HILBERT MODULAR FORMS

GERARD VAN DER GEER

Abstract. These are the lecture notes for my Intercity Talk of October 13, 2005 on Galois Representations associated to Hilbert Modular Forms.

1. Introduction

Hilbert started the study of the Hilbert modular group and the associated modular forms and modular varieties in an effort to extend the theory of complex multiplication to totally real fields (kronecker's Jugendtraum). The group $\mathrm{SL}(2,\mathbb{Z})$ is replaced by the group $\mathrm{SL}(2,O_F)$ with O_F the ring of integers of a totally real field of degree n over $\mathbb{Q}$. To some extent the theory is similar to the case $n = 1$, but there are also big differences. The geometry of the associated varieties is much more complicated and interesting. For example, the compactified Hilbert modular varieties have for $n > 1$ singularities which were first resolved by Hirzebruch in 1971 for the quadratic case.

The existence of Galois representations is also much more involved. The first thought of looking in the cohomology of the associated Hilbert modular variety does not work since there one does not find the desired Galois representations but a tensor product of such.

2. The Hilbert Modular Group

Let F be a totally real field of degree $n = [F : \mathbb{Q}]$ with ring of integers O_F. We let $I = \{\sigma : F \to \mathbb{R}\}$ be the set of real embeddings of F, often identified with $\{1, 2, \ldots, n\}$. For every $\sigma \in I$ we have an associated embedding $\mathrm{GL}(2,F) \to \mathrm{GL}(2,\mathbb{R})$. The group $\mathrm{SL}(2,\mathbb{R})/\{1,-1\}$ is the holomophic automorphism group of the upper half plane $\mathcal{H}$ of the complex plane. We thus get in a natural way an action of $\mathrm{SL}(2,F)$ on the product $\mathcal{H}^n$ of n upper half planes. The discrete subgroup $SL(2,O_F)$ of $\mathrm{SL}(2,\mathbb{R})^n$ acts properly discontinuously on $\mathcal{H}^n$ and the quotient space

$$\mathrm{SL}(2,O_F)\backslash\mathcal{H}^n$$

is a complex analytic variety (not quite a complex manifold because there are elements of finite order), or rather an orbifold since also the action is not effective. It is also non-compact. To compactify it we act just as in the case of $n = 1$ and $\mathrm{SL}(2,\mathbb{Z})$. We add the orbits of the action of $\mathrm{SL}(2,O_F)$ on the “rational” part of the boundary of $\mathcal{H}^n$, namely $\mathbb{P}^1(F)$. The orbits are called the cusps; we add the orbit space $\mathrm{SL}(2,O_F)\backslash\mathbb{P}^1(F)$ to the orbifold $\mathrm{SL}(2,O_F)\backslash\mathcal{H}^n$. The cusps are in 1-1 correspondence with the ideal classes via $\alpha/\beta \mapsto (\alpha,\beta)$. We thus obtain a compact analytic variety, called a Hilbert modular variety. In contrast to the case $n = 1$ these cusps are very singular for $n > 1$.

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The notion of modular form can be easily generalized. If $n > 1$ then a classical Hilbert modular form on $\mathrm{SL}(2, O_F)$ is a holomorphic function $f : \mathcal{H}^n \to \mathbb{C}$ satisfying

$$f(\gamma\, z) = (c_1 z_1 + d_1)^{k_1} \cdots (c_n z_n + d_n)^{k_n} f(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, O_F)$. Here we write $z = (z_1, \ldots, z_n)$ and $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ for the image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ under the ith embedding $\mathrm{SL}(2, O_F) \to \mathrm{SL}(2, \mathbb{R})$. The vector $k = (k_1, \ldots, k_n)$ is called the weight of f. For $n > 1$ no holomorphicity conditions at the cusps are necessary (Koecher's Principle, in fact a version of Hartog's Theorem implies this). If f is a Hilbert modular form then f has a Fourier expansion

$$f = \sum_{\nu \in O_F^\vee, \nu \geq 0} a_\nu e^{2\pi i \mathrm{Tr}(\nu z)},$$

where ν runs over the elements in the inverse different $O_F^\vee$ (= dual of O_F with respect to the trace map) and either $\nu = 0$ or ν is totally positive (positive under all embeddings into $\mathbb{R}$); moreover $\mathrm{Tr}(\nu z)$ stands for $\sum_{i=1}^n \nu_i z_i$. We can also develop f into a Fourier expansion at the other cusps (tranport the cusp to ∞ by an element of $\mathrm{SL}(2, F)$). We call f a cusp form if for all these Fourier expansions the constant term a_0 vanishes. We have spaces $M_k(\Gamma)$ and $S_k(\Gamma)$ of Hilbert modular forms, resp. cusp forms, of weight k for $\Gamma = \mathrm{SL}(2, O_F)$ or for any subgroup Γ of finite index.

