# 2 Vector bundles

Before defining what a vector bundle is, let us study a most important example: the tangent bundle of a manifold. So first we recall what the tangent spaces of a manifold at its points are.

## 2.1 Tangent spaces

Let $k \geq 1$ and let $(X, C_X^k)$ be a $C^k$-manifold. For $x$ in $X$ we want to define its tangent space. There are several ways to do this (which are of course equivalent). For a detailed discussion of all of those I know, see Spivak's book, Volume I, Chapter 3. We will discuss some of them. Intuitively, the tangent space of $X$ at a point $x$ in $X$ is the first order approximation of $X$ at $x$. We need it in order to speak of the derivatives of morphisms of manifolds.

Suppose that we have an atlas $(X, I, n, U, \phi)$ for $X$. Let $x$ be in $X$. A tangent vector $v$ at $x$ will then be a compatible system of pairs $(i, v_i)$, with $v_i$ in $\mathbb{R}^{n_i}$, for the $i$ in $I$ such that $x$ is in $X_i$. The compatibility is defined as follows. Let $i$ and $j$ be in $I$ with $x \in X_i$ and $x \in X_j$. Then the transition isomorphism $\phi_j^{-1}\phi_i$ from $U_{i,j}$ to $U_{j,i}$ has the property that:

$$(D(\phi_j^{-1}\phi_i))(\phi_i^{-1}x) \text{ sends } v_i \text{ to } v_j. \tag{2.1.1}$$

Since for every such pair $(i, j)$ the map $(D(\phi_j^{-1}\phi_i))(\phi_i x)$ is an isomorphism of $\mathbb{R}$-vector spaces from $\mathbb{R}^{n_i}$ to $\mathbb{R}^{n_j}$, it is clear that a compatible system of $(i, v_i)$ is determined by any of its elements, and that such an element can be arbitrary in $\mathbb{R}^{n_i}$. So to give such a compatible system, it is equivalent to give, for one $i$ in $I$ with $X_i \ni x$, an element $v_i$ of $\mathbb{R}^{n_i}$. In particular, the set of such compatible systems, that we call the tangent space of $X$ at $x$ and that we denote $\mathrm{T}_X(x)$, has a natural structure of $\mathbb{R}$-vector space and is, via this construction, isomorphic to the $\mathbb{R}$-vector space $\mathbb{R}^{n_i}$.

Our second description of $\mathrm{T}_X(x)$ uses parametrized curves, and does not require charts. Let $x$ be in $X$. A parametrized curve at $x$ is a differentiable map $c \colon U \to X$ with $U \subset \mathbb{R}$ an open interval containing zero and with $c(0) = x$. We want to define the tangent space at $x$ as the set of equivalence classes of such curves, where $c_1$ and $c_2$ are to be equivalent if and only if they give the same tangent vector. Of course, we do not want to use the previous definition in terms of charts, so we want another way to say that $c_1$ and $c_2$ define the same tangent vector. One way to do this is the following. Let $c \colon U \to X$ be a parametrized curve at $x$, and $f$ in $C_X(V)$ with $V$ an open neighborhood of $x$. Then, after shrinking $U$ if necessary, $c^*f := fc$ is a differentiable function on $U$; let $(fc)'(0)$ be its derivative at 0. Then we say that $c_1$ and $c_2$ are equivalent if for all open neighborhoods $V$ of $x$ in $X$ and all $f$ in $C_X(V)$ we have $(fc_1)'(0) = (fc_2)'(0)$. Now we have some work to do: we have to show that this relation is an equivalence relation, and that the set of equivalence classes is in some natural way (this will be made precise below) an $\mathbb{R}$-vector space and as such isomorphic to the one defined above. The relation is clearly an equivalence relation. Let $c$ be a parametrized curve at $x$, and $i$ in $I$ with $X_i \ni x$. Then $\phi_i^{-1}c$ is defined at 0; we get an element $v_i := (D(\phi_i^{-1}c))0$ of $\mathbb{R}^{n_i}$. One checks immediately that this is a compatible system of $(i, v_i)$ in the sense explained above. If $c_1$ and $c_2$ are equivalent, then they give the same $v_i$ (view $\phi_i^{-1}$ as an $n_i$tuple of functions). Suppose now that $c_1$ and $c_2$ give the same $v_i$. We want to show that $c_1$ and $c_2$ are equivalent. For doing this, we may suppose that $X = U_i$ and that $\phi_i = \mathrm{id}_X$. Then we know that $(Dc_1)0 = (Dc_2)0$ (consider partial derivatives). But then we have, for $f$ in

