \mathrm{GL}_2(\mathbf{R}) & \times & \mathbf{H} & \times & \cdots & \times & \mathbf{H} \\
& x + yi & \mapsto & & \left(\left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right)\right. & , & 1 & , & \cdots & , & \left.1\right).
\end{array}
$$

It is easily seen that there is an isomorphism $X \simeq \mathbf{P}^1\mathbf{C} - \mathbf{P}^1\mathbf{R}$ such that the action of $G(\mathbf{R})$ on $\mathbf{P}^1\mathbf{C}$ is the usual one, and such that $h$ corresponds to $i \in \mathbf{P}^1\mathbf{C}$. The weight homomorphism $w\colon \mathbf{G}_{m,\mathbf{R}} \to G_{\mathbf{R}}$ associated to $h$ is given by $w(r) = (r^{-1}1, 1, \ldots, 1)$ (where $r \in \mathbf{R}$); note that it is *not* defined over $\mathbf{Q}$.

In the remaining part of this sections, we will describe the Shimura variety corresponding to the datum $(G, X)$. We will list some standard facts from the theory of Shimura varieties, see for example [Del1] or [Mil] for the proofs.

First we introduce some notation. Let $G^{\mathrm{ad}}$ be the adjoint group associated to $G$, which, as $G$ is reductive, is the quotient by $G$ by its center. There is the canonical map $G \to G^{\mathrm{ad}}$. Let $G^{\mathrm{ad}}(\mathbf{R})^+$ be the connected component of 1 in the real topology, and let $G(\mathbf{R})_+$ be the inverse image of $G^{\mathrm{ad}}(\mathbf{R})^+$. In fact, $G(\mathbf{R})_+$ is just the connected component of $G(\mathbf{R})$. (Note that an element sits in the connected component if and only if the determinant of its $\mathrm{GL}_2(\mathbf{R})$-part is positive.) Finally, we set $G(\mathbf{Q})_+ = G(\mathbf{Q}) \cap G(\mathbf{R})_+$.

Let $K \subset G(\mathbf{A}^f)$ be a compact open subgroup. Fix a set $\mathcal{C}$ of double coset representatives of $G(\mathbf{Q})\backslash G(\mathbf{A}^f)/K$. It is a finite set (the proof of this uses the strong approximation theorem). The action of $G(\mathbf{R})$ on $X$ factors through $G^{\mathrm{ad}}(\mathbf{R})$. Let $X^+ \subset X$ be the connected component containing $h$, which is isomorphic to the complex upper half plane. Then $G^{\mathrm{ad}}(\mathbf{R})^+$ stabilizes this component. The topological space

$$M_K(\mathbf{C}) := \mathrm{Sh}_K(G, X)(\mathbf{C}) := G(\mathbf{Q})\backslash\big(X \times G(\mathbf{A}^f)/K\big)$$

decomposes into a finite disjoint union

$$M_K(\mathbf{C}) = \bigsqcup_{g \in \mathcal{C}} \Gamma_g\backslash X^+,$$

where $\Gamma_g$ is the image of $\Gamma'_g := gKg^{-1} \cap G(\mathbf{Q})_+$ under the canonical map $G(\mathbf{R}) \to G^{\mathrm{ad}}(\mathbf{R})$. If $\Gamma_g$ is torsion free —which happens if $K$ is small enough — it acts freely on $X^+$ and in that case $\Gamma_g\backslash X^+$ has a unique structure of a complex manifold such that $X^+$ is its universal covering space. In fact, $\Gamma_g\backslash X^+$ has the

structure of a connected Riemann surface with fundamental group $\Gamma_g$. If $\Gamma_g$ also has no non-trivial unipotent elements, the quotient $\Gamma_g \backslash X^+$ is compact. This happens in our situation, as $G(\mathbf{Q}) = B - \{0\}$ and $B$ is a division algebra.

An inclusion $K' \subset K$ induces a morphism $M_{K'} \to M_K$; in this way we obtain a projective system $(M_K(\mathbf{C}))_K$. Multiplication in $G(\mathbf{A}^f)$ from the right by an element $g \in G(\mathbf{A}^f)$ induces a morphism $M_K(\mathbf{C}) \to M_{g^{-1}Kg}(\mathbf{C})$. In this way we obtain a right action of $G(\mathbf{A}^f)$ on the system $(M_K(\mathbf{C}))_K$.