Note that if the weight k is parallel, then $a_\nu = a_{\nu\epsilon}$ for ϵ in a certain subgroup of totally positive units.

The Koecher Principle, actually due to Götzky (1928) in this case, stated above, said that for $n > 1$ the coefficient $a_\nu = 0$ if $\nu \neq 0$ and ν is not totally positive. To see this, suppose that $a_\nu \neq 0$ and $\nu_1 < 0$. Choose a unit $\epsilon \in O_F^*$ with $\epsilon_1 > 1$ and $\epsilon_j < 1$ for $j = 2, \ldots, n$ and $\mathrm{Tr}(\epsilon\nu) < 0$. Then the terms in the Fourier expansion with $\nu = \nu_0 \epsilon^m$ and $z = (i, \ldots, i)$ tend to ∞ if $m \to \infty$ contradicting the convergence. So we see that f is automatically regular at the cusps for $n > 1$.

The following lemma is easy to prove.

Lemma 2.1. *If $M_k(\Gamma) \neq (0)$ then $k = (0, \ldots, 0)$ or all $k_i > 0$.*

Corollary 2.2. *The quotient $\Gamma \backslash \mathcal{H}^n$ has vanishing first cohomology.*

For example, if we have a 1-form $\sum_{i=1}^n f_i dz_i$ then the functions f_i are modular forms of weight $(0, \ldots, 0, 2, 0, \ldots, 0)$ and thus vanish. This remark will be important later.

3. The Adèlic Formulation

Let $\mathbb{A}_F$ be the adèle ring of F and $\mathbb{A}_{F,f}$ the finite part of it. Then $\mathbb{A}_{F,f} = \hat{O}_F \otimes \mathbb{Q} = \varprojlim O_F/N \otimes \mathbb{Q}$, where N runs through the non-zero integral ideals of O_F, and $\mathbb{A}_F = \mathbb{A}_{F,f} \times (F \otimes \mathbb{R})$. Furthermore, let $\mathbb{A}_F^*$ be the idèles of F and $\mathbb{A}_F^*/F^*$ the idèle class group of F. There is a surjective map

$$\mathbb{A}_F^*/F^* \longrightarrow Cl^+,$$

where Cl^+ is the strict ideal class group (non-zero ideals modulo the principal ideals generated by a totally positive element) and the map sends (a_p) to $\prod_{p \nmid \infty} p^{\mathrm{ord}_p(a_p)}$.

We now consider the group $G = R_{F/\mathbb{Q}}\mathrm{GL}(2)$ obtained by Weil restriction from $\mathrm{GL}(2)$. That is, we consider the group $\mathrm{GL}(2,F)$ as a group over $\mathbb{Q}$. Then $G(\mathbb{R}) = \mathrm{GL}(2,\mathbb{R})^n$.

Let $h_0 : \mathbb{C}^* \to G(\mathbb{R})$ be the homomorphism given by

$$x + iy \mapsto \left(\dots, \begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \dots\right)$$

We let K_∞ be the centralizer of h_0. Then $G(\mathbb{R})/K_\infty$ is a manifold of dimension $2n$ with 2^n connected components and via the map

$$G(\mathbb{R})/K_\infty \longleftrightarrow (\mathbb{C} - \mathbb{R})^n, \quad g \mapsto (g_1(i), \dots, g_m(i))$$

we get also a complex structure on it. (Some people might prefer $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$ instead of $\mathbb{C} - \mathbb{R}$.)

The group $G(\mathbb{A})$ with $\mathbb{A} = \mathbb{A}_\mathbb{Q}$, the adele ring of $\mathbb{Q}$, is a topological group. The group $\prod_{p \not| \infty} \mathrm{GL}(2, O_p) \times G(\mathbb{R})^0$ is an open subgroup.

