College Meetkunde/Geometry, 3de en 4de jaar, najaar/Fall 2002, 12 weken/weeks.

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The writing of this syllabus is in progress, as one can see from the title above. In the 'studiegids' it is advertised as follows.

> In this course we will study various kinds of geometrical objects, such as differentiable varieties, analytic varieties and algebraic varieties. These objects occur in many areas of mathematics, ranging from physics (general relativity, conformal field theory, mechanics) and analysis (Lie groups, differential equations) to algebra and number theory (arithmetic algebraic geometry). The course starts with definitions and examples of such varieties and the right kind of maps between them. Then we study tangent bundles, vector bundles, tensor constructions, differential forms, integration. Two goals will be to establish the basic properties of de Rham cohomology, and to give the description of what a space-time is in general relativity. The aim of the course is to provide the students with the fundamentals of geometry, and to prepare for more advanced algebraic geometry. Each week exercises will be handed out, and one hour of the course will be used to discuss them. As the teacher (Edixhoven) is new in Leiden, he does not know too much what kind of audience to expect. So the course will be adapted to its public, meaning that more details will always be given if needed, and that the goals can be adjusted if necessary.

The syllabus is based on one that was used in Rennes in 1996, and which can be downloaded in various formats (ps, ps.gz, pdf) at:

`http://www.maths.univ-rennes1.fr/~edix/cours/dea9697.html`

together with a syllabus by Looijenga that is used in Utrecht:

`http://www.math.ruu.nl/people/siersma/difvar.html`

Regularly, new sections will be handed out, including exercises. These sections will also be put on the webpages for this course. It is probably a good idea to print out Looijenga's syllabus, but not the old Rennes syllabus by Edixhoven (unless one does not care about wasting paper).

Reserved for table of contents.

1 Differentiable varieties

We are going to study various kinds of geometrical objects, such as differentiable varieties, analytic varieties and algebraic varieties. These objects occur in many areas of mathematics, ranging from physics (general relativity, conformal field theory, mechanics) to algebra and number theory (algebraic geometry) and analysis (Lie groups, differential equations). Students taking this course are invited to look for further details or other points of view in text books. Some books I can think of at this moment are: Spivak's series of books (differential geometry), the book by Bott and Tu (differential forms in algebraic topology), Lang's book (differentiable manifolds) and Bourbaki's book (variétés différentielles et analytiques: fascicule de résultats). A good idea is to look in the library under the appropriate AMS subject classifications (differential geometry is 53, algebraic geometry is 14, Lie groups is 22, analytic geometry: 32, geometry: 51, manifolds: 57 and 58).

We begin with differentiable varieties, also called manifolds. These are usually defined in terms of charts. Intuitively, they are objects that "locally" look like $\mathbb{R}^n$ for some n. We begin by making that precise.

1.1 Definition. *Let X be a set. An atlas for X then consists of the following data: a set I, for each i in I a subset X_i of X, an integer $n_i \geq 0$, an open subset U_i of $\mathbb{R}^{n_i}$ and a bijection $\phi_i\colon U_i \to X_i$. These data are required to satisfy the following conditions. Firstly, the X_i cover X, that is, $\cup_i X_i = X$. Secondly, the charts ϕ_i are compatible, in the sense that we will now explain. For i and j in I let $X_{i,j}$ be $X_i \cap X_j$, and let $U_{i,j}$ be $\phi_i^{-1}X_{i,j}$. Then ϕ_i induces a bijection, still denoted ϕ_i, from $U_{i,j}$ to $X_{i,j}$. Saying that ϕ_i and ϕ_j are compatible means that $U_{i,j}$ is open in U_i, $U_{j,i}$ open in U_j, and the bijection $\phi_j^{-1} \circ \phi_i\colon U_{i,j} \to U_{j,i}$ is differentiable.*

Some remarks are in order here. First of all, the differentiability of $\phi_j^{-1} \circ \phi_i$ in the definition can mean various things. When we just say differentiable, we mean in fact infinitely differentiable, that is, the n_j $\mathbb{R}$-valued functions making up $\phi_j^{-1} \circ \phi_i$ are C^∞-functions on $U_{i,j}$. But we could also consider functions of class C^k for some $k \geq 0$; in that case we will say that the atlas is C^k. Note that the terminology "differentiable" is misplaced in the case $k = 0$; in that case we speak of a topological atlas. The second remark concerns the integers n_i.

1.2 Definition. *Let $k \geq 0$ be an integer or ∞. A variety or manifold of class C^k is a set X equipped with a C^k-atlas. Notation: (X, I, n, U, ϕ).*

For X a C^k-variety and x in X, all n_i for i such that X_i contains x are equal; this integer is called the dimension of X at x; we denote it by $\dim_X(x)$, so that we can view $\dim_X$ as a $\mathbb{Z}$-valued function on X. (For $k > 0$ the equality of the n_i is easy to prove (consider derivatives and use linear algebra); for $k = 0$ one needs some algebraic topology.) Most of the time we will just consider the C^∞ case. As usual, defining the objects to study is not too interesting; we should also say what maps between them we want to consider. For example, we want to say what it means that two manifolds are isomorphic.