The main theorem in the theory of Shimura varieties says (in our situation) that $(M_K(\mathbf{C}))_K$ has a canonical model over $F$. This is a projective system of schemes $M_K$ over $F$ on which $G(\mathbf{A}^f)$ still acts. See the literature for the meaning of the word "canonical"; it is not essential for the understanding of this text.

REMARK: *The curve used in the previous section, and more generally Hilbert modular varieties, are of PEL-type. This means that they have a suitable description as moduli spaces of abelian varieties with additional structures. In these situations, the canonical models and the sheaves that one needs can all be defined using this moduli interpretation.*

*In the present case, unfortunately, the fact that the weight homomorphism $w$ is not defined over $\mathbf{Q}$ implies that we cannot expect a description of $M_K(\mathbf{C})$ as a moduli space. (The reason is that if $M_K(\mathbf{C})$ where a moduli space, we could describe $h$ as given by the Hodge structure of an abelian variety; which is rational.) Hence there is no straightforward way to construct the canonical model or the sheaves we need. Therefore, we will adopt an ad-hoc method and use explicit double quotient constructions; and fall back on the general theory of Shimura varieties whenever we need to descend to a number field. There is, however, a method to link $M_K(\mathbf{C})$ to Shimura varieties which do have a moduli interpretation using an imaginary quadratic extension of $\mathbf{Q}$. This explains better the constructions we make; and it also helps to prove things about models. See [Sa] or [Oh] for this.*

# 4  Construction of certain sheaves

Our goal is to define a vector bundle $\mathcal{V}$ and a local system $\mathcal{F}$ on the Shimura curve, both depending on the weight $k$. For this, we start constructing $G(\mathbf{Q})$-equivariant bundles over $\mathbf{P}^1_\mathbf{C}$, which we restrict to $X$. We then take the double quotient, and pass to a model over a number field. The method to do this is described in [Mil] (in much greater generality).

Recall that $L \subset \mathbf{C}$ is a number field for which $G_L \simeq (\mathrm{GL}_{2,L})^I$. We want to produce certain objects with a $G_L$ action on it, and this we will do for each factor $\mathrm{GL}_{2,L}$ seperately.

For $1 \leq i \leq n$, define the $\mathrm{GL}_2(L)$-representation

$$W_i = \mathrm{Sym}^{k_i - 2} L^2 \otimes_L \det^{(w-k_i)/2}.$$

Form the product representation $W = \bigotimes_{i=1}^{n} W_i$ of $(\mathrm{GL}_2(L))^I$. Viewing $W$ as a vector group over $L$, we can form the constant vector bundle $\mathbb{W} = W \times_L \mathbf{P}_L^1 = \bigotimes \mathbb{W}_i$ on $\mathbf{P}_L^1$. This vector bundle is $(\mathrm{GL}_{2,L})^I$-equivariant, if we let $(\mathrm{GL}_{2,L})^I$ act on $\mathbf{P}_L^1$ via the first projection $(\mathrm{GL}_{2,L})^I \to \mathrm{GL}_{2,L}$.

Let $\mathrm{Taut}_{\mathbf{P}_L^1}$ be the tautological line bundle on $\mathbf{P}_L^1$. It can be obtained as a subbundle of the constant bundle $L^2 \times_L \mathbf{P}_L^1$. On $L$-valued points, the fibre above $[a:b] \in (L^2 - \{0\})/L^\times = \mathbf{P}^1 L$ is spanned by $(a,b) \in L^2$. By construction, $\mathrm{GL}_{2,L}$ acts equivariantly.

Define a $(GL_{2,L})^I$-equivariant vector bundle $\mathbb{V}$ on $\mathbf{P}_L^1$ as the product bundle $\mathbb{V} = \bigotimes_{i=1}^{n} \mathbb{V}_i$ with

$$\mathbb{V}_1 = \mathrm{Taut}_{\mathbf{P}_L^1}^{\otimes (k_1 - 2)} \otimes_L \det^{(w-k_1)/2}$$

and $\mathbb{V}_i = \mathbb{W}_i$. There is a canonical embedding $\mathbb{V} \hookrightarrow \mathbb{W}$ of equivariant bundles.

