Galois representations associated to modular forms

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These are notes from a talk given at an intercity seminar arithmetic geometry. The main reference is [1], where more details and further references can be found.

1 Modular forms

1.1 Definitions

Consider the complex upper half plane $\mathfrak{h} := \{z \in \mathbb{C} : \Im z > 0\}$. On it we have an action of $\mathrm{SL}_2(\mathbb{Z})$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}.$$

We can also add cusps to $\mathfrak{h}$. The cusps are the points in $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. We will denote the completed upper half plane by $\mathfrak{h}^*$, so $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{P}^1(\mathbb{Q})$. It is clear how to extend the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{h}$ to an action on $\mathfrak{h}^*$: use the same fractional linear transformation.

We will focus on two subgroups of $\mathrm{SL}_2(\mathbb{Z})$: define for a positive integer N,

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

and, even more interesting,

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N, a \equiv d \equiv 1 \bmod N \right\}.$$

Clearly, $\Gamma_1(N)$ is a subgroup of $\Gamma_0(N)$. It is actually a normal subgroup and

$$\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^* \quad \text{by } \overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \mapsto \overline{d}.$$

Now, let k and N be positive integers. In this talk, a *modular form of weight k and level N* is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ satisfying the following conditions:

- $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(N)$.
- f is holomorphic at the cusps. Roughly, this means that f should not behave too wildly if z approaches a cusp. More precisely, for any matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the function $(cz+d)^{-k} f(\frac{az+b}{cz+d})$ should be bounded in the region $\{z \in \mathbb{C} : \Im z \geq M\}$ for some $M > 0$.

The set of modular forms of weight k and level N is denoted by $M_k(\Gamma_1(N))$. From this notation the reader is able to guess correctly what $M_k(\Gamma_0(N))$ and $M_k(\mathrm{SL}_2(\mathbb{Z}))$ mean. Now, $M_k(\Gamma_1(N))$ is a vector space over $\mathbb{C}$ which is known to be of finite dimension. If f is not only holomorphic at the cusps, but even vanishes at the cusps, by which we mean that for each element of $\mathrm{SL}_2(\mathbb{Z})$ the function $(cz+d)^{-k} f(\frac{az+b}{cz+d})$ should approach 0 if $\Im z$ approaches infinity, then f is called a *cusp form.* The subspace of $M_k(\Gamma_1(N))$ of cusp forms is denoted by $S_k(\Gamma_1(N))$.

Note that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_1(N)$ for all N. If we plug this matrix into the transformation property of a modular form f, then it follows that $f(z+1) = f(z)$. In other words, f is periodic with period 1. Hence f is a holomorphic function of

$$q := e^{2\pi i z}.$$

We therefore have a power series expansion

$$f(z) = \sum_{n \geq 0} a_n(f) q^n,$$

the so-called *q-expansion* of f. The nonexistence of terms with nonnegative exponent is equivalent with f being holomorphic at ∞. If f is a cusp form, then it vanishes at ∞ and hence $a_0(f) = 0$. Be aware of the fact that $a_0 = 0$ does not in general imply that f is a cusp form because there are other cusps than ∞.

Let's give some examples of modular forms of level 1 now. Note that in this case $\Gamma_1(N) = \mathrm{SL}_2(\mathbb{Z})$. There are no nonzero modular forms of odd weight here. This can be seen by plugging in the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which yields the identity $f(z) = (-1)^k f(z)$. If $k \geq 4$ is even, then

$$E_k(z) = -\frac{B_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n$$

is a modular form of weight k, the so-called *normalized Eisenstein series.* Here B_k is the k-th Bernoulli number and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. The lowest weight were we have a cusp form is $k = 12$ (for higher levels, however, there are cusp forms of lower weight):

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24}.$$

This form is called the *discriminant modular form* and generates the space $S_{12}(\mathrm{SL}_2(\mathbb{Z}))$. If we write it out as a series

$$\Delta(z) = \sum_{n \geq 1} \tau(n) q^n,$$

then $\tau(n)$ is called the *Ramanujan tau function.*

1.2 Diamond and Hecke operators

The group $\Gamma_0(N)$ acts on $S_k(\Gamma_1(N))$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Note that this does not define an action of the whole group $\mathrm{SL}_2(\mathbb{Z})$ on $S_k(\Gamma_1(N))$ since the resulting function need not be an element of $S_k(\Gamma_1(N))$ in that case. Now, the action of the subgroup $\Gamma_1(N)$ is trivial so this defines an action of $(\mathbb{Z}/N\mathbb{Z})^*$ on $S_k(\Gamma_1(N))$:

$$\langle d \rangle f := \begin{pmatrix} a & b \\ c & d \end{pmatrix} f$$

only depends on d and not on the other entries of the matrix in $\Gamma_0(N)$. The operator $\langle d \rangle$ is called a *diamond operator.*