1.3 Definition. *Let (X, I, n, U, ϕ) and (Y, J, m, V, ψ) be manifolds. Let f be a map from X to Y. Let x be in X. Then f is called differentiable at x if for every (i, j) such that $x \in X_i$ and $f(x) \in Y_j$ the subset $\phi_i^{-1}((f^{-1}Y_j) \cap X_i) \subset \mathbb{R}^{n_i}$ is open and the map $\psi_j^{-1} f \phi_i$ from $\phi_i^{-1}((f^{-1}Y_j) \cap X_i) \subset \mathbb{R}^{n_i}$ to $\mathbb{R}^{m_j}$ is differentiable at $\phi_i^{-1}(x)$. The map f is called differentiable, or a morphism of manifolds, if it is differentiable at all x in X.*

Note that this definition does not change if we require the openness and differentiability at x only for one pair (i, j). If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms, then $g \circ f\colon X \to Z$ is also a morphism. So we have the category of manifolds: we have objects, morphisms, composition of morphisms, the composition is associative and each object X has an identity morphism id_X. A morphism $f\colon X \to Y$ is called an isomorphism if and only if there exists a morphism $g\colon Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$. Equivalently: a map $f\colon X \to Y$ is an isomorphism if and only it is bijective and f and f^{-1} are differentiable. Let us look at some examples of manifolds.

Let $n \geq 0$. Then $\mathbb{R}^n$ with the atlas consisting of the chart $\mathrm{id}_{\mathbb{R}^n}$ is a manifold, that we will denote by $\mathbb{R}^n$. In the same way, every open subset of some $\mathbb{R}^n$ becomes a manifold. If X and Y are manifolds (we have already dropped the atlas from the notation), then $X \times Y$ is easily equipped with an atlas (take $K := I \times J$, $W_{i,j} := U_i \times V_j$, etc.). We leave it as an exercise to the reader to verify that the two projections pr_X and pr_Y from $X \times Y$ to X and Y are differentiable, and that $(X \times Y, \mathrm{pr}_X, \mathrm{pr}_Y)$ has the following universal property: for Z a manifold and morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ there exists a unique morphism $h\colon Z \to X \times Y$ such that $f = \mathrm{pr}_X h$ and $g = \mathrm{pr}_Y h$. Let V be a finite dimensional $\mathbb{R}$-vector space, say of dimension n. Then an isomorphism of $\mathbb{R}$-vector spaces $\phi\colon \mathbb{R}^n \to V$ makes V into a manifold. It is clear that two such isomorphisms $\phi \neq \phi'$ give different atlases but that id_V is an isomorphism between the manifolds.

For $n \geq 0$ the subset $\mathrm{GL}_n(\mathbb{R})$ of $\mathrm{M}_n(\mathbb{R})$ consisting of invertible n by n matrices with coefficients in $\mathbb{R}$ is an open subset (it is $\det^{-1}(\mathbb{R} - \{0\})$). It is easy to check that the maps $m\colon \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ and $i\colon \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ given by $m(x, y) = xy$ and $i(x) = x^{-1}$ are differentiable (for i, use the formula for x^{-1} in terms of the matrix of cofactors and $\det(x)$). In general, a group G equipped with an atlas such that the multiplication and inversion are differentiable is called a Lie group. We will see more examples soon.

Our next example is in a sense more interesting, because it is not isomorphic to an open subset of $\mathbb{R}^n$ for any n. We consider the circle S^1 in $\mathbb{R}^2$: it is the set of (x, y) such that $x^2 + y^2 = 1$. One way to make an atlas is the following. The projection on the first coordinate gives a bijection from $\{(x, y) \in S^1 \mid y > 0\}$ to the open interval $]-1, 1[$; the same holds for $\{(x, y) \in S^1 \mid y < 0\}$. We also have the projections on the second coordinate from the sets $\{(x, y) \in S^1 \mid x > 0\}$ and $\{(x, y) \in S^1 \mid x < 0\}$. The four inverses of these maps form an atlas. Another atlas is obtained by restricting the map $(\sin, \cos)$ from $\mathbb{R}$ to $\mathbb{R}^2$ to suitable subsets of $\mathbb{R}$. Yet a third atlas is given by projection from points of S^1. For t in $\mathbb{R}$ consider the line through $(t, 0)$ and $(0, 1)$. This line intersects S^1 in $(0, 1)$ and a unique other point: $(2t/(t^2 + 1), (t^2 - 1)/(t^2 + 1))$. This map gives a bijection from $\mathbb{R}$ to $S^1 - \{(0, 1)\}$. Considering lines through $(0, -1)$ gives a second chart. The map id_{S^1} is an isomorphism between all three atlases for S^1 that we have just seen. It is interesting to note what kind of functions we get from the charts and transition maps (i.e., the $\phi_j^{-1}\phi_i$) in these three cases. In the first case the charts and the transition maps are algebraic functions (they are built up from rational functions and square roots). In the second case the charts are given by the transcendental functions sin and cos, but the transition maps are just translations in $\mathbb{R}$. In the third case all functions are rational functions.

We could treat the n-sphere S^n in a similar way (it is defined as the subset of $(x_0, \ldots, x_n)$ in $\mathbb{R}^{n+1}$ such that $x_0^2 + \cdots + x_n^2 = 1$). In particular, the first and third methods we used for $n = 1$ are easily adapted (not the second method, as far as I can see; it has to do with the fact that S^n is simply connected for $n \geq 2$ (well, of course, one has the usual spherical coordinates r, θ and ϕ, but it is not as nice)). But it is more useful to develop a systematic way to make subsets of $\mathbb{R}^n$ that are defined by suitable equations into manifolds. In order to do this we need the implicit function theorem. We will state this theorem in a quite general context, so that it will suffice for the whole course.

