Automorphic theory for GL(2)

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0.1 Introduction

The aim of this lecture is to describe in a rather informal way two results in the automorphic theory for $G = GL(2)$, related to the Langland's functoriality principle: base change for solvable extensions and Jacquet-Langlands correspondence. No proofs will be given, but we will discuss some examples. Everything is contained in the books [2] and [3].

Let F be a number field.

Let $L^2(G_F \backslash G_{\mathbf{A}}, \psi)$ be the Hilbert space of measurable functions such that $f(zg) = \psi(z)f(g), z \in Z_{\mathbf{A}}, g \in G_{\mathbf{A}}$ for a fixed Hecke character ψ of $F^* \backslash F^*_{\mathbf{A}}$, and

$$\int_{Z_{\mathbf{A}} G_F \backslash G_{\mathbf{A}}} |f(g)|^2 |\psi(det(g))|^{-1} dg < \infty$$

where Z is the center of G. Let R_ψ be the representation of $G_{\mathbf{A}}$ on $L^2(G_F \backslash G_{\mathbf{A}}, \psi)$ given by right translation. Then an automorphic form on G is an irreducible unitary representation of $G_{\mathbf{A}}$ occuring in some R_ψ.

Let π be an automorphic representation of G. One can write the (partial) L-function of π as

$$L(s, \pi) = \prod_{v \notin S} L(s, \pi_v) \tag{0.1.0.1}$$

where $L(s, \pi_v) = det(1 - t_{\pi,v} Nv^{-s})^{-1}$ if v is a finite place and S is a finite set of places (where π_v is ramified). The $t_{\pi,v}$ are 2-dimensional complex matrices.

At the infinite places not in S, the local factors $L(s, \pi_v)$ are products of Gamma functions times powers of π.

If π is the automorphic representation corresponding to a classical Hecke newform of weight k and level N, with Fourier coefficients a_n, then the conjugacy class of the matrix $t_{\pi,p}$ is determined by:

$$a_p = p^{(k-1)/2} tr(t_{\pi,p}) \tag{0.1.0.2}$$

and $det(t_{\pi,p}) = 1$, thereby we have:

$$L(s, \pi_p) = (1 - p^{-s-(k-1)/2} a_p + p^{-2s})^{-1} \tag{0.1.0.3}$$

if $p \nmid N$.

At the infinite place, π_∞ is the discrete series representation $\sigma(k-1, 0)$ and the local factor is:

$$L(s, \pi_\infty) = (2\pi)^{-s-(k-1)/2} \Gamma(s + (k-1)/2) \tag{0.1.0.4}$$

In general, the matrices $t_{\pi,v}$ (at finite places) are defined by means of the *Satake isomorphism* (cf.[1]). One can also consider the complete L-function:

$$L(s, \pi) = \prod_v L(s, \pi_v) \tag{0.1.0.5}$$

For the definition of the local factors at the "bad" primes (primes in S), we refer to [4], §1.6.

Now let E be a finite cyclic extension of F and $G' = R_{E/F} GL(2, E)$.

The base-change lift of π to G' is an automorphic representation π' on G' with corresponding matrices $t_{\pi',v}$ given by:

$$t_{\pi',w} = t_{\pi,v} \tag{0.1.0.6}$$

if $v = \prod_{\gamma \in \Gamma} w^\gamma$ splits in E, and

$$t_{\pi',v} = t^{\ell}_{\pi,v} \tag{0.1.0.7}$$

if v is inert.

A precise definition will be given later.

The set $\{\pi_v, v \ unramified\}$ suffices to determine π, because of the following theorem (multiplicity one theorem):

Theorem 0.1.0.1. *([2]) If π_1, π_2 are two irreducible automorphic representations which are isomorphic at all but finitely many places, then $\pi_1 \simeq \pi_2$.*

The equations 0.1.0.6, 0.1.0.7 imply that $L(s, \pi') = \prod_{\xi \in \hat{\Gamma}} L(s, \pi \otimes \xi)$. Nevertheless, it would be desirable to have some information even at the ramified places.