Let $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ be a character. Then we define the subspace $S_k(N, \varepsilon)$ of $S_k(\Gamma_1(N))$ as

$$S_k(N, \varepsilon) := \{f \in S_k(\Gamma_1(N)) : \langle d \rangle f = \varepsilon(d) f \quad \text{for all } d \in (\mathbb{Z}/N\mathbb{Z})^*\}$$

and call it the ε-eigenspace of $S_k(\Gamma_1(N))$. Note that if ε is the trivial character, then $S_k(N, \varepsilon) = S_k(\Gamma_0(N))$. It is a fact that

$$S_k(\Gamma_1(N)) = \bigoplus_{\varepsilon : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*} S_k(N, \varepsilon).$$

Now, we will define Hecke operators. Let p be a prime number, not dividing N. Let $M_p \in M_2(\mathbb{Z})$ be any matrix of determinant p that is of the form $\left(\begin{smallmatrix} ap & b \\ Np & p \end{smallmatrix}\right)$. Define

$$S := \left\{ \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix} : j \in \{0, \ldots, p-1\} \right\} \cup \{M_p\}. \tag{1}$$

Later, in section 2.2, we will see where this set of matrices comes from. For $f \in S_k(\Gamma_1(N))$, we define

$$T_p f(z) := p^{k-1} \sum_{\gamma \in S} (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right).$$

The operator T_p is called a *Hecke operator.* It follows from the transformation property of f that this definition of T_p is independent of the choice of M_p. One can show that $T_p f \in S_k(\Gamma_1(N))$ for $f \in S_k(\Gamma_1(N))$ and also that $T_p f \in S_k(N, \varepsilon)$ for $f \in S_k(N, \varepsilon)$. Furthermore, the Hecke operators commute with each other and with the diamond operators:

$$T_p T_q = T_q T_p \quad \text{and} \quad T_p \langle d \rangle = \langle d \rangle T_p$$

for all d, p, q coprime to N.

One can decompose $S_k(\Gamma_1(N))$ into subspaces that are simultaneous eigenspaces for the T_p and $\langle d \rangle$ operators. Actually, we already mentioned above that we can do this decomposition for the diamond operators, and we can extend this decomposition to eigenspaces for the Hecke operators as well. In particular, an eigenform always lies in a space $S_k(N, \varepsilon)$ and we call ε the character of the eigenform.

If a simultaneous eigenspace for the diamond and Hecke operators has dimension 1, then it has a generator $f = \sum a_n q^n$ with $a_1 = 1$. Such an f is called a *newform.* This terminology is explained as follows. If $M \mid N$ and $d \mid N/M$, then we have a map

$$\alpha_d : S_k(\Gamma_1(M)) \to S_k(\Gamma_1(N)) : f(z) \mapsto f(d \cdot z).$$

Now, if $M \neq N$ and f is an eigenform in $S_k(\Gamma_1(M))$, then the $\alpha_d(f)$ all lie in the same eigenspace of $S_k(\Gamma_1(N))$, so this eigenspace has dimension greater than 1. We call this an *old* eigenspace because it can be constructed from a modular form that already appeared in a lower level than N. It can be shown that the *new* eigenspaces, that is the ones that cannot be constructed this way, have dimension 1. This phenomenon is refered to as *multiplicity one.*

A newform f has the property that $T_p f = a_p f$ for all primes $p \nmid N$, i.e. the eigenvalue of T_p is exactly the coefficient a_p of the q-expansion. Actually one can also define T_p for $p \mid N$, and then for a newform the relation $T_p f = a_p f$. The point however is that if we allow these T_p then we have no decomposition into eigenspaces anymore. So the coefficients of a modular form are not just things that exist by analysis, but have an actual meaning as eigenvalues of interesting operators. Furthermore, the coefficients of a newform satisfy

$$\begin{array}{rcll} a_{mn} & = & a_m a_n & \text{if } \gcd(m,n)=1, \\ a_{p^{r+1}} & = & a_p a_{p^r} - \varepsilon(p) p^{k-1} a_{p^{r-1}} & \text{if } p \nmid N, \\ a_{p^r} & = & a_p^r & \text{if } p \mid N. \end{array}$$

The coefficients a_n of a newform f generate a field $K_f := \mathbb{Q}(a_1, a_2, \ldots)$ over $\mathbb{Q}$ which is known to be a number field and contains all the values of ε. We can view K_f as an abstract number field (i.e. not embedded in $\mathbb{C}$) and ε as a character with values in $\mu(K_f)$, the group of roots of unity of K_f. This view is justified by the fact that for any embedding $\sigma : K_f \hookrightarrow \mathbb{C}$ the function $\sigma f = \sum \sigma(a_n) q^n$ is a newform, with character $\sigma\varepsilon$.