1.4 Theorem. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let X, Y and Z be normed K-vector spaces, with Y complete. Let U be an open subset of $X \times Y$, $k \geq 1$ and $f\colon U \to Z$ a C^k-map. Let (x,y) be in U such that the derivative $(D_2 f)(x,y)\colon Y \to Z$ of f with respect to the second variable is an isomorphism of topological vector spaces (i.e., it is bijective and its inverse is continuous). Then there exist open neighborhoods V of x in X and W of y in Y such that $V \times W \subset U$ and for every v in V there exists a unique w in W with $f(v,w) = f(x,y)$. The map $g\colon V \to Y$ thus defined is C^k. Moreover, if X and Y are finite dimensional and f analytic, then g is analytic.*

For a proof the reader is referred to the standard text books, or to course notes from analysis courses. In the complex case, i.e., $K = \mathbb{C}$, we say that $f\colon U \to Z$ is C^k if it is so when we view X, Y and Z as $\mathbb{R}$-vector spaces. When we want to talk about differentiability in the complex sense, we will allways explicitly say so. The reason for this terminology is that a function $f\colon U \to \mathbb{C}$, with $U \subset \mathbb{C}^n$ open, is analytic if and only if it is C^1 in the complex sense (see section 2).

Let us now consider the following situation. We have positive integers n and m, we have an open subset U of $\mathbb{R}^n$ and a C^k map f from U to $\mathbb{R}^m$, for some $k \geq 1$. Let X be the set of zeroes of f: $X := \{x \in U \mid f(x) = 0\}$. We want to equip X with an atlas, in some natural way (for example, the charts should be C^k-maps to $\mathbb{R}^n$). It turns out that at least some conditions have to be satisfied for this to be possible. For example, consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = xy$. Then X is the union of the two coordinate axes; consequently, X, with its induced topology, cannot be a C^0-manifold, since no neighborhood of $(0,0)$ is homeomorphic to an open interval in $\mathbb{R}$. Note that $(0,0)$ is special, since both partial derivatives of f vanish at that point, i.e., f has derivative zero at $(0,0)$. So, in the situation above, we assume that for all x in X the derivative $(Df)x$ is surjective (i.e., f is a submersion at all x in X). Let now x be in X. Let V be the kernel of the linear map $(Df)x\colon \mathbb{R}^n \to \mathbb{R}^m$. Let $V^\perp$ be the orthogonal of V (for the standard inner product on $\mathbb{R}^n$). We view $\mathbb{R}^n$ as the product $V \times V^\perp$, and hence U as an open subset of $V \times V^\perp$. In this situation we can apply Theorem 1.4, since $(D_2 f)x$ is an isomorphism from $V^\perp$ to $\mathbb{R}^m$. We get an open subset $V' \subset V$ and a C^k-map $g\colon V' \to V^\perp$ such that x is in $V' \times V^\perp$ and for all v in V' we have $f(v, g(v)) = 0$. Hence the map $\phi\colon V' \to X$ defined by $\phi(v) = (v, g(v))$ is a chart at x. We can obviously cover X with such charts (for example, take one chart for each x). These charts are compatible because ϕ^{-1} is just the orthogonal projection on V. Note that $\dim_X(x) = \dim V = n - m$.

1.5 Exercise. Suppose that we have $U \subset \mathbb{R}^n$ open, and C^k-maps $f\colon U \to \mathbb{R}^m$ and $f'\colon U \to \mathbb{R}^{m'}$, defining the same X and both submersions at all x in X. Then show that any two atlases obtained from the construction above are such that id_X is an isomorphism between them.

We can now easily give some more examples of Lie groups: the classical matrix groups.

1.6 Example. Let $n \geq 1$. The group special linear group $\mathrm{SL}_n(\mathbb{R})$ is defined as the kernel of the morphism of groups $\det\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*$. We have to show that for all x in $\mathrm{SL}_n(\mathbb{R})$ the derivative $(D\det)x$ is non-zero. So we have to "compute" $\det(x + \varepsilon y)$ for small ε in $\mathbb{R}$ and any y in $\mathrm{M}_n(\mathbb{R})$. We have:

$$\det(x+\varepsilon y) = \det(x(1+\varepsilon x^{-1}y)) = \det(x)\det(1+\varepsilon x^{-1}y) = 1 + \varepsilon \mathrm{tr}(x^{-1}y) + O(\varepsilon^2), \tag{1.6.1}$$

with $\mathrm{tr}(x^{-1}y)$ the trace of $x^{-1}y$. It follows that $((D\det)x)y = \mathrm{tr}(x^{-1}y)$. This cannot be zero for all y, since x^{-1} is invertible.

1.7 Example. Let again $n \geq 1$. The orthogonal group $\mathrm{O}_n(\mathbb{R})$ is the subgroup of x in $\mathrm{GL}_n(\mathbb{R})$ that preserve the standard scalar product of $\mathbb{R}^n$, i.e., the x such that $x^t x = 1$. The special orthogonal group is the subgroup $\mathrm{SO}_n(\mathbb{R})$ of x in $\mathrm{O}_n(\mathbb{R})$ with $\det(x) = 1$. We consider the map $f\colon \mathrm{GL}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})^+$ given by $f(x) = x^t x - 1$, where $\mathrm{M}_n(\mathbb{R})^+$ denotes the set of real symmetric n by n matrices. We have to show that this map is submersive at all x in $\mathrm{O}_n(\mathbb{R})$. For x in $\mathrm{O}_n(\mathbb{R})$, y in $\mathrm{M}_n(\mathbb{R})$ and small ε in $\mathbb{R}$ we have (with $z := x^{-1}y$):

$$(1.7.1)\qquad \begin{aligned} f(x+\varepsilon y) &= f(x(1+\varepsilon z)) = (1+\varepsilon z)^t x^t x(1+\varepsilon z) - 1 = (1+\varepsilon z^t)(1+\varepsilon z) - 1 = \\ &= \varepsilon(z^t + z) + O(\varepsilon^2). \end{aligned}$$