The second matter of this talk, the Jacquet-Langlands correspondence, is how automorphic representations change when we modify the group G at a finite number of places, namely when we take as G' the multiplicative group of a quaternion algebra over F. More details on this in section 4.

First, we look at the easiest example of base change.

0.2 Base change for GL(1)

Let $\Pi(F)$ be the set of automorphic representations of the idele classgroup of a global field F (that is, the Hecke characters). Let E be a finite cyclic extension of F. The group $\Gamma = Gal(E/F)$ acts on $\Pi(E)$ by:

$$\chi^{\gamma}(x) = \chi(\gamma x) \tag{0.2.0.8}$$

To the extension E/F there are associated $[E : F]$ Hecke characters of finite order, by class-field theory. Let $\hat{\Gamma}$ denote the set of such Hecke characters.

Let us denote by $N_{E/F}$ the adelic norm.

Theorem 0.2.0.2. *The map $\chi \to \chi \circ N_{E/F}$ defines a bijection between $\Pi(F)$ and $\Pi(E)^{\Gamma}$*

$$L_E(s, \chi \circ N_{E/F}) = \prod_{\xi \in \hat{\Gamma}} L_F(s, \chi\xi) \tag{0.2.0.9}$$

$$\epsilon(\chi \circ N_{E/F}) = \prod_{\xi \in \hat{\Gamma}} \epsilon(\chi\xi) \tag{0.2.0.10}$$

This follows easily from the exact sequence

$$1 \to C_E^{1-\sigma} \to C_E \to C_F \to \Gamma \to 1$$

where C denotes the adele group or the multiplicative group. The equations 0.1.0.6, 0.1.0.7 are obviously verified. It follows from the definition that the base-change map is transitive (with respect to extensions of fields).

0.3 Solvable base change for GL(2)

Let E/F be a finite cyclic extension of number fields, of prime degree ℓ, with Galois group generated by σ. Let $G_F = GL(2, F)$, $G_E = GL(2, E)$. Let π be an irreducible automorphic representation of G, then we can decompose it as: $\pi = \otimes_v \pi_v$, the product being over all the places of F. If a place v of F splits in E, then we define the base-change of π_v as $BC(\pi_v) = \pi_v^{\otimes \ell}$. If v doesn't split, we make the following definition:

Definition 0.3.0.3. If π_v is the principal series representation $\pi(\mu, \nu)$, then $BC(\pi_v) = \pi(\mu_{E/F}, \nu_{E/F})$ (the principal series defined by the base-change of the characters μ, ν).

If not, define $BC(\pi)$ as any irreducible admissible representation π' of G_E, invariant for σ and such that

$$\chi_{\pi'}(g) = \chi_{\pi}(x) \tag{0.3.0.11}$$

whenever $Ng := g^{\sigma^{\ell-1}} ... g^{\sigma} g$ is conjugate in G_E to a regular semisimple element $x \in G_F$.

We explain the second part of this definition in the Appendix 3.

We have defined the base-change for local representations, we extend it to a global definition, by the decomposition $\pi = \otimes_v \pi_v$.

Theorem 0.3.0.4. *([3]) There is a map $\pi \mapsto \pi_{E/F}$ from the irreducible automorphic representations of $GL(\mathbf{A}_F)$ to the irreducible automorphic representations of $G_{E/F}(\mathbf{A}_F)$, such that:*

- *$\pi_{E/F}$ is the base-change of π;*
- *a cuspidal π_0 is in the image of the map if and only if $\pi_0^\sigma \simeq \pi_0$.If so, it comes from a cuspidal representation;*
- *if $E \supset F \supset K$, then $(\pi_{E/F})^{\gamma'} = (\pi^\gamma)_{E/F}$ for any $\gamma \in Gal(F/K)$ image of $\gamma' \in Gal(E/K)$.*
- *if π_1 and π_2 have the same base-change, then $\pi_1 \simeq \pi_2 \otimes \xi$ for some Hecke character ξ of F.*

Note that: $L(\pi_{E/F}) = \prod_{\xi\in\hat{\Gamma}} L(\pi \otimes \xi)$ and $\epsilon(\pi_{E/F}) = \prod_{\xi\in\hat{\Gamma}} \epsilon(\pi \otimes \xi)$

Remarks (1) The base-change of a cuspidal representation is not necessarily cuspidal: if $\ell = 2$, some supercuspidals can lift to principal series.