1.3 Galois representations

It is a conjecture of Ramanujan and Petersson that for a newform f of weight k, the inequality

$$|a_p| \leq 2p^{(k-1)/2}$$

holds for all primes $p \nmid N$. Later, Serre refined this conjecture to a more delicate conjecture, which was eventually proved by Deligne for weights $k \geq 2$ and Deligne and Serre for $k = 1$. In this section we will state this proven conjecture of Serre.

Let f be a newform and let K_f be the coefficient field of f. Choose a rational prime ℓ and a prime λ of K_f lying over ℓ. Then there is a continuous representation

$$\rho = \rho_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_{f,\lambda})$$

satisfying a set of properties that we are about to describe. Here, as usual, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ has the Krull topology and $\mathrm{GL}_2(K_{f,\lambda})$ has the λ-adic topology.

First of all, ρ is unramified outside $N\ell$. This means that each finite extension of $\mathbb{Q}$ that sits in the fixed field $\overline{\mathbb{Q}}^{\ker(\rho)}$ of $\ker(\rho)$ is unramified above each prime $p \nmid N\ell$. Knowing this, for each prime $p \nmid N\ell$ we can do the following. Choose an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. This induces an embedding $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We surject $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ onto $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ in the obvious way. Now, $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ contains the element Frob_p that sends x to x^p for all x. We can lift Frob_p to $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and from there put it into $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By abuse of language we will call this element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ also Frob_p. Note that $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is not uniquely determined but depends on two choices, namely the lift to $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and the embedding $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We want $\rho(\mathrm{Frob}_p)$ to be well-defined up to conjugation in $\mathrm{GL}_2(K_{f,\lambda})$. This is implied by the fact that ρ is unramified outside $N\ell$ together with the fact that different embeddings of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ into $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ lead to conjugate subgroups of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In particular the trace and the determinant of $\rho(\mathrm{Frob}_p)$ are well-defined, hence also its characteristic polynomial. Now comes the most interesting property of ρ:

$$\begin{array}{rcl} \mathrm{tr}(\rho(\mathrm{Frob}_p)) & = & a_p, \\ \det(\rho(\mathrm{Frob}_p)) & = & \varepsilon(p)p^{k-1}. \end{array}$$

So the characteristic polynomial of $\rho(\mathrm{Frob}_p)$ is equal to $X^2 - a_p X + \varepsilon(p)p^{k-1}$. The interesting thing is that the complex roots of this polynomial have absolute value equal to $p^{(k-1)/2}$. This is a consequence of a famous result of Deligne, namely the Riemann Hypothesis part in his proof of the Weil Conjectures. It is clear that this implies the conjecture of Ramanujan and Petersson, as the trace is the sum of the roots of the characteristic polynomial.

In the case $k > 2$, the construction of ρ and the proof that it satisfies the above mentioned properties is quite technical and uses étale cohomology, therefore we will not give it in these notes. However, if $k = 2$ it is a lot easier than in the general case. In the remainder of these notes we will explain how to construct $\rho_{f,\lambda}$ if f is a newform of weight 2 and some level N.

2 Modular curves

2.1 Preliminaries

One can divide out the group action of $\Gamma_1(N)$ on $\mathfrak{h}$ to get the so-called *modular curve*

$$Y_1(N) = \Gamma_1(N)\backslash\mathfrak{h}.$$

We can add the cusps to $Y_1(N)$ to compactify it and obtain the modular curve

$$X_1(N) = \Gamma_1(N)\backslash\mathfrak{h}^*.$$

The topology that we use in $\mathfrak{h}^*$ is not the inherited topology of $\mathbb{C}$. We do take the inherited topology of $\mathbb{C}$ on $\mathfrak{h}$, but as a basis of open neighbourhoods of the cusps we take the following sets:

$$\{\gamma\left(\{z : \Im z > M\} \cup \{\infty\}\right) : \gamma \in \mathrm{SL}_2(\mathbb{Z}), M \in \mathbb{R}_{>0}\}$$