It follows that $((Df)x)y = z^t + z$,which clearly shows that $(Df)x$ is surjective. So $\mathrm{O}_n(\mathbb{R})$ is now a manifold. Our computation above also shows that it has everywhere the same dimension, namely $(n^2 - n)/2 = \dim_{\mathbb{R}}(\mathrm{M}_n(\mathbb{R}))^-$, the dimension of the $\mathbb{R}$-vector space of anti-symmetric n by n matrices. For every x in $\mathrm{O}_n(\mathbb{R})$ we have $1 = \det(x^t x) = \det(x)^2$, hence $\det(x) = \pm 1$. There are x in $\mathrm{O}_n(\mathbb{R})$ with $\det(x) = -1$, hence we have a short exact sequence:

$$(1.7.2)\qquad 1 \to \mathrm{SO}_n(\mathbb{R}) \to \mathrm{O}_n(\mathbb{R}) \to \{\pm 1\} \to 1.$$

This sequence is split: send -1 to the diagonal matrix $\mathrm{diag}(-1, 1, \ldots, 1)$, for example. If n is odd, we even have a splitting with image in the center: send -1 to -1; hence for odd n we have an isomorphism of Lie groups from $\mathrm{O}_n(\mathbb{R})$ to $\mathrm{SO}_n(\mathbb{R}) \times \mathbb{Z}/2\mathbb{Z}$. For even n there is not such a splitting, and one cannot do better than say that $\mathrm{O}_n(\mathbb{R})$ is isomorphic to the semi-direct product $\mathrm{SO}_n(\mathbb{R}) \times_\alpha \mathbb{Z}/2\mathbb{Z}$, with $\alpha\colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathrm{SO}_n(\mathbb{R}))$ the morphism of groups that sends -1 to the inner automorphism given by conjugation by $\mathrm{diag}(-1, 1, \ldots, 1)$.

1.8 Example. Again, $n \geq 1$. The symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ is the subgroup of $\mathrm{GL}_n(\mathbb{R})$ that preserves the "standard" alternating bilinear form on $\mathbb{R}^{2n}$ that is given by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, in which the coefficients are n by n matrices. One computes easily that $\mathrm{Sp}_{2n}(\mathbb{R})$ is the subset of $\mathrm{GL}_{2n}(\mathbb{R})$ of the x that satisfy $x^t\psi x = \psi$. So we consider the map f from $\mathrm{GL}_{2n}(\mathbb{R})$ to $M_{2n}(\mathbb{R})^-$ given by $f(x) = x^t\psi x - \psi$. Let x be in $\mathrm{Sp}_{2n}(\mathbb{R})$, y in $M_{2n}(\mathbb{R})$ and ε in $\mathbb{R}$. Put $z := x^{-1}y$. Then we have:

$$(1.8.1)\qquad f(x+\varepsilon y) = f(x(1+\varepsilon z)) = (1+\varepsilon z^t)\psi(1+\varepsilon z) - \psi = \varepsilon(z^t\psi + \psi z) + O(\varepsilon^2).$$

This shows that $((Df)x)y = z^t\psi + \psi z$, so we have to show that the map $z \mapsto z^t\psi + \psi z$ from $\mathrm{M}_{2n}(\mathbb{R})$ to $\mathrm{M}_{2n}(\mathbb{R})^-$ is surjective. To do this, we compute its kernel (this is interesting anyway, since this kernel is what is called the Lie algebra of $\mathrm{Sp}_{2n}(\mathbb{R})$). So write z as a two by two matrix of n by n matrices: $z = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then a short computation gives:

$$(1.8.2)\qquad z^t\psi + \psi z = 0 \iff (c^t = c \text{ and } b^t = b \text{ and } d = -a^t).$$

It follows that the kernel has dimension $2n^2+n$. The dimension of $\mathrm{M}_{2n}(\mathbb{R})^-$ equals $((2n)^2-(2n))/2$. Linear algebra then implies that our map $z \mapsto z^t\psi + \psi z$ is surjective.

1.9 Exercise. Let $n \geq 1$. Show that the Lie groups $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$ and $\mathrm{Sp}_{2n}(\mathbb{R})$ are connected (the last one is more difficult to do). Show also that $\mathrm{Sp}_{2n}(\mathbb{R})$ is contained in $\mathrm{SL}_{2n}(\mathbb{R})$.

1.10 Exercise. Let $n \geq 1$. Show that $\mathrm{GL}_n(\mathbb{C})$ is a manifold (i.e., make it into one, in the right way). The unitary group $\mathrm{U}_n(\mathbb{R})$ is the subgroup of $\mathrm{GL}_n(\mathbb{C})$ consisting of those x that preserve the standard scalar product on $\mathbb{C}^n$ (the one that sends (v,w) to $v_1\overline{w_1} + \cdots + v_n\overline{w_n}$), or, equivalently, the x with $\overline{x}^t x = 1$. Make $\mathrm{U}_n(\mathbb{R})$ into a manifold. Compute its dimension. Show that it is connected and compact. Do the same things for its subgroup $\mathrm{SU}_n(\mathbb{R})$ consisting of those x with $\det(x) = 1$.