(2) There are examples of non-cuspidal representations of G_E, fixed by Γ, which are not lifts.

(3) The theorem was proved by H.Saito in the holomorphic Hilbert modular case, for trivial conductor and cyclic tamely ramified totally real fields.

The construction of an *admissible* representation filling the desired properties is relatively easy; in fact:

1. At a finite place v, such that $E_v = E \otimes F_v$ is still cyclic, if $\pi_v = \pi(\mu, \nu)$ is the principal series representation associated to two characters μ, ν of F_v^*, then one defines $\pi_{E_v/F_v} = \pi(\mu_{E_v/F_v}, \nu_{E_v/F_v})$.
2. If $E_v = F_v^\ell$, then $\pi_{E_v/F_v} = \pi_v^{\otimes \ell}$.
3. If $\ell = 2$ and π_v is the Weil representation (cf.§4) associated to a character ω of E^*, then π_{E_v/F_v} is the principal series representation associated to ω and $\bar{\omega}$.
4. if π_v is the special representation associated to ν, μ, then π_{E_v/F_v} is the special representation associated to $\mu_{E_v/F_v}, \nu_{E_v/F_v}$

Therefore, the definition of the map is, in most cases, simple and natural. The most difficult point is to prove the *automorphy* of the representation. An interesting result in this direction is the so-called *converse theorem*:

Theorem 0.3.0.5. *([2]) Let π be an irreducible representation of $GL(2, \mathbf{A}_F)$, such that:*

- *π_v is infinite-dimensional for all v.*
- *$L(s, \pi \otimes \omega), L(s, \tilde{\pi} \otimes \omega^{-1})$ are entire and bounded in vertical strips, for every Hecke character ω of F ($\hat{\pi}$ is the contragredient).*
- *they satisfy the functional equation $L(s, \pi \otimes \omega) = \epsilon(s, \pi \otimes \omega) L(1 - s, \hat{\pi} \otimes \omega^{-1})$.*

Then π is automorphic and cuspidal.

We don't define the ϵ factor in the functional equation above, but it can be found in [2], especially theorems 14.2, 2.18 and page 97.

Clearly, to prove a functional equation is often a difficult task, nevertheless in some cases it can be done. Indeed, Jacquet used the converse theorem to prove the theorem 0.3.0.4 for quadratic extensions. Its strategy was simply to translate in the automorphic setting of [2] the original proof by Doi and Naganuma (see example below).

In general, one reduces the proof to an equality between the "characters" (which are actually distributions) of the representations, and then uses Arthur-Selberg's trace formula. See [6] for an overview of the proof.

Example. One of the first examples of base change lift was given by Doi and Naganuma (after an idea of Shimura): let $F = \mathbf{Q}(\sqrt{d})$ be a real quadratic field with associated character χ. Take a cusp form $f(z) = \sum a_n q^n$ of weight k and define a family of algebraic numbers $(\mathfrak{c}(\mathfrak{a}))$ indexed by the integral ideals of F, as follows:

$$\mathfrak{c}(\prod \mathfrak{p}^{e_i}) = \prod \mathfrak{c}(\mathfrak{p}^{e_i}) \tag{0.3.0.12}$$

$$\mathfrak{c}(\mathfrak{p}^e) = \mathfrak{c}(\mathfrak{p})\mathfrak{c}(\mathfrak{p}^{e-1}) - N\mathfrak{p}^{k-1}\mathfrak{c}(\mathfrak{p}^{e-2}) \tag{0.3.0.13}$$