By base change we get bundles $\mathbb{V}(\mathbf{C}) \hookrightarrow \mathbb{W}(\mathbf{C})$ on $\mathbf{P}^1\mathbf{C}$ equivariant under the action by $G(\mathbf{C}) \simeq (\mathrm{GL}_2(\mathbf{C}))^I$. For the descent argument later on it is important to remember that these equivariant bundles are definable over $L$.
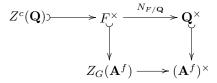
Before we look at bundles over our Shimura curve, we need a small intermezzo on Hodge structures. Let us first consider certain Hodge decompositions of $W \otimes_L \mathbf{C}$. For $z = x + iy \in \mathbf{C}^\times$, put $h_g(z) = g\left(\begin{smallmatrix} x & -y \\ y & x \end{smallmatrix}\right)g^{-1} \in \mathrm{GL}_2(\mathbf{R})$. Let $\mathrm{GL}_2(\mathbf{R})$ act via $\mathrm{GL}_2(\mathbf{C})$ on $W_1 \otimes_L \mathbf{C}$, and trivially on $W_i \otimes_L \mathbf{C}$ for $i > 1$. For $p, q \in \mathbf{Z}$, define the subspace $W_g^{p,q}$ as the space of $w \in W \otimes_L \mathbf{C}$ for which $h_g(z)w = z^p \bar{z}^q w$ for all $z \in \mathbf{C}^\times$.

PROPOSITION. *Put $\alpha = (w - k_1)/2$ and let $s = gi \in \mathbf{P}^1\mathbf{C}$. If $p$ lies outside $[\alpha, \alpha + k_1 - 2]$, or if $p + q \neq w - 2$, then $W_g^{p,q} = 0$. Furthermore, $W_g^{\alpha, \alpha + k_1 - 2}$ is the fibre of $\mathbb{V}(\mathbf{C})$ above $s$ viewed as a subspace of $\mathbb{W}_s = W$.*

PROOF: We can assume $n = 1$, thus ignoring all Sym-terms except the first. The eigenvalues of $h_g(z)$ in $\mathrm{Aut}(W \otimes_L \mathbf{C})$ are $(x+iy)^p(x-iy)^{w-2-p}$ with $p \in [\alpha, \alpha + k_1 - 2]$ and corresponding eigenvectors $\left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right)^{\otimes p} \otimes \left(\begin{smallmatrix} -i \\ 1 \end{smallmatrix}\right)^{\otimes (w-2-p)}$ (recall that we have twisted the natural action by $\det^{(w-k_1)/2}$). Q.E.D.

Recall that $X$ is a $G(\mathbf{R})$-conjugacy class of maps $\mathbf{S} \to G_{\mathbf{R}}$. So if $L \subset \mathbf{R}$, we would naturally get a variation of real Hodge structure on the constant sheaf on $X$ defined by $W \otimes_L \mathbf{R}$. Unfortunately, this is not the case and we have to do a little trick. Let $W'$ be $W$ considered as an $L \cap \mathbf{R}$-vector spaces. It has a natural $G(L \cap \mathbf{R})$-action on it. To each map $h: \mathbf{S} \to G_{\mathbf{R}}$ corresponds a real Hodge structure on $W' \otimes_{\mathbf{R} \cap L} \mathbf{R}$. Fix such an $h \in X \subset \mathbf{P}^1\mathbf{C}$. The decomposition $\mathbf{C} \cdot \left(\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right) \otimes \mathbf{C} \cdot \left(\begin{smallmatrix} -i \\ 1 \end{smallmatrix}\right) = \mathbf{C} \oplus \overline{\mathbf{C}} \simeq \mathbf{C} \otimes_{\mathbf{R}} \mathbf{R}^2$ induces an isomorphism $W \otimes_L \mathbf{C} \oplus \overline{W \otimes_L \mathbf{C}} \xrightarrow{\sim} W' \otimes_{L \cap \mathbf{R}} \mathbf{C}$. The Hodge structure is now described by the previous proposition; in particular $(W'_{\mathbf{C}})_h^{\alpha, \alpha + k_1 - 2} = \mathbb{V}_h \oplus \overline{\mathbb{V}_h}$.