We have already quite a few examples at our disposal, and it seems a good moment to do something about the foundations again. The reader is certainly aware that up to now each time we had various atlases on one set X, they had the property that id_X was an isomorphism between them. There should be a much more natural way to express this. In fact, we should replace the atlases by something else, giving us an equivalent category (this will be made precise). For example, when one studies groups, it is very clumsy to deal only with groups in terms of generators and relations. So we look for an object associated to an atlas for X such that two atlases such that id_X is an isomorphism between them give exactly the same object. We do this by considering a topology on X and the notion of differentiable functions on open subsets of X.

Let X be a set equipped with a C^k-atlas (some $k \geq 0$) (I, n, U, ϕ). Then we can define a topology on X by saying that a subset V of X is open if and only if for all i in I the subset $\phi_i^{-1}(V \cap X_i)$ is open in $\mathbb{R}^{n_i}$. (The verification that this works is left to the reader, and also that of the following assertion.) A subset V of X is open if and only if for all x in V there exists an i such that the subset $\phi_i^{-1}(V \cap X_i)$ of $\mathbb{R}^{n_i}$ is open. Suppose now that $V \subset X$ is open. Then a function $f\colon V \to \mathbb{R}$ is called of class C^k if and only if for all i the function $f \circ \phi_i$ on $\phi_i^{-1}(V \cap X_i)$ is C^k. The reader will verify that $f\colon V \to \mathbb{R}$ is C^k if and only if for all x in V there exists i such that $f \circ \phi_i$ on $\phi_i^{-1}(V \cap X_i)$ is C^k. The set of C^k functions on V will be denoted by $C_X^k(V)$; it is clearly an $\mathbb{R}$-algebra, usually of infinite dimension. For an open subset W contained in V we have the restriction map $\mathrm{res}(V,W)\colon C_X^k(V) \to C_X^k(W)$, that sends a function f on V to its restriction $f|_W$ to W. It is clear that for Z an open subset contained in W we have $\mathrm{res}(V,Z) = \mathrm{res}(W,Z)\mathrm{res}(V,W)$. Such a collection of sets $C_X^k(V)$ and maps $\mathrm{res}(V,W)$ is what one calls a presheaf on the topological space X, denoted C_X^k (a concise reference for the notions of presheaf and sheaf is Hartshorne's book "Algebraic Geometry" (GTM 52, Springer)). A very important property of this presheaf C_X^k is the following direct consequence of the local nature of a function being C^k: if we have an open subset V of X, and a covering of V by open subsets V_j with the j in some set J, and for each j an element f_j of $C_X^k(V_j)$ such that for all j and j' we have $\mathrm{res}(V_j, V_j \cap V_{j'})f_j = \mathrm{res}(V_{j'}, V_j \cap V_{j'})f_{j'}$, then there exists a unique f in $C_X^k(V)$ such that $\mathrm{res}(V, V_j)(f) = f_j$ for all j. (I apologize for the long sentence.) In general, a presheaf that satisfies this glueing condition will be called a sheaf. For the moment, the notions of sheaf and presheaf are just convenient for us for notational matters; we won't do anything complicated with sheaves for some time.

Given a set X with an atlas, the object we associate to it is the pair (X, C_X^k), consisting of a topological space and a sheaf of rings on it. Such objects are called ringed spaces, and all geometrical objects we will consider in this course will be ringed spaces. Let us now look at what it means for a map to be a morphism in terms of these sheaves.

1.11 Proposition. *Let X and Y be manifolds, and let $f\colon X \to Y$ be a map (of sets). Then f is a morphism of manifolds if and only if f is continuous and for each open U in Y and $g \in C_Y(U)$ the function gf is in $C_X(f^{-1}U)$.*

Proof. Suppose that f is a morphism of manifolds, i.e., f is differentiable. Then it follows directly from the definitions, and the fact that compositions of differentiable maps are differentiable, that f is continuous and that for each open U in Y and $g \in C_Y(U)$ the function gf is in $C_X(f^{-1}U)$.

Suppose now that f is continuous and that for each open U in Y and g in $C_Y(U)$ the function gf is in $C_X(f^{-1}U)$. Then $X_i \cap f^{-1}Y_j$ is open in X_i, hence $\phi_i^{-1}(X_i \cap f^{-1}Y_j)$ is open in U_i, hence in $\mathbb{R}^{n_i}$. We have to show that the map $\psi_j^{-1} f \phi_i$ from $\phi_i^{-1}(X_i \cap f^{-1}Y_j)$ to $V_j \subset \mathbb{R}^{m_j}$ is differentiable. It is equivalent to show that the m_j coordinate functions $x_k \psi_j^{-1} f \phi_i$ of this map are differentiable. Now $x_k \psi_j^{-1}$ is in $C_Y(Y_j)$, hence $x_k \psi_j^{-1} f$ is in $C_X(f^{-1}Y_j)$. It follows that $x_k \psi_j^{-1} f \phi_i$ is differentiable. □