$C_X(V)$, that

$$(fc_1)'0 = ((Df)x)((Dc_1)0) = ((Df)x)((Dc_2)0) = (fc_2)'0. \tag{2.1.2}$$

This shows that, as a set, the set of equivalence classes of $c$ is the same as $\mathrm{T}_X(x)$ as above, so we will use the same notation for both. The $\mathbb{R}$-vector space structure on $\mathrm{T}_X(x)$ has the following interpretation (we denote the class of a curve $c$ by $[c]$): $[c_1]+[c_2]=[c_3]$ if and only if for all $V$ and $f$ we have $(fc_1)'(0)+(fc_2)'(0)=(fc_3)'(0)$. Likewise: $a[c_1]=[c_2]$, for $a$ in $\mathbb{R}$, if and only if $a(fc_1)'(0)=(fc_2)'(0)$ for all $f$. So indeed we have a description of $\mathrm{T}_X(x)$ that does not use charts.

From the previous description it is just a small step to the third and last one. But in order to define it, it is really convenient to use the notions of germ and stalk. So here follows a short intermezzo.

Suppose that $X$ is a topological space, $F$ a sheaf (of sets) on $X$ and $x$ an element of $X$. The stalk $F_x$ of $F$ at $x$ is then defined as the direct limit $\varinjlim F(U)$ taken over all open neighborhoods $U$ of $x$ (see for example in Hartshorne's book for the notion of direct limit). Concretely, this means that $F_x$ is the set of equivalence classes of pairs $(U,s)$, with $U$ an open neighborhood of $x$ and $s$ in $F(U)$, for the following equivalence relation. Two such pairs $(U_1,s_1)$ and $(U_2,s_2)$ are equivalent if and only if there is an open neighborhood $V$ of $x$ contained in $U_1\cap U_2$, such that the elements $s_1|_V$ and $s_2|_V$ are equal. The elements of $F_x$ are called germs of sections of $F$, and the class $s_x$ of $(U,s)$ is called the germ of $s$ at $x$. If $F$ is a sheaf of $\mathbb{R}$-algebras, then $F_x$ is an $\mathbb{R}$-algebra.

Let us now go back to our tangent spaces. So $X$ is again a $C^k$-manifold with $k\geq 1$, and $x$ is in $X$. For $c$ a parametrized curve at $x$, we get a map $\partial_c\colon C^k_{X,x}\to\mathbb{R}$ defined by: $\partial_c(f)=(fc)'0$ (note that this makes sense). This map satisfies the following properties:

1. it is $\mathbb{R}$-linear;

2. $\partial_c(fg)=f(x)\partial_c(g)+g(x)\partial_c(f)$ (Leibniz's or product rule);

3. $\partial_c(f)=0$ if $(\phi^*f)y=o(||y||)$, for $\phi$ a chart at $x$ such that $\phi^{-1}(x)=0$.

For $A$ a commutative ring, $B$ a commutative $A$-algebra and $M$ a $B$-module, an $A$-linear derivation from $B$ to $M$ is an $A$-linear map $D\colon B\to M$ satisfying $D(bb')=D(b)b'+b'D(b)$ for all $b$ and $b'$ in $B$. The set $\mathrm{Der}_A(B,M)$ of such maps is clearly a $B$-module: $(bD+D')b'=bD(b')+D'(b')$. Hence the first two of the properties of $\partial_c$ above say that $\partial_c$ is in $\mathrm{Der}_{\mathbb{R}}(C^k_{X,x},\mathbb{R})$.

We let $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,x},\mathbb{R})$ be the set of maps $\partial\colon C^k_{X,x}\to\mathbb{R}$ that satisfy:

1. $\partial$ is $\mathbb{R}$-linear;

2. $\partial(fg)=f(x)\partial(g)+g(x)\partial(f)$;

3. $\partial_c(f)=0$ if $(\phi^*f)y=o(||y||)$, for $\phi$ a chart at $x$ such that $\phi^{-1}(x)=0$.