Let K be a compact open subgroup of $G(\mathbb{A}_f)$ with $\mathbb{A}_f = \mathbb{A}_{\mathbb{Q},f}$, the finite adele ring of $\mathbb{Q}$. Then we define

$$X_K = G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K.$$

To describe it we consider the determinant map

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K \xrightarrow{\det} \mathbb{A}_F^*/F^* \det(G(\mathbb{R})^0 K).$$

The receiving end is a sort of class group. Strong approximation for $\mathrm{SL}(2)$ tells us that if $g_1, \dots, g_m$ are elements of $G(\mathbb{A}_f)$ such that the $\det(g_i)$ are a complete set of representatives of $\mathbb{A}_F^*/F^* \det(G(\mathbb{R})^0 K)$ then we have

$$G(\mathbb{A}) = \sqcup_{j=1}^m G(\mathbb{Q}) g_j G(\mathbb{R})^0 K.$$

We thus get

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K = \sqcup_{j=1}^m \Gamma_j \backslash \mathcal{H}^n,$$

where

$$\Gamma_j = g_j G(\mathbb{R})^0 K g_j^{-1} \cap G(\mathbb{Q}).$$

It follows that X_K is a union of classical Hilbert modular varieties. In particular for the choice $K = K_0 = \prod_{p \not| \infty} \mathrm{GL}(2, O_p)$ we get $h^+ = \# Cl^+$ components of the form $\mathrm{GL}^+(O_F \oplus a)\backslash\mathcal{H}$ with a running through Cl^+. Here GL^+ means that the determinant is totally positive.

Note that X_K is an orbifold. Actually we have a whole tower (projective system) of orbifolds X_K for K running over the compact open subgroups of $G(\mathbb{A}_f)$. The group $G(\mathbb{A}_f) = \mathrm{GL}(2, \mathbb{A}_{F,f})$ acts continuously, meaning that if K and L are compact open subgroups and $g \in G(\mathbb{A}_f)$ such that $g^{-1}Kg \subseteq L$ then we have a morphism

$$j_{L,K} : X_K \to X_L$$

given by right multiplication by g. These maps $j_{L,K}$ satisfy the identities

(1) $j_{M,L}(g) j_{L,K}(h) = j_{M,K}(gh)$;
(2) $j_{K,K}(g) = \mathrm{id}$ if $g \in K$;
(3) if K is a normal subgroup of L then j defines an action of L/K on X_K with quotient X_L.

We can set $X = \varprojlim_K X_K$ and then $X_K = X/K$.

4. Modular Forms

Suppose we start with $K = K_0$. Then X_K is the union of h^+ components $\mathrm{GL}^+(O_F \oplus a)\backslash\mathcal{H}^n$. If we take for a given weight $(k_1, \dots, k_n)$ a Hilbert modular form f_a on each component then the tuple $(\dots, f_a, \dots)$ with $a \in Cl^+$ determines a function $\tilde{f}$ on $G(\mathbb{A})$ by setting

$$\tilde{f}(\gamma g_j g) = (f_{|k} g_\infty)(i, \dots, i)$$

if we use the decomposition $G(\mathbb{A}) = \sqcup_{j=1}^m G(\mathbb{Q}) g_j G(\mathbb{R})^0 K_0$ and write the corresponding elements as $\gamma g_j g$ accordingly. Moreover, $f_{|k} g_\infty$ stands for the usual slash operator. This may motivate the adèlic definition of Hilbert modular forms.

Consider $(\mathbb{C} - \mathbb{R})^n \times G(\mathbb{A}_f)$. We have an action from the left of the group $\mathrm{GL}(2, F)$ on $(\mathbb{C} - \mathbb{R})^n \times G(\mathbb{A}_f)$ given by