We are now ready to formulate a new, improved definition of the category of manifolds. Note that a morphism of manifolds $f\colon X \to Y$ induces, for every open U in Y, a morphism of $\mathbb{R}$-algebras $f^*(U)\colon C_Y(U) \to C_X(f^{-1}U)$. In the language of sheaves, this is just a morphism of sheaves from C_Y to f_*C_X. For $f\colon X \to Y$ a morphism of topological spaces and F a sheaf on X, f_*F is the sheaf defined by $(f_*F)(U) = F(f^{-1}U)$. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair (f, ϕ) with f a continuous map from X to Y and ϕ a morphism of sheaves from $\mathcal{O}_Y$ to $f_*\mathcal{O}_X$. Let (f, ϕ) be a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ and (g, γ) a morphism from $(Y, \mathcal{O}_Y)$ to $(Z, \mathcal{O}_Z)$. Then $k := gf\colon X \to Z$ is continuous, and for every open $U \subset Z$ we have a morphism of rings $\kappa(U)$ from $\mathcal{O}_Z(U)$ to $\mathcal{O}_X(k^{-1}U)$ obtained by composing $\gamma(U)\colon \mathcal{O}_Z(U) \to \mathcal{O}_Y(g^{-1}U)$ and $\phi(g^{-1}U)\colon \mathcal{O}_Y(g^{-1}U) \to \mathcal{O}_X(f^{-1}g^{-1}U)$. In the case where X, Y and Z are obtained from manifolds as above, these maps are just the maps that do pullback of functions. Anyway, one easily verifies that κ is a morphism of sheaves (i.e., the $\kappa(U)$ are compatible with te restriction maps) so that (k, κ) is a morphism of ringed spaces. This composition of morphisms gives us a category: the category of ringed spaces. So now we also have the notion of isomorphisms between ringed spaces.

But in fact all this is not exactly what we need at this moment. Our ringed spaces (X, C_X) are somehow special: the sheaf C_X is a sheaf of $\mathbb{R}$-valued functions. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are topological spaces with sheaves of $\mathbb{R}$-valued functions, we define a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ to be a continuous map $f\colon X \to Y$ such that for all $U \subset Y$ open and all g in $\mathcal{O}_Y(Y)$ the function $f^*g := gf$ is in $\mathcal{O}_X(f^{-1}U)$. One checks immediately that this also gives us a category (let us call it the category of topological spaces with a sheaf of $\mathbb{R}$-valued functions), hence also a notion of isomorphism. For $(X, \mathcal{O}_X)$ a topological space with a sheaf of $\mathbb{R}$-valued functions, and $U \subset X$ open, we have the ringed space $(U, \mathcal{O}_X|_U)$, with $\mathcal{O}_X|_U(V) = \mathcal{O}_X(V)$ for all open $V \subset U$; this topological space with a sheaf of $\mathbb{R}$-valued functions is called the open subspace U of X. We can now state our improved definition of manifolds.

1.12 Definition. *Let $k \geq 0$. A C^k-manifold is a topological space with a sheaf of $\mathbb{R}$-valued functions (X, C_X), that is locally isomorphic (in the category of topological spaces with sheaves of $\mathbb{R}$-valued functions) to some (U, C_U^k), with U an open subset of some $\mathbb{R}^n$ and C_U^k the sheaf of C^k-functions on U. (Note that n may vary.) A morphism from a C^k manifold (X, C_X) to a C^k-manifold (Y, C_Y) is a morphism in the category of topological spaces with a sheaf of $\mathbb{R}$-valued functions.*

Let us now see what it means that this definition is equivalent to the older one. So let, for a moment, **Man** and **Man**$'$ denote the categories of C^k-manifolds in the old and the new sense, respectively. Then we have a functor F from **Man** to **Man**$'$ that sends a set X with a C^k-atlas to the space (X, C^k_X) as explained above, and sends a morphism $f\colon X \to Y$ in **Man** to the morphism (f, f^*) in **Man**$'$. By saying that our two definitions are equivalent, we mean that F is induces an equivalence of categories from **Man** to **Man**$'$. By definition, this means that there is a functor G from **Man**$'$ to **Man** such that GF and FG are isomorphic to $\mathrm{id}_{\mathbf{Man}'}$ and $\mathrm{id}_{\mathbf{Man}}$, respectively. General category nonsense says that such a G exists if and only if F is full and faithful (i.e, for all X and Y in **Man** F gives a bijection from $\mathrm{Hom}_{\mathbf{Man}}(X, Y)$ to $\mathrm{Hom}_{\mathbf{Man}'}(F(X), F(Y))$) and essentially surjective (i.e., every $(X, \mathcal{O}_X)$ in **Man**$'$ is isomorphic to the image under F of some object of **Man**). Proposition 1.11 means that F is full and faithful. It remains to show that F is essentially surjective. So let $(X, \mathcal{O}_X)$ be an object of **Man**$'$. Then we can cover X by open sets X_i such that the $(X_i, \mathcal{O}_X|_{X_i})$ are isomorphic to some (U, C^k_U) with U open in some $\mathbb{R}^n$. These isomorphisms form an atlas on X for which one easily verifies that it induces the sheaf $\mathcal{O}_X$.

We can give another interpretation of the manifold structure on sets of zeroes of equations that we defined just after Theorem 1.4. The situation is the following: we have an open subset U of some $\mathbb{R}^n$, a C^k-map (some $k \geq 1$) $f\colon U \to \mathbb{R}^m$ such that $(Df)x$ is surjective for all x in $X := f^{-1}0$. Note that X is a closed subset of U. Let Z be any closed subset of U. Then we get a sheaf C^k_Z of $\mathbb{R}$-algebras on Z by:

$$C^k_Z(V) := \{g\colon V \to \mathbb{R} \,|\, \exists h \in C^k_U(V') \text{ with } V' \subset U \text{ open, } V = V' \cap Z \text{ and } g = h|_V\}.$$

It is left to the reader to show that for X as above, this space (X, C^k_X) is in fact the manifold obtained by the construction mentioned above. (I realize that in this last part I really skipped a lot of details.)