To emphasize the third condition, we use the notation $\mathrm{Der}'$ in stead of Der. Since $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,x},\mathbb{R})$ is closed under $\mathbb{R}$-linear combinations, it is an $\mathbb{R}$-vector space.

By definition, we have an injective $\mathbb{R}$-linear map from $\mathrm{T}_X(x)$ to $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,x},\mathbb{R})$, sending $[c]$ to $\partial_c$. We claim that this map is an isomorphism. To prove that, it suffices to show that every element of $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,x},\mathbb{R})$ is of the form $\partial_c$. Since everything is defined intrinsically, and the question is local, we may suppose that $X$ is an open subset of $\mathbb{R}^n$ and that $x=0$. Let $\partial$ be in

$\mathrm{Der}'_{\mathbb{R}}(C^k_{X,0},\mathbb{R})$. We claim that $\partial = \sum_{i=1}^n \partial(x_i)\partial_i$, with $\partial_i$ be the element of $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,0},\mathbb{R})$ that sends $f$ to its $i$th partial derivative at 0. The elements $\partial_i$ are linearly independent because of the relations $\partial_i x_j = \delta_{i,j}$ (where $x_j$ is the $j$th coordinate function, and $\delta_{i,j}$ the Kronecker symbol). To prove the identity, let $f$ be in $C^k_{X,0}$. Rewriting the equality:

$$\int_0^1 \left(\frac{d}{dt}f(tx)\right)dt = f(x) - f(0),$$

with $x$ in some neighborhood of 0, gives:

$$f(x) = f(0) + \sum_{i=1}^n x_i \int_0^1 (D_i f)(tx)dt = f(0) + \sum_{i=1}^n \partial_i(f)x_i + \sum_{i=1}^n x_i g_i(x) = f(0) + \sum_{i=1}^n \partial_i(f)x_i + g(x),$$

with $D_i$ the $i$th partial derivative, and $g_i(x) = \int_0^1((D_i f)(tx) - (D_i f)(0))dt$. As $g$ is the difference of two $C^k$-functions, it is $C^k$. Since the $D_i f$ are $C^{k-1}$-functions, the $g_i$ are $C^{k-1}$. As $g_i(0) = 0$, we see that $g(x) = o(||x||)$. Applying $\partial$ to the last identity gives the desired result:

$$\partial = \sum_{i=1}^n \partial(x_i)\partial_i.$$

Now we know that the $\partial_i$ form a basis of $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,0},\mathbb{R})$. It follows that our map from $\mathrm{T}_X(x)$ to $\mathrm{Der}'_{\mathbb{R}}(C_{X,x},\mathbb{R})$ is an isomorphism.

In the case $k = \infty$, the $g_i$ above are $C^\infty$-functions, hence, without assuming that $\partial$ satisfies property 3, one can apply $\partial$ to the $g_i$, and get that $\partial(x_i g_i) = 0$ for all $i$, and hence the same conclusion: $\partial = \sum_{i=1}^n \partial(x_i)\partial_i$. Hence in the $C^\infty$-case we see that $\mathrm{T}_X(x) = \mathrm{Der}_{\mathbb{R}}(C^\infty_{X,x},\mathbb{R})$.

Let now $f\colon X \to Y$ be a morphism of $C^k$-manifolds, with $k \geq 1$, and $x$ in $X$. Then we get an $\mathbb{R}$-linear map $\mathrm{T}_f(x)\colon \mathrm{T}_X(x) \to \mathrm{T}_Y(fx)$ as follows. In the first description of tangent spaces, the map is $(D(\psi_j^{-1}f\phi_i))(\phi_i^{-1}x)$ (in the notation of Definition 1.3). In the second description it is the map that sends $[c]$ to $[fc]$. In the last description it is the map that sends $\partial$ to $\partial \circ f^*$, with $f^*\colon C_{Y,fx} \to C_{X,x}$ given by $g \mapsto gf$. It is left to the reader to show that these three maps are compatible with respect to the identifications between the various kinds of tangent spaces. The map $\mathrm{T}_f(x)$ is called the tangent map of $f$ at $x$, or also the derivative of $f$ at $x$. One can say that the main purpose of defining tangent spaces is just to have these tangent maps. In the same way, the main purpose of defining the tangent bundle is to have a tangent map $\mathrm{T}_f$ for all $x$ at once.