Fix a connected component $\Gamma \backslash X^+$ of $M_K(\mathbf{C})$, where as above $X^+ \subset X \subset \mathbf{P}^1\mathbf{C}$ is a connected component of $X$ and $\Gamma$ is the image of $\Gamma' = gKg^{-1} \cap G(\mathbf{Q})_+$ under the canonical map $G(\mathbf{R})_+ \to G^{\mathrm{ad}}(\mathbf{R})^+$. Let $Z^c(\mathbf{Q})$ be the subgroup of

the center $F^\times$ of $G(\mathbf{Q})$ given as the kernel of the norm. Consider the following diagram

$$\begin{array}{ccc}
Z^c(\mathbf{Q}) \rightarrowtail F^\times & \xrightarrow{\ N_{F/\mathbf{Q}}\ } & \mathbf{Q}^\times \\
\downarrow & & \downarrow \\
Z_G(\mathbf{A}^f) \longrightarrow & & (\mathbf{A}^f)^\times
\end{array}$$

where the top row is exact, the square is commutative and the lower map is continuous. The set $\mathbf{Q}^\times$ is a discrete subset of $(\mathbf{A}^f)^\times$. Therefore, using the diagram we see that there exists an open neighbourhood $U \subset G(\mathbf{A}^f)$ of 1 such that $U \cap Z(\mathbf{Q}) = Z^c(\mathbf{Q})$. By shrinking $K$ if necessary, we may assume that $K$ is contained in $U$. As a consequence, the kernel of $\Gamma' \to \Gamma$ is contained in $Z^c(\mathbf{Q})$.

Now note that $Z^c(\mathbf{Q})$ acts trivially on the vector spaces $V$ and $W$ that where the starting points of the constructions we made so far.. Therefore, they carry an action of $\Gamma$. But for $K$ small enough, $\Gamma$ is the fundamental group of $\Gamma \backslash X^+$. So the holomorphic vector bundle $\mathbb{V}(\mathbf{C})|_{X^+}$ can be divided out by $\Gamma$ to give a bundle on $\Gamma \backslash X^+$. Doing so for all components, we get a vector bundle $\mathcal{V}_K(\mathbf{C})$ on $M_K(\mathbf{C})$.

There is a correspondence between local systems and vector bundles with connection (see [Del2]). In particular, the constant sheaf on $\mathbf{P}^1(\mathbf{C})$ associated to $W \otimes_L \mathbf{C}$ corresponds to $\mathbb{W}(\mathbf{C})$ with the trivial connection. In the same way as before, we get a vector bundle $\mathcal{W}_K(\mathbf{C})$ on $M_K(\mathbf{C})$ with a flat connection $\nabla$. In turn this corresponds to a local system $\mathcal{F}_{K,\mathbf{C}}$. As the map $\mathbb{V}(\mathbf{C}) \hookrightarrow \mathbb{W}(\mathbf{C})$ is equivariant, it induces an embedding $\mathcal{V}_K(\mathbf{C}) \hookrightarrow \mathcal{W}_K(\mathbf{C}) = \mathcal{F}_{K,\mathbf{C}} \otimes_\mathbf{C} \mathcal{O}_{M_K(\mathbf{C})}$ of vector bundles.

In fact, there is another way to make $\mathcal{F}_{K,\mathbf{C}}$. The fundamental groups $\Gamma$ of $\Gamma \backslash X^+$ acts on $W$, hence gives a locally constant sheaf $\mathcal{F}_K$ of $L$-vector spaces on $M_K(\mathbf{C})$. Of course $\mathcal{F}_{K,\mathbf{C}} = \mathcal{F}_K \otimes_L \mathbf{C}$.

We can algebraize this second construction. Let $\lambda$ be a prime of $L$. Note that the representations $\Gamma \to \mathrm{Aut}_L(W)$ we use are continuous if we use the $\lambda$-adic topology on the right and the profinite topology on the left. So we can complete and obtain representations of the étale fundamental groups of the connected components of $M_{K,\mathbf{C}}$. These define a constructible $\lambda$-adic sheaf $\mathcal{F}_{\lambda,K,\mathbf{C}}$ on $(M_K \times_L \mathbf{C})_{\mathrm{et}}$ (c.f. SGA 5 VI 1). The analytification of $\mathcal{F}_{\lambda,K,\mathbf{C}}$ is $\mathcal{F}_K \otimes_L L_\lambda$.