$$\gamma\,(z, g) = (\gamma(z), \gamma g)$$

and an action on the right of $G(\mathbb{A}_f)$ given by

$$(z, g)\, g' \mapsto (z, gg').$$

We consider functions $f : (\mathbb{C} - \mathbb{R})^n \times G(\mathbb{A}_f) \to \mathbb{C}$ that are holomorphic in the first factor ($(\mathbb{C} - \mathbb{R})^n$) and are locally constant on $G(\mathbb{A}_f)$. (The last condition means essentially that such a function comes from a congruence subgroup.) We shall call such functions simply *holomorphic* for short. The two actions (of $\mathrm{GL}(2, F)$ and $G(\mathbb{A}_f)$) induce actions on such functions depending on a weight $k = (k_1, \dots, k_n, w)$ where we shall restrict us to the case $w \geq k_i \geq 2$ and $k_i \equiv w (\mathrm{mod}\, 2)$. The action of $G(\mathbb{Q})$, or $\mathrm{GL}(2, F)$ is given as:

$$(\gamma^{(k)})^* f = \gamma^* f : (z, g) \mapsto \frac{(\det \gamma)^{(w+k-2)/2}}{(cz + d)^k} f(\gamma z, \gamma g)$$

while the action of $g' \in G(\mathbb{A}_f)$ is given as

$$(g')^* f : (z, g) \mapsto f(z, gg').$$

We define the space of modular forms of weight k on K as

$$M^{(k),K} = \{f : f \text{holom. on } (\mathbb{C} - \mathbb{R})^n \times G(\mathbb{A}_f),\ (\gamma^{(k)})^* f = f,\ g^* f = f\}.$$

for all $\gamma \in G(\mathbb{Q})$ and all $g \in K$ and

$$M^{(k)} = \cup_K M^{(k),K}.$$

We also have spaces of cusp forms $S^{(k),K}$ and $S^{(k)}$. (The extra weight w is useful for dealing with (Tate) twists, but can be safely ignored here by putting $w = 2$.) (although $w = 2$ is not compatible with your conditions above ($w \geq k_i$)).

The level of a general modular form f could be defined to be the largest subgroup of $G(\mathbb{A}_f)$ that fixes it, i.e., to be the stabiliser of f for the action by $G(\mathbb{A}_f)$ on $M^{(k)}$; we note that the level is then an open subgroup. For K a compact open subgroup we could then call $M^{(k),K}$ the space of forms of level (dividing) K. But it is a fact that, up to the action of $G(\mathbb{A}_f)$, such a stabiliser contains a subgroup of the form $K_1(N)$, where N is a non-zero ideal of O_F, to be defined in a minute.

The subgroup $K_1(N)$, for N a non-zero ideal of O_F, is defined as follows: let $L = \hat{O_F} \oplus D^{-1}\hat{O_F} \subset \mathbb{A}_{F,f}^2$ be an adelic lattice, where $D^{-1} = O_F^\vee$ is the inverse different. Then we put

$$K_1(N) = \{g \in G(\mathbb{A}_f) : gL = L, g\begin{pmatrix}1\\0\end{pmatrix} \equiv \begin{pmatrix}1\\0\end{pmatrix} (\mathrm{mod} NL)\}.$$

With this notation, we call $M^{(k),K_1(N)}$ the space of modular forms of weight k and level N. See Edixhoven's talk on Hilbert modular forms and local Langlands for the notion of newform of level N.

A modular form f of level N has a Fourier development. Consider

$$F(z,b) = f(z, \begin{pmatrix}1 & b\\0 & 1\end{pmatrix} g).$$

It satisfies $F(z+\beta, b+\beta) = F(z,b)$, hence as a continuous function on the compact group $\mathbb{A}_F/F$ has a Fourier expansion in terms of characters of $\mathbb{A}_F/F$. Note that the dual group is F, and all characters are of the form $x \mapsto \psi(ax)$ ($a \in F$) for some fixed non-trivial additive character ψ. So we can write

$$f(z, \begin{pmatrix}1 & b\\0 & 1\end{pmatrix} g) = \sum_{a\in F} c_a(g,f)\psi(a,bz).$$

If $d \in \mathbb{A}_F^*$ then $c_1(\begin{pmatrix}(1 & 0\\0 & d\end{pmatrix}, f)$ depends only on the fractional ideal $m = (d^{-1}) = d^{-1}O_F \cap F$. For $m \subset O_F$ an integral ideal we put

$$c(m,f) = c_1\begin{pmatrix}(1 & 0\\0 & d\end{pmatrix}, f)$$

where $d \in \mathbb{A}_{F,f}^*$ is such that $(d^{-1}) = m$.