**2.1.3 Exercise.** Let $n$ and $m$ be positive integers, and $k \geq 1$. Let $U \subset \mathbb{R}^n$ be open, and $f\colon U \to \mathbb{R}^m$ be a $C^k$-map. As in Section 1, let $X := f^{-1}0$, suppose that $f$ is a submersion at all $x$ in $X$, and consider $X$ as a $C^k$-manifold. Let $i\colon X \to U$ be the inclusion map. Show that for $x$ in $X$ the map $\mathrm{T}_i(x)$ identifies $\mathrm{T}_X(x)$ with $\ker(\mathrm{T}_f(x))$.

## 2.2 Vector bundles, the tangent bundle

Let $X$ be a $C^k$-manifold with $k \geq 1$. We want to make an object $\mathrm{T}_X$, called the tangent bundle of $X$, that combines all the $\mathrm{T}_X(x)$. As a set, $\mathrm{T}_X$ is just the disjoint union of all the $\mathrm{T}_X(x)$, $x \in X$. But in order to have something useful, for example second derivatives of morphisms of manifolds, we need to equip $\mathrm{T}_X$ with the structure of a manifold, reflecting the fact that it is a disjoint union of vector spaces. The notion of vector bundle is made exactly for doing this. Note that we have a canonical map $p\colon \mathrm{T}_X \to X$, such that the fibre $p^{-1}\{x\}$ over $x$ is $\mathrm{T}_X(x)$. The following definition is meant to be a warming up for what comes after it.

**2.2.1 Definition.** *Let $k \geq 0$ and $p\colon E \to X$ be a morphism of $C^k$-manifolds. Then $p$ is called a $C^k$-fibration if for every $x$ in $X$ there exists an open neighborhood $U$, a $C^k$-manifold $F$ and an isomorphism $\phi\colon F \times U \to p^{-1}U$ such that $p \circ \phi$ is the projection $F \times U \to U$. The triple $(U, F, \phi)$ is called a trivialization over $U$. A fibration is called trivial if there exists such an isomorphism with $U = X$.*

**2.2.2 Remark.** The reader is supposed to understand what a product such as $F \times U$ is. See Section 1.

**2.2.3 Example.** The Möbius strip with its map to the circle is a non-trivial fibration with fibre the open interval $]-1,1[$. Another example is the Hopf fibration $S^3 \to S^2$ with fibre $S^1$, obtained from the identifications of $\mathbb{P}^1(\mathbb{C})$ with $S^2$, of $(\mathbb{C}^2 - \{0\})/\mathbb{R}^*_{>0}$ with $S^3$, and $\mathbb{C}^*/\mathbb{R}^*_{>0}$ with $S^1$.

Roughly speaking, a vector bundle is a fibration in which all the fibres are vector spaces (say over $\mathbb{R}$), such that there are local trivializations compatible with the vector space structures.

**2.2.4 Definition.** *Let $k \geq 0$ and $X$ a $C^k$-manifold. A real vector bundle over $X$ is a five tuple $(E, p, 0, +, \cdot)$ with $p\colon E \to X$ a $C^k$-fibration, $(0, +, \cdot)$ the structure of $\mathbb{R}$-vector space on all fibres of $p$, such that for all $x$ in $X$ there exists a local $C^k$-trivialization $\phi\colon F \times U \to p^{-1}U$, with $U \ni x$, respecting the vector space structures. A complex vector bundle is defined analogously.*

**2.2.5 Remark.** If one wants to give a set theoretic meaning to the triple $(0, +, \cdot)$ above, it is the following. The element $0$ is a section of $p$, i.e., it is a map from $X$ to $E$ such that $p \circ 0 = \mathrm{id}_X$. The element $+$ is a map from the fibred product $E \times_X E$ to $E$. The fibred product $E \times_X E$ is the subset of elements $(e_1, e_2)$ of $E \times E$ with $p(e_1) = p(e_2)$, i.e., it is the set of pairs of elements of $E$ that lie in the same fibre over $X$. The map $+$ is then of course the sum map of the vector space structure on the fibres. The element $\cdot$ is a map from $\mathbb{R} \times E \to E$ that gives the multiplication in the fibres.