The last step is to pass from the complex setting to models over number fields.

Choose $\mathbf{Z}_\ell$-sheaves corresponding to $\mathcal{F}_{\lambda,K,\mathbf{C}}$ for all $K$. Then for every $n \geq 1$ the mod-$\ell^n$ reduction of this sheaf is constant for $K$ small enough. Hence $\mathcal{F}_{\lambda,\mathbf{C}}$ is constant in the limit over all $K$, hence it descends to the contant $\lambda$-adic sheaf $\mathcal{F}_\lambda$ on $M = \varprojlim_K M_K$ over $F$. The final step now is to note that for $K$ small, $M$ is a Galois cover of $M_K$.

The descent for vector bundles is a lot harder. Recall that $V$ is defined over $L$. Choose $F \hookrightarrow L$. There is a canonical way to construct a vector bundle $\mathcal{V}_K$ over $M_K \times_F L$, which is a model of $\mathcal{V}_K(\mathbf{C})$ (this last bundle is algebraic by

GAGA). The construction of these models is sketched (in much greater generality) in [Mil]; the basic ideas of the proof are the same as in the proof of the existence of a canonical model of a Shimura variety.

All these constructions again work in the context of projective systems with a $G(\mathbf{A}^f)$-action. We will not botter to spell this out. What we eventually need is that the cohomologies of the sheaves we have constructed form direct systems equipped with $G(\mathbf{A}^f)$-actions.

## 5  Hodge theory

The next theorem is the analogue of the Eichler-Shimura isomorphism.

THEOREM.   *For every $K$, there is an isomorphism*

$$\mathrm{H}^1(M_K(\mathbf{C}), \mathcal{F}_K^k \otimes_L \mathbf{C}) \simeq$$

$$\mathrm{H}^0(M_K(\mathbf{C}), \mathcal{V}_K^k \otimes_{\mathcal{O}_{M_K(\mathbf{C})}} \Omega^1_{M_K(\mathbf{C})}) \oplus \overline{\mathrm{H}^0(M_K(\mathbf{C}), \mathcal{V}_K^k \otimes_{\mathcal{O}_{M_K(\mathbf{C})}} \Omega^1_{M_K(\mathbf{C})})}.$$

*These isomorphisms are compatible with the direct systems we obtain we let let $K$ vary; as well as with the $G(\mathbf{A}^f)$-action on these systems.*

PROOF:   In the proof, we will erase $K$ and some $\mathbf{C}$'s from the notations, put $\mathcal{O} = \mathcal{O}_{M(\mathbf{C})}$ and $\Omega^1 = \Omega^1_{M(\mathbf{C})}$.

By the properties of a Shimura variety, the real local system $\mathcal{F}'$ carries a variation of real Hodge structure, which in the fibre above $h \in X^+$ is the one defined by $h$ and $W' \simeq \mathcal{F}'_h$ as above.

The complex local system $\mathcal{F}' \otimes_{\mathbf{R}} \mathbf{C} = \mathcal{F} \otimes_{\mathbf{R} \cap L} \mathbf{C}$ is associated with the vector bundle $\mathcal{W}' = \mathcal{F}' \otimes_{\mathbf{R}} \mathcal{O}$ and the connection

$$\mathcal{W}' \quad \xrightarrow{\nabla} \quad \mathcal{W}' \otimes_{\mathcal{O}} \Omega^1.$$

We regard this as a complex $R^\bullet$ of sheaves of $\mathbf{C}$-vector spaces. The complex $R^\bullet$ is a resolution of $\mathcal{F}' \otimes_{\mathbf{R}} \mathbf{C}$ and therefore

$$\mathrm{H}^1(M(\mathbf{C}), \mathcal{F}' \otimes_{\mathbf{R}} \mathbf{C}) \simeq \mathrm{H}^1(M(\mathbf{C}), R^\bullet),$$

where on the right we use hypercohomology.