$$\mathfrak{c}(\mathfrak{p}) = 2a_{p^2} - a_p^2 \tag{0.3.0.14}$$

when $\mathfrak{p} = (p)$ and

$$\mathfrak{c}(\mathfrak{p}) = \mathfrak{c}(\bar{\mathfrak{p}}) = a_p \tag{0.3.0.15}$$

if p splits or ramifies. Then there is a Hilbert modular form g on F with associated Dirichlet series $\sum \mathfrak{c}(\mathfrak{a}) N(\mathfrak{a})^{-s} = L(f, s) L(f \otimes \chi, s)$. This equality follows from the equations 3.0.8-3.0.11. To prove the automorphy and cuspidality of g, Doi and Naganuma prove the conditions of theorem 3.0.4, by the standard Rankin-Selberg method (cf.[.

Remark: many ideas of the proof of theorem 3.0.3 are already contained in the beautiful and self-contained paper by Shintani [5], where he proves a similar result for irreducible representations of general linear groups over finite fields.

Sketch of the proof for holomorphic Hilbert modular forms.

Let F be a totally real field, of prime degree ℓ and conductor a prime number q. Let $S_k^F(\Gamma)$ be the space of Hilbert cusp forms of scalar weight k, with respect to a congruence group Γ. We will assume that Γ is the full Hilbert modular goup, that is, our forms have level 1. Let τ be the operator on $S_k^F(\Gamma)$ such that $\tau f(z_1, ..., z_l) = f(z_2, ..., z_l, z_1)$ and define

$$\mathcal{S}_k = \{f \in S_k^F(\Gamma); T\tau f = \tau T f \; \forall T \in R_F\} \tag{0.3.0.16}$$

where R_F is the Hecke algebra of level prime to q, acting on $S_k^F(\Gamma)$ (call $\mathcal{T}$ this representation). Then it is not hard to see that R_F acts on $\mathcal{S}_k$ and we call $\mathcal{T}_{\mathcal{S}}$ this representation. An easy but important remark is that

$$Tr\mathcal{T}_{\mathcal{S}} = Tr(\tau\mathcal{T}) \tag{0.3.0.17}$$

Notice that $\tau^\ell = 1$ and that its action on q-expansions is simply $\mathfrak{c}(\mathfrak{a}) \mapsto \mathfrak{c}(\mathfrak{a}^\sigma)$. The equations 0.3.0.12, 0.3.0.13, 0.3.0.14 define by duality a homomorphism of Hecke algebras

$$\lambda : R_F \longrightarrow R_{\mathbf{Q}} \tag{0.3.0.18}$$

where $R_{\mathbf{Q}}$ is the Hecke algebra on $\mathbf{Q}$, of prime-to-q level. Explicitly, to a Hecke operator at $\mathfrak{p}$ we associate the Hecke operator at $N\mathfrak{p}$ and then we extend this rule to a homomorphism by an equation similar to 0.3.0.13. Through composition with λ, we have a representation $\mathcal{T}_1$ of R_F on the space of cuspforms of level 1 and weight k. For every Dirichlet character modulo q, we get also a representation $\mathcal{T}_\chi$ of R_F on the space of cuspforms of weight k, character χ and level q. Then the main result of Saito is:

$$Tr(\mathcal{T}_{\mathcal{S}}) = Tr(\mathcal{T}_1) + \frac{1}{2}\sum_{\chi} Tr(\mathcal{T}_\chi) \tag{0.3.0.19}$$

This is in fact a variant of Selberg's trace formula, and it implies a direct sum decomposition for the corresponding spaces of modular forms. In particular, this proves the existence of the base-change lift. Moreover, it says that if two cusp forms (on $\mathbf{Q}$) of weight k and level 1 have the same base-change, then they are equal up to a multiplicative factor. But by 0.3.0.19, cusp forms of level q and character χ (of conductor q) can also be lifted to elements of $\mathcal{S}$: their lifts are different from the lifts from the cusp forms of level 1.

Remarks. (1) The proof in the general case is essentially the same, but one has to use the trace formula in a more abstract formulation.