It turns out that $X_1(N)$ has a model $X_1(N)_{\mathbb{Q}}$ over $\mathbb{Q}$. If $N > 3$, then for each field $K/\mathbb{Q}$ we have the following moduli-description of the K-valued points of $Y_1(N)_{\mathbb{Q}} \subset X_1(N)_{\mathbb{Q}}$:

$$Y_1(N)_{\mathbb{Q}}(K) \cong \{(E, P) : E \text{ elliptic curve over } K,\ P \in E(K) \text{ torsion point of order } N\} /_{\cong_K}. \quad (2)$$

If $K = \mathbb{C}$ then we can make this description more concrete: the point $\Gamma_1(N)\tau$ corresponds to the pair

$$\left(\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}), 1/N \bmod (\mathbb{Z}\tau + \mathbb{Z})\right).$$

We can attach to the modular curve $X_1(N)$ its jacobian

$$J_1(N) := \mathrm{Jac}(X_1(N)),$$

which is also defined over $\mathbb{Q}$. As a group,

$$J_1(N) \cong (\text{divisors of degree 0 on } X_1(N))/(\text{principal divisors on } X_1(N)).$$

We have a very nice description of $J_1(N)$ as a complex torus. Let g be the genus of $X_1(N)$. The space $S_2(\Gamma_1(N))$ is isomorphic to the space of holomorphic differentials on $X_1(N)$ by $f \mapsto f\frac{dq}{q}$. In particular it is of dimension g. Choose a basis $f_1, \ldots, f_g$ of $S_2(\Gamma_1(N))$. Then define a lattice $\Lambda \subset \mathbb{C}^g$ by

$$\Lambda = \left\{ \int_\gamma \left(f_1 \frac{dq}{q}, \ldots, f_g \frac{dq}{q} \right) : [\gamma] \in H_1(X_1(N), \mathbb{Z}) \right\},$$

so we integrate our basis over all closed paths in $X_1(N)$. Now, it can be shown that

$$J_1(N) \cong \mathbb{C}^g/\Lambda.$$

This isomorphism is obtained as follows. We can write a degree 0 divisor as $\sum_i (P_i - Q_i)$. This is mapped to

$$\sum_i \int_{Q_i}^{P_i} \left(f_1 \frac{dq}{q}, \ldots, f_g \frac{dq}{q} \right) \bmod \Lambda.$$

This map is well-defined in the sense that the result in $\mathbb{C}^g/\Lambda$ does not depend on the chosen divisor nor on the chosen paths from Q_i to P_i. It is called the *Abel-Jacobi isomorphism.*

2.2 Diamond and Hecke operators

Not only on $S_k(\Gamma_1(N))$ but also on $J_1(N)$ we can define diamond and Hecke operators. We can give a very neat description in terms of the moduli interpretation (2). On points (E, P) we define

$$\begin{aligned} \langle d\rangle(E,P) &= (E, dP), \\ T_p(E,P) &= \sum_{C\subset E}(E/C, P \bmod C), \end{aligned}$$

where this sum is taken over all cyclic subgroups C of E of order p. As endomorphisms of $J_1(N)$, the diamond and Hecke operators are defined over $\mathbb{Q}$. Over $\mathbb{C}$ these operators are defined by exactly the same matrices that are used in defining them on $S_k(\Gamma_1(N))$. In particular this clearifies the choice of S in (1).

The operators $\langle d\rangle$ and T_p induce an action on the integral homology

$$H_1(J_1(N),\mathbb{Z}) \cong H_1(X_1(N),\mathbb{Z}) \cong \Lambda,$$

which we will also denote by $\langle d\rangle$ and T_p. We have a pairing

$$H_1(X_1(N),\mathbb{Z}) \times \left(S_2(\Gamma_1(N)) \oplus \overline{S_2(\Gamma_1(N))}\right) \to \mathbb{C} : (\gamma,\omega) \mapsto \int_\gamma \omega$$

(in this notation a modular form is identified with the corresponding differential on $X_1(N)$). If we tensor the left factor of the pairing with $\mathbb{C}$ it becomes a perfect pairing. A particularly nice property is that

$$\int_{T_p\gamma} f\frac{dq}{q} = \int_\gamma T_p f\frac{dq}{q}.$$

Furthermore, on $S_2(\Gamma_1(N))$ there is a perfect pairing for which $\langle d\rangle$ and T_p are self-adjoint. This implies that the decomposition of $S_2(\Gamma_1(N))$ into eigenspaces induces a decomposition of $H_1(X_1(N),\mathbb{C})$ into eigenspaces, but beware that here the eigenspaces have twice the dimension of their counterparts in $S_2(\Gamma_1(N))$. Since the eigenvalues of the diamond and Hecke operators are algebraic numbers, we can find the full decomposition already in $H_1(X_1(N),\overline{\mathbb{Q}})$. For a newform f, the eigenvalues of the operators $\langle d\rangle$ and T_p are in K_f, so we can find its corresponding 2-dimensional eigenspace inside $H_1(X_1(N), K_f)$.