The variation of Hodge structure on $\mathcal{F}'$ puts a filtration on $R^\bullet$:

$$
\begin{array}{ccccc}
R^\bullet : & \mathcal{W}' & \stackrel{\nabla}{\longrightarrow} & \mathcal{W}' \otimes_{\mathcal{O}} \Omega^1_{\mathcal{O}} \\
& \| & & \| & \\
\mathrm{Fil}^{\alpha} R^\bullet : & \mathrm{Fil}^{\alpha} \mathcal{W}' & \longrightarrow & \mathrm{Fil}^{\alpha-1} \mathcal{W}' \otimes_{\mathcal{O}} \Omega^1 \\
& \cup & & \| & \\
\mathrm{Fil}^{\alpha+1} R^\bullet : & \mathrm{Fil}^{\alpha+1} \mathcal{W}' & \longrightarrow & \mathrm{Fil}^{\alpha} \mathcal{W}' \otimes_{\mathcal{O}} \Omega^1 \\
& \cup & & \cup & \\
& \vdots & & \vdots & \\
& \cup & & \cup & \\
\mathrm{Fil}^{\alpha+k_1-2} R^\bullet : & \mathrm{Fil}^{\alpha+k_1-2} \mathcal{W}' & \longrightarrow & \mathrm{Fil}^{\alpha+k_1-3} \mathcal{W}' \otimes_{\mathcal{O}} \Omega^1 \\
& \cup & & \cup & \\
\mathrm{Fil}^{\alpha+k_1-1} R^\bullet : & 0 & \longrightarrow & \mathrm{Fil}^{\alpha+k_1-2} \mathcal{W}' \otimes_{\mathcal{O}} \Omega^1 \\
& \cup & & \cup & \\
\mathrm{Fil}^{\alpha+k_1} R^\bullet : & 0 & \longrightarrow & 0,
\end{array}
$$

where $\alpha = (w - k_1)/2$. The fact that $\nabla(\mathrm{Fil}^i \mathcal{W}') \subset \mathrm{Fil}^{i-1} \mathcal{W}' \otimes_{\mathcal{O}} \Omega^1$ is Griffiths transversality.

This filtration induces a spectral sequence

$$E_1^{p,q} = \mathrm{H}^{p+q}(M(\mathbf{C}), \mathrm{Gr}^p R^\bullet) \quad \Rightarrow \quad \mathrm{H}^{p+q}(M(\mathbf{C}), R^\bullet).$$

Here $\mathrm{Gr}^p R^\bullet$ is the complex

$$\mathrm{Gr}^p R^0 \quad \stackrel{\nabla}{\longrightarrow} \quad \mathrm{Gr}^p R^1.$$

If $\alpha < p < \alpha + k_1 - 1$ the map $\nabla$ appearing here is an isomorphism. If we had a moduli interpretation, this would be the Kodaira-Spencer isomorphism; here one needs to check it by hand. So for these $p$, the complex is homotopy equivalent to the zero complex and hence $E_1^{p,q} = 0$ for every $q$.

Now let $p = \alpha + k_1 - 1$. As $\mathrm{Fil}^{p-1} \mathcal{W}' = \mathrm{Gr}^{p-1} \mathcal{W}' \simeq \mathcal{V} \oplus \overline{\mathcal{V}}$, we see that

$$E_1^{p,1-p} = \mathrm{H}^0(M(\mathbf{C}), \mathcal{V} \otimes_{\mathcal{O}} \Omega^1) \oplus \overline{\mathrm{H}^0(M(\mathbf{C}), \mathcal{V} \otimes_{\mathcal{O}} \Omega^1)}.$$

By Serre duality $E_1^{\alpha,-\alpha+1}$ is isomorphic to $E^{p,1-p}$.

Also $E_1^{\alpha,-\alpha} = E_1^{p,2-p} = 0$, which one sees by proving $\mathrm{H}^1(M(\mathbf{C}), \mathcal{V} \otimes_{\mathcal{O}} \Omega^1) = 0$ (again by Serre duality, this suffices). For this one can use induction over the $k_i$'s, where in the case of a line bundle there is the usual degree argument.