After these generalities (general nonsense), let us consider some more examples: projective spaces and Grassmannians.

1.13 Example. Let k be a field, and V a k-vector space. Then we define the projective $\mathbb{P}(V)$ space associated to V to be the set of 1-dimensional subspaces of V (i.e., the set of lines through the origin). Clearly, we have a bijection $(V - \{0\})/k^* \to \mathbb{P}(V)$ induced by the map that sends $v \neq 0$ to the subspace kv. For $n \geq 0$ we define $\mathbb{P}^n(k)$ to be $\mathbb{P}(k^{n+1})$. An element in $\mathbb{P}^n(k)$ will be denoted as $(a_0 : a_1 : \cdots : a_n)$ with $a_i \in k$ not all zero; in this notation we have $(a_0 : \cdots : a_n) = (b_0 : \cdots : b_n)$ if and only if there exists $\lambda \in k^*$ with $b_i = \lambda a_i$ for all i. Clearly, $\mathbb{P}^0(k)$ is just the one point (1). We can describe $\mathbb{P}^1(k)$ as follows: it is the disjoint union of the set $\{(a : 1) \,|\, a \in k\}$, that we can identify with k, and the point $(1 : 0)$ "at infinity". In the same way, $\mathbb{P}^n(k)$ is easily seen to be the disjoint union of k^0, $k^1, \ldots, k^n$. If we want to equip $\mathbb{P}^n(k)$ with an atlas (say for $k = \mathbb{R}$ or $\mathbb{C}$), then these disjoint unions are not so useful: we need "open subsets". So we consider the covering of $\mathbb{P}^n(k)$ by the sets $\mathbb{P}^n(k)_i := \{a_0 : \cdots : a_n \,|\, a_i \neq 0\}$, for $0 \leq i \leq n$. For each i we have a bijection ϕ_i from k^n to $\mathbb{P}^n(k)_i$, sending $(x_1, \ldots, x_n)$ to the point $(a_0 : \cdots : a_n)$ such that $a_i = 1$, $a_j = x_j$ for $j > i$ and $a_j = x_{j+1}$ for $j < i$. The inverse of ϕ_i sends $(a_0 : \cdots : a_n)$ to the $n$tuple of a_j/a_i with $j \neq i$. So, in the usual notation for charts, we have $U_i = k^n$ for all i. For $j > i$ the subset $U_{i,j}$ of $U_i = k^n$ is the set $\{x \in k^n \,|\, x_j \neq 0\}$, which is an open subset if $k = \mathbb{R}$ or $k = \mathbb{C}$. The map $\phi_j^{-1}\phi_i$ from $U_{i,j}$ to k^n sends $(x_1, \ldots, x_n)$ to the $n$tuple consisting of the x_l/x_j with $l < i$, $1/x_j$, and the x_l/x_j with $j \neq l \geq i$. Clearly, these maps are differentiable if $k = \mathbb{R}$ or $\mathbb{C}$.

1.14 Exercise. Show that the $\mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{C})$ are connected and compact. Show that $\mathbb{P}^1(\mathbb{R})$ is isomorphic to S^1 and that $\mathbb{P}^1(\mathbb{C})$ is isomorphic to S^2.

1.15 Example. For k a field, V a k-vector space and $d \geq 0$, let $\mathrm{Gr}_d(V)$ be the set of d-dimensional subspaces of V. Define $\mathrm{Gr}_{d,n}(k) := \mathrm{Gr}_d(k^n)$. Note that $\mathrm{Gr}_1(V) = \mathbb{P}(V)$ and $\mathrm{Gr}_{1,n}(k) = \mathbb{P}^{n-1}(k)$. For $k = \mathbb{R}$ or $\mathbb{C}$, we want to make the $\mathrm{Gr}_{d,n}(k)$ into manifolds. We will do this in two ways: first by making charts, then by considering $\mathrm{Gr}_{d,n}(k)$ as a quotient of $\mathrm{GL}_n(k)$ by a certain subgroup (in this case we can define charts, or a topological space with a sheaf of functions).

Let $n \geq 0$, $d \geq 0$ and a field k be given. Let V be a k-vector space of dimension n. Let x be in $\mathrm{Gr}_d(V)$, i.e., x is a d-dimensional subspace of V. Choose a subspace y of V such that $V = x \oplus y$. Let $\mathrm{Gr}_d(V)_x$ be the subset $\{z \mid z \cap y = \{0\}\}$ of $\mathrm{Gr}_d(V)$. For each z in $\mathrm{Gr}_d(V)_x$ the projection from z to x along y is an isomorphism. It follows that we have a bijection ϕ_x from $\mathrm{Hom}_k(x, y)$ to $\mathrm{Gr}_d(V)_x$ sending f to $\mathrm{im}(\mathrm{id}_x + f)$. Note that $x = \phi_x(0)$. We view the set $\mathrm{Hom}_k(x, y)$ as a k-vector space in the usual way. We leave it to the reader to study the maps ϕ_x and to verify that, for $k = \mathbb{R}$ or $\mathbb{C}$, they form an atlas. It might be a good idea to look first at what happens when one takes various y's for one x, and then to take one y for various x's. Anyway, it is not necessary to do this, since the next method we use shows that the charts are compatible.