(2) The hypothesis of solvability of the extension E/F makes simpler the comparison between the conjugacy classes in G_E and in G_F. It seems hopeless to have a uniform treatement for any Galois extension E/F.

0.4 The Jacquet-Langlands correspondence

In this case we take a quaternion algebra D over F and consider the reductive group $G' = D^*$. Recall that a place v of F is said *unramified* in D if $D_v := D \otimes_F F_v$ is isomorphic to $M(2, F_v)$ and *ramified* if D_v is still a quaternion algebra. Moreover, the number of ramified places is finite. One can define automorphic forms on $G'(\mathbf{A}_F)$, exactly as for $GL(2)$. If F is totally real and D is unramified at some places at infinity, then such a form gives a function f on a product $\mathcal{H}^r$ of upper half planes, such that $f(\gamma z) = j_\gamma(z)^k f(z)$ for any γ in a discrete subgroup Γ of $G'_{\mathbf{R}}$. But a major difference with the $GL(2)$-case is that the quotient $\Gamma \backslash \mathcal{H}^r$ is compact. In particular, such automorphic forms have no Fourier expansions (the groups Γ do not contain any unipotent element).

Since $G_v \simeq G'_v$ for almost all v, it is natural to expect some strong relation between the automorphic forms on the two groups.

Theorem 0.4.0.6. *There is a map JL from the cuspidal automorphic representations of G to the cuspidal automorphic representations of G' inducing a bijection between cusp forms on G, which are special or supercuspidal for each ramified v, and automorphic forms on G' of dimension > 1.*

We recall how L-functions are attached to automorphic forms on quaternion algebras:

Theorem 0.4.0.7. *Let $\pi' = \otimes_v \pi'_v$ an automorphic form on $G'_{\mathbf{A}}$. Choose a Schwarz-Bruhat function f_v on D_v for every v. Let*

$$\zeta(f, \pi'_v, s) = \int_{G'_v} f(x) \pi'_v(x) |Nx|_v^{s+1/2} dx$$

Then there exist Euler factors $L(s, \pi'_v)$ and constants $\epsilon(s, \pi'_v)$ such that:

$$\frac{\zeta(s, f_v, \pi'_v)\epsilon(s, \pi'_v)}{L(s, \pi'_v)} = \frac{\zeta(1-s, \hat{f}_v, \tilde{\pi}'_v)}{L(1-s, \tilde{\pi}'_v)}$$

where $\hat{f}_v$ denotes a Fourier transform (see below).

Theorem 0.4.0.8. *(Jacquet,Langlands)*
One has the equalities: $L(s, \pi) = L(s, JL(\pi))$ and $\epsilon(s, \pi) = \epsilon(s, JL(\pi))$.

To be precise, the map JL depends on the choices of the isomorphisms $G_v \simeq G'_v$ at the unramified places. One can fix these isomorphisms in the following way. Take a maximal order $\mathfrak{O}$ in D and suppose v unramified and non-archimedean. Let $\mathfrak{O}_v$ be the O_v-module in D_v generated by $\mathfrak{O}$. Then the choice of an isomorphism $\mathfrak{O}_v \simeq M_2(O_v)$ gives an isomorphism $\theta_v : D_v \simeq M_2(F_v)$. If $\mathfrak{O}'$ is a different maximal order, then $\mathfrak{O}_v \simeq \mathfrak{O}'_v$ for almost all v, so that the family of isomorphisms $G_v \simeq G'_v$ is canonical so far as almost all v are concerned.

Although the proof of theorem 0.4.0.6 is quite involved and requires Arthur's trace formula, the map can be described easily, by means of the Weil representation. We should mention a different proof, by Shimizu, which doesn't use the trace formula.

To recall what the Weil representation is, we change the notation and let F be a local field, D a quaternion algebra over F. Let ω be an irreducible representation of D^* on a space U and $\mathcal{S}_U = \mathcal{S} \otimes U$, where $\mathcal{S}$ is the space of Bruhat-Schwartz functions on D. Elements of $\mathcal{S}_U$ are regarded as functions on D with values in U, whose coordinates are Schwartz functions on D.