It is certainly possible to equip the sets above with the structure of manifolds, in a natural way. The maps $0$, $+$ and $\cdot$ are then morphisms of manifolds.

Now that we know what a vector bundle is, let us construct tangent bundles. So let $X$ be a $C^k$-manifold with $k \geq 1$. As a set, $\mathrm{T}_X$ is the disjoint union of the $\mathrm{T}_X(x)$, for $x$ in $X$. The map $p$ from $\mathrm{T}_X$ to $X$ is the unique map such that $p^{-1}x = \mathrm{T}_X(x)$ for all $x$. Suppose that we have an atlas $(X, I, n, U, \phi)$. Then we get an atlas for $\mathrm{T}_X$ as follows. For $i$ in $I$, let $\mathrm{T}_{X,i} := p^{-1}X_i$. Put $V_i := \mathbb{R}^{n_i} \times U_i$ and define $\psi_i\colon V_i \to \mathrm{T}_{X,i}$ by: $\psi_i(v, x) = [(i, v)] \in \mathrm{T}_X(x)$, where $[(i, v)]$ denotes the compatible system corresponding to $(i, v)$ as in our first description of $\mathrm{T}_X(\phi_i x)$. These $\psi_i$ are easily seen to form a $C^{k-1}$-atlas, since

$$\psi_j^{-1}\psi_i\colon \mathbb{R}^{n_i} \times U_{i,j} \to \mathbb{R}^{n_j} \times U_{j,i}, \quad (v, x) \mapsto (((D(\phi_j^{-1}\phi_i))(x))v, (\phi_j^{-1}\phi_i)x). \tag{2.2.6}$$

The $C^{k-1}$-manifold thus obtained does not depend on the choice of the atlas (verification left to the reader). It remains now to show that the five tuple $(\mathrm{T}_X, p, 0, +, \cdot)$ is a vector bundle, i.e, that it has local trivializations as in Definition 2.2.4. But such trivializations are given by our maps $\psi_i$.

Suppose that $f\colon X \to Y$ is a morphism of $C^k$-manifolds, with $k \geq 1$. Let $\mathrm{T}_f\colon \mathrm{T}_X \to \mathrm{T}_Y$ be the map that is $\mathrm{T}_f(x)$ on $\mathrm{T}_X(x)$. Then $\mathrm{T}_f$ is a morphism of $C^{k-1}$-manifolds, it induces a commutative diagram:

$$\begin{array}{ccc} \mathrm{T}_X & \xrightarrow{\mathrm{T}_f} & \mathrm{T}_Y \\ \downarrow^{p_X} & & \downarrow^{p_Y} \\ X & \xrightarrow{f} & Y \end{array} \tag{2.2.7}$$

and it is $\mathbb{R}$-linear on the fibres. This motivates the following definition.

**2.2.8 Definition.** *Let $f\colon X \to Y$ be a morphism of $C^k$-manifolds, and let $p_E\colon E \to X$ and $p_F\colon F \to Y$ be $C^l$-vector bundles (with $l \leq k$ of course). A $C^l$-morphism from $E$ to $F$ over $f$ is then a morphism of $C^l$-manifolds $g\colon E \to F$ with $p_F g = f p_E$ that is a morphism of vector spaces on the fibres.*

It is clear that morphisms of vector bundles can be composed, and that we get a category of vector bundles. So finally we can say that associating to a manifold its tangent bundle, and to morphisms their derivative, is a functor T from the category of manifolds to the category of vector bundles.

## 2.3 Vector bundles as sheaves of modules

In this section we are interested in vector bundles over a fixed manifold $X$. A morphism of vector bundles over $X$ is a morphism as in (2.2.7) with $f = \mathrm{id}_X$.