5. Hecke Operators

The right way of dealing with Hecke operators is to start with the *local* Hecke algebra, namely the $\mathbb{Q}$-algebra of $\mathbb{Q}$-valued locally constant functions on $G(\mathbb{Q}_p)$ which are invariant under the action of K on both left and right. But for lack of time we take here a more pedestrian point of view.

The group $G(\mathbb{A}_f)$ acts on the space $S^{(k)}$: if $gKg^{-1} \subseteq L$ then we have $g^* : S^{(k),gKg^{-1}} \to S^{(k),L}$. We also have a trace map: if $K \supseteq L$ then $\mathrm{Tr} : S^{(k),L} \to S^{(k),K}$ is given by $f \mapsto \sum_{g\in K/L} g^*f$.

For a subset $T \subset G(\mathbb{A}_f)$ stable under the compact open subgroup K acting both on the left and the right we have a map

$$T^* : S^{(k),K} \to S^{(k),K}, \quad f \mapsto \sum_{g\in T/K} g^* f.$$

If T has the special form $T = KgK$ then this map T^* can be identified with

$$S^{(k),K} \xrightarrow{g^*} S^{(k),K\cap gKg^{-1}} \xrightarrow{\mathrm{Tr}} S^{(k),K}.$$

If our compact open K is sufficiently neat, e.g. $K = K_pK^p$, where $K_p = \mathrm{GL}_{O_{F_p}}(L)$ for the O_{F_p}-lattice $L = O_{F_p}^2$, then the element $g = \begin{pmatrix}\pi & 0\\0 & 1\end{pmatrix}$ defines an operator $T(p)$. Similarly, the element $g = \begin{pmatrix}\pi & 0\\0 & \pi\end{pmatrix}$ gives rise to an operator $R(p)$. Here p is a prime of F and π is a uniformiser in O_{F_p}.

We thus get Hecke operators similarly to the case $n = 1$. We have a formal identity much as in the case $n = 1$

$$\sum T(m)N(m)^{-s} = \prod_p (1 - T(p)N(p)^{-s} + R(p)N(p)^{1-2s})^{-1}.$$

If f is a normalized eigen form (cusp form) then the eigenvalue under $T(m)$ is $c(m, f)N(m)$. We then have a formal L-series

$$L(f, s) = \prod_p (1 - c(p, f)N(p)^{-s} + \chi_f(p)N(p)^{1-2s})^{-1}.$$

6. Galois Representations

What do we want? If f is a normalized cusp form of level N, an eigenform of the Hecke algebra with field $L = L_f$ of eigenvalues and λ is a finite prime of L then we wish to have at our disposal a λ-adic Galois representation

$$\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}(2, L_\lambda)$$

such that ρ is unramified outside $N\,l$ with l the rational prime of $\mathbb{Q}$ lying under λ and such that

$$\det(1 - \rho(\mathrm{Frob}_p)T) = L_p(f, T) = 1 - c(p, f)N(p)T + \chi_f(p)N(p)^{w-1}T^2.$$

for p a finite prime of F not dividing l and p not dividing N, although we would like to get rid of this last restriction and allow p that divide N but not l. Here Frob_p is a geometric Frobenius at p.

Let us first look at the simple case $k = 2$, $n = 1$ and $f \in S_2(\Gamma_0(N))$. Then we know that we get the l-adic representation from the Tate module $T_l(J_0(N))$, where $J_0(N)$ is the Jacobian of the modular curve $X_0(N)$. However, as we saw above, if $n > 1$ then the first cohomology of a Hilbert modular variety vanishes, and we cannot use the first cohomology as in the case $n = 1$ to construct the l-adic representation. Actually, it is conjectured (by T. Oda, see [8, 7]) that given f there exists an abelian variety A defined over F such that

$$\text{the part of } H^n(X_K) \text{ determined by } f = H^1(A_1) \otimes \cdots \otimes H^1(A_n),$$

where A_i is the complex abelian variety obtained from the i-th embedding $F \to \mathbb{R}$. In generalization of Shimura-Tamiyama-Weil one conjectures that for an abelian variety A defined over F with real multiplication by a field of degree $(1/2)\dim(A)$ and conductor N there exists a Hilbert modular form f of weight 2 and level N such that the desired Galois representation of f is found on the Tate module of this abelian variety.