Theorem 0.4.0.9. *(Weil, cf.[2],chapter 1)*

There exists a unique admissible representation r of $GL(2,F)$ on $\mathcal{S}_U$ such that:

1. $r(\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix})\phi(x) = |t|_F^{1/2}\phi(tx)$
2. $r(\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix})\phi(x) = \psi_F(t\nu(x))\phi(x)$
3. $r(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix})\phi(x) = -\hat{\phi}(x')$
4. $r(\begin{pmatrix} a & \\ & 1 \end{pmatrix})\phi(x) = |a|_F^{1/2}\omega(h)\phi(xh)$, *if* $a = \nu(h)$
5. $r = \pi(\omega)^{deg(\omega)}$, *with* $\pi(\omega)$ *irreducible.*
6. $L(s,\omega) = L(s,\pi(\omega)), L(s,\tilde{\omega}) = L(s,\pi(\tilde{\omega})), \epsilon(s,\omega) = \epsilon(s,\pi(\omega))$.
7. *all the supercuspidal and the special representations of G are obtained in this way.*

Notation: $\hat{\phi}$ is the Fourier transform (with respect to a fixed additive character and the self-dual Haar measure of F), ι is the canonical involution of D, ν is the reduced norm.

Properties 1-4 describe completely r, in particular we see that it depends on the choice of ω and an additive character ψ_F of F. Then one defines JL locally at every place: if v is ramified one takes the Weil representation, otherwise one uses the fixed isomorphism $G_v \simeq G'_v$. Therefore we know how to associate, at each place, an admissible representation of G_v to an admissible representation of G'_v. The problem, as usual, is to prove the automorphy. A key point is the following lemma:

Lemma 0.4.0.10. *If ω is as before, then*

$$\chi_\omega(b) = -\chi_{\pi(\omega)}(b) \qquad (0.4.0.20)$$

for any b with distinct eigenvalues in a quadratic extension of F.

This statement makes sense, because the multiplicative group of any separable quadratic extension of F embeds as a Cartan subgroup in G and in G'. These embeddings are unique up to inner automorphism, by the Skolem-Noether theorem.

Remark The theory of Weil representations works more generally when D is one of the following: a) $F \oplus F$, b) a quaternion algebra, c) a separable quadratic extension, d) $M(2, F)$.

Example . Let $F = \mathbf{Q}$ and f a cusp newform for $GL(2)$ of level $\Gamma_0(p)$, of weight k and which is a Hecke eigenform. Suppose it has trivial character. One can show that it (its automorphic representation) comes from an automorphic representation of the unique (up to isomorphism) quaternion algebra D over $\mathbf{Q}$ which is ramified at p and ∞. We want to describe explicitly the correspondence at the infinite place. There is an isomorphism of $D \otimes \mathbf{R}$ with the algebra of complex matrices $\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}\}$. Fixing such an isomorphism we get a representation ρ_1 of D^* into $GL_2(\mathbf{C})$. Let $\rho_n = \rho_1^{\otimes n}$, the n-th symmetric tensor product. Then every finite-dimensional representation of D^* is equivalent to $|det|^r \otimes \rho_n$ for some r, n. Then to such a representation corresponds the discrete series $\sigma(\mu_1, \mu_2)$ of $GL(2, \mathbf{R})$, with:

$$\mu_1(x) = |x|^{r+n+1/2} \tag{0.4.0.21}$$

$$\mu_2(x) = |x|^{r-1/2} sgn(x)^n \tag{0.4.0.22}$$

and such a representation is associated to our cuspform f if and only if $n = k - 2$ and $r = 1 - k/2$.

Moreover, this is also the Weil representation associated to the character $\omega(z) = (z\bar{z})^{r-1/2} z^{n+1}$ of $\mathbf{C}^*$.

Remarks . (1) Analogous theorems were proved by Rogawski for the unitary groups. His results were then used by himself and Blasius, to give a motivic construction of the Galois representations associated to Hilbert modular forms.