Let $p\colon E \to X$ be a real $C^l$-vector bundle over a $C^k$-manifold $X$ (with $k \geq l$, of course). In practice one is very often more interested in the sections of $E$ then in $E$ itself. A $C^l$-section of $E$ over an open subset $U$ of $X$ is a morphism of $C^l$-manifolds $s\colon U \to E$ such that $ps = \mathrm{id}_U$. The set of $C^l$-sections of $E$ over $U$ will be denoted $E(U)$. (For example, $\mathrm{T}_X(X)$ is the set of $C^{k-1}$-vector fields on $X$.) If $V \subset U$ is an inclusion of open subsets of $X$, then we have a restriction map from $E(U)$ to $E(V)$. These restriction maps clearly make the system of $E(U)$'s into a sheaf that we will still denote by $E$. Consider a set $E(U)$. It has the structure of $\mathbb{R}$-vector space, and as such it is usually infinite dimensional. But it also has the structure of a module over the $\mathbb{R}$-algebra $C^l_X(U)$. This structure is compatible with the restriction maps, hence the following definition says that the sheaf $E$ is a sheaf of $C^l_X$-modules.

**2.3.1 Definition.** *Let $X$ be a $C^k$-manifold, and $l \leq k$. A sheaf of $C^l_X$-modules (or just a $C^l_X$-module) is then a sheaf $\mathcal{M}$ on $X$ together with the structure, for all open $U$ in $X$, of $C^l_X(U)$-module on $\mathcal{M}(U)$, compatible with the restriction maps. A morphism of $C^l_X$-modules is a morphism of sheaves such that on each open subset $U$ of $X$ it gives a morphism of $C^l_X(U)$-modules.*

Associating to a vector bundle its sheaf of sections is a functor from the category of $C^l$-vector bundles to the category of $C^l_X$-modules. We will show that this functor induces an equivalence of categories from the category of $C^l$-vector bundles to the full subcategory of $C^l_X$-modules that are "locally free of finite rank".

If $\mathcal{M}$ is a $C^l_X$-module and $U \subset X$ an open subset, then the restriction $\mathcal{M}|_U$ is a $C^l_U$-module. For $\mathcal{M}_1$ and $\mathcal{M}_2$ two $C^l_X$-modules we define a presheaf $\mathcal{M}_1 \oplus \mathcal{M}_2$ by: $(\mathcal{M}_1 \oplus \mathcal{M}_2)U = \mathcal{M}_1(U) \oplus \mathcal{M}_2(U)$. This presheaf is in fact a sheaf (exercise), of $C^l_X$-modules (trivial), with canonical morphisms from $\mathcal{M}_1$ and $\mathcal{M}_2$ to it, such that it is the direct sum of $\mathcal{M}_1$ and $\mathcal{M}_2$ in the category of $C^l_X$-modules (exercise). The same then works of course for arbitrary finite direct sums. A finite direct sum of $C^l_X$-modules has projection morphisms to its components, making it into the direct product of those (exercise).

Suppose now that $\mathcal{M}$ is a $C^l_X$-module, and that we have global sections $m_1, \ldots, m_r$ of it. Then we get a morphism from $(C^l_X)^r := \oplus_{i=1}^r C^l_X$ to $\mathcal{M}$ such that for all open $U \subset X$ and all $f_1, \ldots, f_r$ in $C^l_X(U)$ the element $(f_1, \ldots, f_r)$ of $(C^l_X)^r(U)$ is sent to $f_1 m_1 + \cdots + f_r m_r$ in $\mathcal{M}(U)$ (we omit the restriction maps). The sequence $(m_1, \ldots, m_r)$ is said to be a basis of $\mathcal{M}$ if this morphism is an isomorphism. A $C^l_X$-module $\mathcal{M}$ is called free of rank $r$ if it is isomorphic to $(C^l_X)^r$, or, equivalently, if it has a basis with $r$ elements. A $C^l_X$-module $\mathcal{M}$ is called locally free of rank $r$ (with $r$ a locally constant function on $X$) if every $x$ in $X$ has an open neighborhood $U$ such that $\mathcal{M}|_U$ is a free $C^l_U$-module of rank $r(x)$.