So in any case, conjecturally, even for weight 2 we do not get the Tate module and the λ-adic representation, but a tensor product that we cannot disentangle. So we cannot find the desired representation in the cohomology of a Hilbert modular variety. Instead people have used tricks to get around this difficulty. For example, one has used the theory of lifting modular forms and looked at Shimura curves associated to quaternion algebras split at one infinite place and ramified at all other $n-1$ places. This is what Carayol did in [2]. Blasius and Rogawski have looked at modular forms on $U(2, 1)$ and used liftings to Hilbert modular forms, [1]. Wiles used congruences between modular forms to construct the desired respresentations. But we do not yet have an elegant way of constructing these representations.

Here is the result. If f is an eigenform for the Hecke operators then we let $L = L_f$ be the number field obtained by adjoining the eigenvalues of the Hecke operators $T(p)$ and $R(p)$.

Theorem 6.1. *Let f be a normalized eigenform of level N and field of eigenvalues L. Moreover, let $\theta : T \to L$ the character on the Hecke algebra that associates to an operator its eigenvalue. Then there exists a continous representation*

$$\rho = \rho_{f,\lambda} : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}(2, L_\lambda)$$

unramified outside Nl and such that for a prime p of F not dividing Nl we have

$$\mathrm{tr}(\rho(\mathrm{Frob}_p)) = \theta(T(p))$$

and

$$\det(\rho(\mathrm{Frob}_p)) = \theta(R(p))N(p).$$

There is also the improvement saying that if $p|N$, $p \not| l$ and if $\sigma \in D_p$ (=decomposition group at p) lies above Frob_p then if $\theta(T(p)) \neq 0$ we have

$$\mathrm{tr}(\rho(\sigma) = \theta(T(p)) + \chi(\sigma)N(p)\theta(T(p))^{-1}$$

and

$$\det(\rho(\sigma) = \chi(\sigma)N(p),$$

with χ the continuous character extending the map $\mathrm{Frob}_p \mapsto \theta(R(p))$.

This last property can be better expressed by saying that the Frobenius semi-simplification of the Weil-Deligne group is prescribed by the Langlands correspondence.

7. Modular Interpretation

We give a sketch of the modular interpretation of these Hilbert modular varieties. We refer here to [7], and also to [5], especially the first sections of the appendix there.

Adèlically, it would be best to look at adèlic lattices. We restrict ourselves here to classical lattices over $\mathbb{C}$.

Let A be a complex torus of dimension n. We say that A has real multiplication by F if an embedding $F \to \mathrm{End}(A) \otimes \mathbb{Q}$ is given and $\dim(A) = [F : \mathbb{Q}]$. Suppose that we actually have an embedding of the ring of integers $O_F \to \mathrm{End}(A)$. Then our torus is an n-dimensional complex vector space divided by some lattice that is a projective O_F-module of rank 2, say $\Lambda = O_F \oplus a$ for some integral ideal a. An isomorphism $\Lambda \cong O_F \oplus a$ determines a $\mathbb{C}$-vector space isomorphism $V \cong \mathbb{C}^n$ such that a maps to $\mathbb{R}^n$ in the natural way. Then O_F is mapped to $\mathbb{C}^n$ as $(\alpha_1, \ldots, \alpha_n) \mapsto (\alpha_1 z_1, \ldots, \alpha_n z_n)$ for some $(z_1, \ldots, z_n) \in \mathcal{H}^n$. We write $A = \mathbb{C}^n/O_F z \oplus a$.

Then we can find a Riemann form on A, namely take a suitable multiple of

$$E(\zeta, \eta) = \sum_{i=1}^{n} \mathrm{Im}(z_i)^{-1}\mathrm{Im}(\zeta_i \overline{\eta}_i).$$

So A is an abelian variety by the criterion of Lefschetz. (We assumed that A has multiplication by O_F, but we can handle other orders as well, but for simplicity of exposition stick to this case.)

If $\mu : O_F \to \mathrm{End}(A)$ is given we say that A has real multiplication by O_F (assuming that $\dim(A) = [F : \mathbb{Q}]$. The dual abelian variety $A^t = \mathrm{Pic}^0(A)$ has then also real multiplication, say by $\mu^t : O_F \to \mathrm{End}(A^t)$.