(2) Let $K = \mathbf{Q}(\sqrt{-d})$ be imaginary quadratic with character χ and ψ a Dirichlet character of $\mathbb{Q}$ and conductor N. Put $\xi = \psi \circ N_K$. Then the series

$$f(z) = \sum \xi(\mathfrak{a}) exp(2\pi i N_K(\mathfrak{a}) z) \tag{0.4.0.23}$$

is a modular form of weight 1 on K and its L-series is $L(s, \psi)L(s, \psi\chi)$. It is not cuspidal. This is a very classical example of modular form constructed by the Weil representation; in this case, the quadratic extension K takes the place of a quaternion algebra (i.e. we are constructing theta series using a binary quadratic form instead of a quadratic one).

(3) On the other hand, one can use the theory of Weil representations also to construct some automorphic forms on quaternion algebras. Indeed, if K is a quadratic subfield of D and χ a Hecke character of K such that χ_v does not factor through the norm for any ramified v, then one can define an automotphic

form $\pi(\chi)$ on G, as follows: if v is a place of F split in K, say $v = ww'$, then $\pi_v(\chi)$ is the principal series representation $\pi(\chi_w, \chi_{w'})$; otherwise $\pi_v(\chi)$ is the Weil representation associated to χ_w and it is supercuspidal. Then, $\pi = \otimes_v \pi_v$ is an automorphic form in the image of the map JL. Therefore, one can obtain automorphic forms on D from any suitable Hecke character on any quadratic subfield of D.

An application of the Jacquet-Langlands correspondence

Theorem 0.4.0.11. *(Zhang) Let F be totally real, f a Hilbert newform of weight 2 and level N, with trivial character. Then there exists a Shimura curve X_N and an abelian subvariety A_f of $Jac(X_N)$, such that:*

- *A_f is defined over F;*
- *$dim(A_f) = [O_f : \mathbf{Z}]$;*
- *$L(s, A_f) = \prod_{\sigma : F \hookrightarrow \mathbf{C}} L(s, f^\sigma)$*

0.5 Appendix 1: the classification of admissible representations of GL(2)

Recall that an admissible representation of a p-adic group is defined by: (1) the stabilizer of any point is an open compact subgroup; (2) the subspace fixed by any open compact subgroup is finite-dimensional.

We give the list of all the irreducible admissible representations of $GL(2, F)$, F local non-archimedean.

- Principal series

$$\rho(\mu, \nu) = \{f \text{smooth} : G \to \mathbf{C}; f(ang) = \mu(u)\nu(v)|u/v|^{1/2} f(g)\} \tag{0.5.0.24}$$

  where $a = \begin{pmatrix} u & \\ & v \end{pmatrix}$. It is irreducible unless $\mu\nu^{-1} = |\ |^{\pm 1}$, otherwise there is an infinite-dimensional, irreducible subquotient $\sigma(\mu, \nu)$, called *special.*

- Weil representations, defined in the text.
- Exceptional representations, which occur only when the residual characteristic is 2.

The last two items are all supercuspidal. Recall that π is supercuspidal if for any vector v in the space of π there is an ideal I of F such that:

$$\int_I \pi(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}) v dx = 0$$

There is a similar classification when F is archimedean, but in that case there are no supercuspidal representations.

0.6 Appendix 2: functoriality principle

Let G be any reductive group. Recall that automorphic cuspidal representations π of G correspond to continuous irreducible representations

$$\sigma(\pi) : W_F \to G(\mathbf{C}) \tag{0.6.0.25}$$

of the Weil group W_F, with the property that

$$trace(\sigma(\pi)(Frob_v)) = trace(t(\pi_v)) \tag{0.6.0.26}$$

for almost all primes v of F. Here $t(\pi_v)$ is the Langlands class of π_v.