Let us now study $\mathrm{Gr}_{d,n}(k)$ from a different point of view. The group $\mathrm{GL}_n(k)$ acts on the vector space k^n, hence on $\mathrm{Gr}_{d,n}(k)$: an element g sends x to gx, the image of x under g. It is easy to see that $\mathrm{GL}_n(k)$ acts transitively on $\mathrm{Gr}_{d,n}(k)$ (for a given x, choose a basis for x and extend it to a basis of k^n). Let x_0 be the subspace $ke_1 + \cdots + ke_d$ of k^n, where e denotes the standard basis of k^n. The stabilizer $P := \mathrm{GL}_n(k)_x$ of x is the subgroup of $\mathrm{GL}_n(k)$ consisting of those g such that $g_{i,j} = 0$ for all (i, j) with $j \leq d < i$. Hence we get a bijection from $\mathrm{GL}_n(k)/P$ to $\mathrm{Gr}_{d,n}(k)$. Suppose now that $k = \mathbb{R}$ or $\mathbb{C}$. It suffices to equip $\mathrm{GL}_n(k)/P$ with a differentiable structure in order to do so for $\mathrm{Gr}_{d,n}(k)$. As already said above, we can make an atlas for $\mathrm{GL}_n(k)/P$, but we can also make $\mathrm{GL}_n(k)/P$ into a topological space with a sheaf of functions, directly. Since we have already seen numerous atlases, let us first construct the ringed space. For the sake of notation, let $G := \mathrm{GL}_n(k)$, $X := G/P$ and $\pi \colon G \to X$ be the quotient map. We equip X with the quotient topology: a subset U of X is open if and only if $\pi^{-1}U \subset G$ is open. Then we equip the topological space X with the sheaf of functions that are P-invariant. Note that for $U \subset X$ open, P acts on $C_G(\pi^{-1}U)$ by the formula $(pf)(g) = f(gp)$. For a set S with a P-action, let S^P be the set of elements fixed by P. The we define a sheaf C_X on X by: $C_X(U) := C_G(\pi^{-1}U)^P$. Of course, now we have to verify that the topological space with sheaf of functions (X, C_X) is a manifold. To do this, we have to show that every point has an open neighborhood that it isomorphic to some (U, C_U) with U some open subset of some $\mathbb{R}^n$, and C_U the sheaf of differentiable functions. This is of course almost the same as to make an atlas. Let g be in G. Then the translate gP of P is the orbit of g under P. Let T be the subspace of $\mathrm{M}_n(k)$ consisting of the m with $m_{i,j} = 0$ if $i \leq d$ or $j > d$. Then $\mathrm{M}_n(k)$ is the direct sum of T and the tangent space of P at 1. Let $U_g := T$ and $\phi_g \colon U_g \to X$ be given by $\phi_g(t) = \pi(g(1+t))$ (note that $1+t$ is in G). One verifies that ϕ_g induces an isomorphism between (T, C_T) and $(X_g, C_X|_{X_g})$ with $X_g = \pi(g(1+T)P)$ open since $(1+T)P$ is exactly the set of g in G with $\det((g_{i,j})_{1 \leq i,j \leq d}) \neq 0$. What makes this method work is the fact that the map $T \times P \to G$, $(t, p) \to (1+t)p$, is an isomorphism from $T \times P$ to the open subset $(1+T)P$ of G. That this is so follows from the simple computation $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ ta & tb+d \end{smallmatrix}\right)$.

Let us finish by noting that the charts obtained here and above are in fact the same, so we have now also shown that the charts above are compatible.

1.16 Remark. We note that our definition of manifold is somewhat more general than the one mostly encountered in textbooks, mainly for two reasons that both concern the underlying topological space. Often, manifolds are required to be Hausdorff (also called *separated*), i.e., distinct points lie in disjoint open neighborhoods. The second difference is a finiteness property: one often demands that a manifold has a countable basis for its topology (or, weaker, that it is paracompact (we will encounter this notion later)). We will state these types of assumptions when they are needed. Another difference is that most texts demand the dimension of a manifold, which is a locally constant function in our terminology, to be actually constant. Even worse, some texts demand manifolds to be connected.

1.17 Exercise. Give an example of a C^∞-manifold of which the dimension is not bounded.

1.18 Definition. *Let $k \geq 0$. A morphism of C^k-manifolds $f\colon X \to Y$ is called an* open immersion *if it induces an isomorphism from X to an open submanifold of Y. More explicitly, this means that fX is open in Y, that f induces a homeomorphism from X to fX (equipped with the topology induced from that of Y), and that for every open subset U of fX the map f^* is an isomorphism from $C^k_Y(U)$ to $C^k_X(f^{-1}U)$.*

1.19 Exercise. Manifolds can be "glued" along open subsets. Let I be a set, and $k \geq 0$. For every i in I, let X_i be a C^k-manifold. For every i and j in I, let $X_{i,j}$ be an open subset of X_i, and let $f_{i,j}$ be an isomorphism from $X_{i,j}$ to $X_{j,i}$. Show that there is a C^k-manifold X, and open immersions g_i from X_i in X, such that X is the union of the g_iX_i, and such that for all i and j one has $g_j \circ f_{i,j} = g_i$ on $X_{i,j}$.

1.20 Exercise. Give an example of a C^∞-manifold that it not Hausdorff. (See Looijenga's syllabus if necessary.)

1.21 Exercise. Give an example of a C^∞-manifold that does not admit a countable basis for its topology.