2.3 Tate modules

Let ℓ be a prime number. For all positive integers n we consider $J_1(N)[\ell^n]$, the group of ℓ^n-torsion points on $J_1(N)$. This is a free $\mathbb{Z}/\ell^n\mathbb{Z}$-module of rank $2g$, equipped with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For all n there is a map

$$J_1(N)[\ell^{n+1}] \xrightarrow{\cdot\ell} J_1(N)[\ell^n]$$

and we can take the projective limit

$$T_\ell(J_1(N)) := \varprojlim J_1(N)[\ell^n].$$

This $T_\ell(J_1(N))$ has a natural $\mathbb{Z}_\ell$-module structure and is called a *Tate module*. It has an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\langle d\rangle$ and T_p on it. One can show that for $p \nmid N\ell$ the following relation holds in $\mathrm{End}_{\mathbb{Z}_\ell} T_\ell(J_1(N))$:

$$T_p = \mathrm{Frob}_p + \langle p\rangle \mathrm{Ver}_p = \mathrm{Frob}_p + \langle p\rangle p \,\mathrm{Frob}_p^{-1}.$$

This relation is called the *Eichler-Shimura relation.*

Now,

$$J_1(N)[\ell^n] \cong \frac{1}{\ell^n}\Lambda/\Lambda \cong \Lambda/\ell^n\Lambda \cong H_1(X_1(N),\mathbb{Z}/\ell^n\mathbb{Z}).$$

Hence also

$$T_\ell(J_1(N)) \cong H_1(X_1(N), \mathbb{Z}_\ell).$$

The advantage of using Tate modules instead of homology is that we have a Galois action. The price we pay for having this action is very low: we only have to replace the coefficient module $\mathbb{Z}$ of the homology by $\mathbb{Z}_\ell$, where we can even choose the ℓ we like best.

We have a decomposition of $T_\ell(J_1(N)) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}$ into Hecke eigenspaces in the same way as we have with $H_1(X_1(N), \overline{\mathbb{Q}})$. Let f be a newform in $S_2(\Gamma_1(N))$, with coefficient field K_f. Let $\lambda \mid \ell$ be a prime of K_f lying above ℓ. The decomposition into eigenspaces, together with multiplicity one, shows that

$$W_{f,\lambda} := \{v \in T_\ell(J_1(N)) \otimes_{\mathbb{Z}_\ell} K_{f,\lambda} : T_p(v) = a_p(f)v \text{ for all } p \text{ and } \langle d \rangle v = \varepsilon(d)v \text{ for all } d\}$$

is a 2-dimensional subspace of $T_\ell(J_1(N)) \otimes_{\mathbb{Z}_\ell} K_{f,\lambda}$. Here, of course, the diamond and Hecke operators act via the left factor of the tensor product and the scalar multiplication is defined via the right factor.

Using the fact that $\langle d \rangle$ and T_p are defined over $\mathbb{Q}$, one can show that $W_{f,\lambda}$ is invariant under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. In particular, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $W_{f,\lambda}$ gives a 2-dimensional representation

$$\rho_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}_{K_{f,\lambda}}(W_{f,\lambda}) \cong \mathrm{GL}_2(K_{f,\lambda}).$$

This last isomorphism invokes a choice of basis and is therefore not canonical. One can show that

$$\mathrm{tr}(\rho(\mathrm{Frob}_p)) = a_p \quad \text{and} \quad \det(\rho(\mathrm{Frob}_p)) = \varepsilon(p)p^{k-1}, \tag{3}$$

hence this gives the Galois representation associated to f. Be aware of the easily made mistake that (3) would immediately follow from the Eichler-Shimura relation: the characteristic polynomial of $\rho(\mathrm{Frob}_p)$ need not equal the minimal polynomial.

References

[1] F. Diamond and J. Im, *Modular forms and modular curves*, Seminar on Fermat's Last Theorem (Toronto, ON, 1993-1994), CMS Conf. Proc., **17**, Amer. Math. Soc., Providence, RI, 1995, 39-133.