Langlands' functoriality principle asserts that whenever there is a reductive F-group G' with a homomorphism

$$^LG \longrightarrow^L G' \tag{0.6.0.27}$$

there exists a corresponding mappping of automorphic representations from G to G' compatible with Langlands classes for almost all v. We have examined two examples where this principle applies: solvable base change and Jacquet-Langlands correspondence.

Definition 0.6.0.12. Let π (resp. π') be an automorphic representation of $G(\mathbf{A}_F)$ (resp.$G(\mathbf{A}_E)$). Then π' is a base change lift of π if for each place v of F and each $w|v$ of E, the Langlands parameter of π'_w equals the restriction to W_{E_v} of the Langlands parameter of π_v.

The L-group of $GL(1, F)$ is $^LG = GL(1, \mathbf{C}) \times \Gamma_F$ and the L-group of the Weil restriction $R_{E/F}G$ is $\prod_{\tau \in \Gamma_E \backslash \Gamma_E} \tau(^LG)$. There is then a natural map

$$^LG \to^L R_{E/F}G \tag{0.6.0.28}$$

and Langlands functoriality principle predicts the existence of a map $\Pi(F) \to \Pi(E)$ well-behaved with respect to L-functions. Such a map was given in theorem 2.0.2.

In the case of $GL(2)$, we have a natural map:

$$^LG = GL(2, \mathbf{C}) \times \Gamma_F \longrightarrow^L G_{E/F} = GL(2, \mathbf{C})^\Gamma \times \Gamma_F \tag{0.6.0.29}$$

which corresponds to the base-change lift given in §3.

0.7 Appendix 3: characters

Let H_v be one of the groups G_v, G'_v of §3. It is known that for an admissible representation $\tau : H_v \to GL(V)$, and an element f in the Hecke algebra (i.e. a locally constant function with compact support) the operator

$$v \mapsto \tau(f)v := \int_{Hv} f(g)\tau(g)vdg$$

is of trace-class (it is of finite rank). The trace determines the representation: if τ and τ' are inequivalent representations, then their traces are linearly independent ([2],lemma 7.1). Moreover, one can prove ([2], §7) that there is a locally integrable function χ on G_{F_v} such that:

$$Tr\tau(f) = \int_{H_v} f(g)\chi(g)dg \tag{0.7.0.30}$$

for any f. We call χ the character of τ. For $H_v = G_v$, we have the following description: if $\tau = \pi(\mu_1, \mu_2)$ and g is semisimple with distinct eigenvalues α, β in F, then:

$$\chi_\tau(g) = (\mu_1(\alpha)\mu_2(\beta) + \mu_2(\beta)\mu_1(\alpha))|\alpha\beta/(\alpha-\beta)^2|^{1/2}$$

and $\chi_\tau(g) = 0$ if $\alpha \notin F$ or $\beta \notin F$. If $\alpha = \beta$, it is undefined. We can express the trace more familiarly in the form:

$$Tr\pi(f) = \int_{GL(2,O_F)} K(k,k)dk$$

where $K(k_1,k_2) = \int_{A_F \times U_F} f(k_1^{-1}diag(\alpha,\beta)uk_2)\mu_1(\alpha)\mu_2(\beta)|\alpha\beta^{-1}|^{1/2}dadu$.
If τ is supercuspidal, then:

$$\int_{Z_v \backslash H_v} (\tau(x^{-1}\gamma x)u, u)dx = d(\tau)^{-1}\chi_\tau(\gamma^{-1}) \tag{0.7.0.31}$$

if u is any unit vector in the space of τ and γ lies in a (separable) quadratic extension E of F_v, where E is a field. If E is not a field, then the integral is zero. The formal degree $d(\tau)$ is defined by the Schur orthogonality relations:

$$\int_{Z_v \backslash H_v} (\tau(g)u_1, \tilde{u_1})\overline{(\tau(g)u_2, \tilde{u_2})}dg = \frac{1}{d(\tau)}(u_1, u_2)(\tilde{u}_2, \tilde{u}_1) \tag{0.7.0.32}$$

At the archimedean primes, the existence of χ_π is a theorem of Harish-Chandra.

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