The tangent space $T_{A,0}$ to A at the origin is an O_F-module and $T_{A,0}$ is free of rank 1 over $O_F \otimes \mathbb{C}$.

We shall say that a polarization $\lambda : A \to A^t = \mathrm{Pic}^0(A)$ is O_F-linear if

$$\mu^t \lambda = \lambda \mu.$$

Instead of fixing one specific polarization we look at a whole module of polarizations:

$$P(A, \mu) = \{\lambda : A \to A^t : \lambda \text{is } O_F\text{-linear and symmetric}\}$$

and

$$P^+(A, \mu) = \{\lambda \in P(A, \mu) : \lambda \text{ is a polarization}\}.$$

We view $P^+(A, \mu)$ as the "positive cone" in $P(A, \mu)$. The lattice $H_1(A, \mathbb{Z})$, a projective O_F-module of rank 2, is isomorphic to $O_F \oplus a$ for some ideal a. Then the polarization module $P(A, \mu)$ can be identified with $\wedge^2_{O_F}(H_1(A, \mathbb{Z})^\vee)$, where the $^\vee$ denotes taking the $\mathbb{Z}$-dual of the O_F-module. We thus see that (P, P^+) can be viewed as an element of Cl^+.

We now look at all abelian varieties A of dimension n with real multiplication by O_F, with lattice $H_1(A, \mathbb{Z}) \cong O_F \oplus a$ and with polarization module (P, P^+). It is described by a quotient

$$\mathrm{GL}^+_{O_F}(O_F \oplus a) \backslash \mathrm{GL}^+_{O_F \otimes \mathbb{R}}(\Lambda \otimes \mathbb{R})/k,$$

where k is the centralizer of a fixed complex structure on $\Lambda \otimes \mathbb{R}/\Lambda$. The plusses come from the fact that we need to preserve the cone of polarizations P^+ inside P. So these spaces are

$$GL^+(O_F \oplus a) \backslash \mathcal{H}^n,$$

and we get back the h^+ components of X_{K_0}.

If one wants to construct the X_K as schemes (or better stacks) then one needs this modular interpretation. The various modular interpretations are compared in [13].

Some people (I think Rapoport in 1977 started this) use the name Hilbert-Blumenthal instead of the name Hilbert which was used for over fifty years. As I indicate in the introduction of my book I do not see any compelling reason to change the name Hilbert modular.

References

[1] D. Blasius, J. Rogawski: Galois representations for Hilbert modular forms. *Bull. Amer. Math. Soc.* **21** (1989), 65–69.

[2] H. Carayol: Sur les représentations l-adiques associées aux formes modulaires de Hilbert. *Ann. Sci. École Norm. Sup. (4)* **19** (1986), no. 3, 409–468.

[3] P. Deligne: Travaux de Shimura. Sém. Bourbaki 389. SLNM 244.

[4] P. Deligne, G. Pappas: Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant. *Comp. Math.* **90** (1994), 59–79.

[5] B. Edixhoven: On the Andre-Oort conjecture for Hilbert modular surfaces. math.NT/9911272.

[6] E. Freitag: Hilbert Modular Forms. Springer-Verlag, Berlin

[7] G. van der Geer: Hilbert Modular Surfaces. Springer Verlag 1987.

[8] T. Oda: Periods of Hilbert modular surfaces. Progress in Mathematics, 19. Birkhuser, Boston, Mass., 1982.

[9] R. Taylor: On Galois representations associated to Hilbert modular forms. *Invent. Math.* **98**, 265–280 (1989).

[10] R. Taylor: On Galois representations associated to Hilbert modular forms, II. Conference on Elliptic Curves and Modular Forms. Eds J. Coates, S.T. Yau (1995), 265–280.

[11] T. Saito: Modular forms and p-adic Hodge theory. *Invent. Math* **129** (1997), 607–620.

[12] T. Saito: Hilbert modular forms and p-adic Hodge theory. See Saito's webpage at the University of Tokyo.

[13] I. Vollaard: On the Hilbert-Blumenthal moduli problem. *Journal of the Inst. of Math. Jussieu* **4** (2005), 653–683.

[14] A. Wiles: On ordinary λ-adic representations asociated to modular forms. *Invent. Math.* **94** (1988), 529–573.

Faculteit Wiskunde en Informatica, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands

E-mail address: geer@science.uva.nl