For the other pairs $(p,q)$, always $E_1^{p,q} = 0$ for dimension reasons. It follows that the spectral sequence degenerates at $E_1$. So

$$\mathrm{H}^1(M(\mathbf{C}), \mathcal{F}' \otimes_{\mathbf{R}} \mathbf{C}) = \left( \mathrm{H}^0(M(\mathbf{C}), \mathcal{V} \otimes_{\mathcal{O}} \Omega^1) \oplus \overline{\mathrm{H}^0(M(\mathbf{C}), \mathcal{V} \otimes_{\mathcal{O}} \Omega^1)} \right)^2.$$

The theorem now follows using $\mathcal{F} \oplus \overline{\mathcal{F}} \simeq \mathcal{F}' \otimes_{\mathbf{R}} \mathbf{C}$.   Q.E.D.

# 6 Using the Jacquet-Langlands correspondence

Consider the $G(\mathbf{A}^f)$-representations

$$S'^k_L = \varinjlim_{K} \mathrm{H}^0(M_{K,L}, \mathcal{V}^k_K \otimes_{\mathcal{O}_{M_{K,L}}} \Omega^1_{M_K/L})$$

and $S'^k_{\mathbf{C}} = S'^k_L \otimes_L \mathbf{C}$. The last one decomposes into a product of irreducible representations, which are called automorphic representations (of fixed weight $k$) of $(B \otimes_F \mathbf{A}^f_F)^{\times}$. Each factor $\pi'$ is a restricted tensor product $\bigotimes'_{\mathfrak{p}} \pi'_{\mathfrak{p}}$ indexed by the primes $\mathfrak{p}$ of $F$. Each factor $\pi'_{\mathfrak{p}}$ is a smooth irreducible representation of $B^{\times}_{\mathfrak{p}}$.

The Jacquet-Langlands correspondence was explained in an earlier talk. It says that there is a bijection

$$\left\{\begin{array}{l}\text{cuspidal automorphic represen-}\\ \text{tations } \pi \text{ of } \mathrm{GL}_2(\mathbf{A}^f_F) \text{ such that}\\ \pi_v \text{ is not a principal series}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{automorphic rep-}\\ \text{resentations of}\\ (B \otimes_F \mathbf{A}^f_F)^{\times})\end{array}\right\},$$

which maps $\pi$ to an irreducible component $\pi'$ of $S'^k_{\mathbf{C}}$ and which "respects the local Langlands correspondence". In particular, let $\mathfrak{p}$ be a prime of $F$ that, if $n$ is even, is not equal to $v$. Choose an isomorphism $\mathrm{GL}_2(F_{\mathfrak{p}}) \simeq B^{\times}_{\mathfrak{p}}$. Then $\pi_{f,\mathfrak{p}}$ and $\pi'_{f,\mathfrak{p}}$ are isomorphic.

From the correspondence it follows that the irreducible components of $S'^k_{\mathbf{C}}$ are indexed by the newforms $f$ of weight $k$ that, if $n$ is even, has the right behaviour at $v$. It also follows that each factor $\pi'_f$ is defined over $L(f)$ and that $\pi'_f = \pi'_g$ if and only if $f = g$.

The $L_{\lambda}$-vector space

$$H^k_{\lambda} = \varinjlim_{K} \mathrm{H}^1(M_{K,\overline{F},\mathrm{et}}, \mathcal{F}^k_{\lambda,K})$$

has actions of $\mathrm{Gal}(\overline{F}/F)$ and $G(\mathbf{A}^f)$ on it and these actions commute. Therefore, the space

$$W_{f,\lambda'} = \mathrm{Hom}_{G(\mathbf{A}^f)-\mathrm{repr.}}(\pi'_f \otimes_{L(f)} L(f)_{\lambda'}, H^k_{\lambda} \otimes_{L_{\lambda}} L(f)_{\lambda'}).$$

carries a $\mathrm{Gal}(\overline{F}/F)$-action. There is also the complex vector space

$$W_{f,\mathbf{C}} = \mathrm{Hom}_{G(\mathbf{A}^f)-\mathrm{repr.}}(\pi'_f \otimes_{L(f)} \mathbf{C}, H^k \otimes_L \mathbf{C});$$

it of course carries no natural Galois action, but by the above theorem combined with the multiplicity-one result we know that it has dimension 2. To compare these two spaces, we employ the comparison theorems for étale cohomology. It follows that $W_{f,\lambda'}$ is of dimension 2.

The representation $W_{f,\mathbf{C}}$ is the dual of the representation $\rho_{f,\lambda}$ we wanted to construct.

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