**2.3.2 Exercise.** Let $E$ be a real $C^l$-vector bundle on $X$. Show that its sheaf of sections $E$ is locally free of finite rank as $C^l_X$-module. Show that for $U \subset X$ open and $e_1, \ldots, e_r$ in $E(U)$ the sequence $(e_1, \ldots, e_r)$ is a basis for $E|_U$ if and only if for all $x$ in $U$ the sequence $(e_1(x), \ldots, e_r(x))$ is a basis for the fibre $E(x)$ of $E$ at $x$.

**2.3.3 Theorem.** *Let $X$ be a $C^k$-manifold, and let $l \leq k$. The functor that sends a $C^l$-vector bundle to its sheaf of sections, viewed as a $C^l_X$-module, is an equivalence from the category of $C^l$-vector bundles to the full subcategory of the category of $C^l_X$-modules consisting of the $C^l_X$-modules that are locally free of finite rank. A quasi-inverse of this functor is described in the proof below.*

**Proof.** We will first describe a functor $G$ from the category of locally free $C^l_X$-modules of finite rank to the category of $C^l$-vector bundles on $X$ and then show that it is a quasi-inverse of the functor $F$ mentioned in the Theorem (i.e., $FG$ and $GF$ are isomorphic to the identity funtors of the two categories in question).

So let $\mathcal{M}$ be a locally free $C^l_X$-module of finite rank $r$. Let $x$ be in $X$. We consider the stalk $C^l_{X,x}$ of $C^l_X$ at $x$ and the stalk $\mathcal{M}_x$ of $\mathcal{M}$ at $x$. It follows from the definition of stalk that $\mathcal{M}_x$ is a $C^l_{X,x}$-module. Since $\mathcal{M}$ is locally free of rank $r$, its stalk $\mathcal{M}_x$ is a free $C^l_{X,x}$-module of rank $r(x)$. Let $C^l_{X,x} \to \mathbb{R}$ be the map that sends $f$ to $f(x)$. It is a morphism of $\mathbb{R}$-algebras; let $m_x \subset C^l_{X,x}$ be its kernel. Then we define:

$$E(x) := \mathcal{M}_x / m_x \mathcal{M}_x = \mathbb{R} \otimes_{C^l_{X,x}} \mathcal{M}_x, \quad E := \coprod_{x \in X} E(x).$$

By construction, we get a map $p\colon E \to X$, such that $p^{-1}x = E(x)$. The $E(x)$ are clearly $\mathbb{R}$-vector spaces. Let $U$ be an open subset of $X$ on which $\mathcal{M}$ is trivial; let $m := (m_1, \ldots, m_d)$ be a basis of $\mathcal{M}|_U$. For $x$ in $U$ and $s$ in $\mathcal{M}(U)$, let $s(x)$ be the image of $s$ in $E(x)$. Then for all $x$ in $U$, the $m_i(x)$ form an $\mathbb{R}$-basis of $E(x)$. Hence we get a bijection:

$$\phi_{U,m}\colon \mathbb{R}^d \times U \to p^{-1}U, \quad (\lambda, x) \mapsto (\sum_i \lambda_i m_i, x).$$

This bijection gives $p^{-1}U$ the structure of a $C^l$-manifold. One checks that this structure does not depend on the choice of the basis $m$, since if $m'$ is another basis, one has a (unique) element $g$ in $\mathrm{GL}_d(C^l_X(U))$ such that $m'_i = g m_i$ for all $i$. It is now clear that $p\colon E \to X$, with the $\mathbb{R}$-vector space structures on the $E(x)$, is a $C^l$-vector bundle. From the construction it is also clear that a morphism between $C^l_X$-modules that are locally free of finite rank induces a morphism of $C^l$-vector bundles on $X$. We have thus defined our functor $G$.

To finish the proof, we have to show that $GF$ (resp., $FG$) is isomorphic to the identity functor of the category of $C^l$-vector bundles (resp., the category of locally free $C^l_X$-modules of finite rank).

Let $E$ be a $C^l$-vector bundle on $X$. Then $F(E)$ is the sheaf of sections of $E$. From the definition of $F(E)$ and the definition of stalk, it follows that we have a map $F(E)_x \to E(x)$ that sends $s$ to $s(x)$. This map of $C^l_{X,x}$-modules is surjective and its kernel is $m_x F(E)_x$ (use a local trivialization of $E$ at $x$). Hence $E(x)$ is canonically isomorphic to $F(E)_x/m_x F(E)_x$, i.e., to $(GF(E))(x)$. It is left to the reader to verify that this fibre wise isomorphism between $E$ and $(GF)E$ is an isomorphism of $C^l$-vector bundles, and that it is functorial.

Let $\mathcal{M}$ be a locally free $C^l_X$-module of finite rank. Let $U$ be an open subset of $X$, and $s$ in $\mathcal{M}(U)$. For $x$ in $U$, let $s(x)$ be the image of $s$ in $(G\mathcal{M})(x)$. Then $x \mapsto s(x)$ is a $C^l$-section of $G\mathcal{M}$ over $U$ (the verification of this, which can be done locally, is left to the reader). Hence we have a map from $\mathcal{M}(U)$ to $(FG\mathcal{M})U$. It is again left to the reader to check that these maps define an isomorphism of $C^l_X$-modules, and that this isomorphism is functorial. $\square$

# 3 Tensor constructions

In multi-linear algebra, there are constructions that associate, to a given collection of vector spaces, a vector space. For example, to a $k$-vector space $V$ ($k$ a field) one can associate its dual $V^\vee := \mathrm{Hom}_k(V, k)$. For $k$-vector spaces $V$ and $W$, one has $\mathrm{Hom}_k(V, W)$ and $V \otimes_k W$. We will show that the constructions in the examples carry over to vector bundles, and to their sheaves of sections. Before we do that, we recall some facts about tensor products, the symmetric algebra and the exterior algebra, mainly for free modules over a ring (that is assumed to be commutative, as usual). As a reference for multi-linear algebra one can consult any algebra book, for example Lang's "Algebra", Bourbaki, or Jacobson's "Basic algebra I and II".

## 3.1 Multi-linear algebra

Let $A$ be a commutative ring. For $A$-modules $M$ and $N$ we have the $A$-module $M \otimes_A N$, called the tensor product of $M$ and $N$ over $A$. This $A$-module $M \otimes_A N$ is defined as follows: we have a universal $A$-bilinear map $M \times N \to M \otimes_A N$, denoted by $(m, n) \mapsto m \otimes n$. (This means that for all $A$-bilinear maps $b\colon M \times N \to P$, there exists a unique $A$-linear map $\overline{b}\colon M \otimes_A N \to P$ such that $b(m, n) = \overline{b}(m \otimes n)$ for all $(m, n)$.) If $M$ and $N$ are free, with bases $m_i$, $i \in I$, and $n_j$, $j \in J$, then $M \otimes_A N$ is free and $m_i \otimes n_j$, $(i, j) \in I \times J$, is a basis. For an $A$-module $M$ we define $M^\vee$ to be the $A$-module $\mathrm{Hom}_A(M, A)$.

**3.1.1 Proposition.** *Let $A$ be a commutative ring, and $M$ and $N$ $A$-modules. Then we have an $A$-linear map $M^\vee \otimes_A N \to \mathrm{Hom}_A(M, N)$ that sends $l \otimes n$ to $m \mapsto l(m)n$. If $M$ is free of finite rank, then this map is an isomorphism of $A$-modules.*

**Proof.** Since $m \mapsto l(m)n$ is bilinear in $l$ and $n$, the required map exists and is unique. Assume now that $M$ is free of some rank $r$. To prove that the map is an isomorphism, we may suppose that $M = A^r$, because the map is functorial in $M$. But then we may identify $M^\vee$ with $A^r$, via the dual basis of the standard basis. Hence $M^\vee \otimes_A N = A^r \otimes_A N = N^r$. On the other hand, $\mathrm{Hom}_A(A^r, N) = N^r$. We leave it to the reader to see that our map is this identification. $\square$

### 3.1.2 The tensor algebra

Let $A$ be a commutative ring, and $M$ an $A$-module. For $i \geq 0$ let $\mathrm{T}^i(M) := M^{\otimes i}$ be the $i$th tensor power of $M$. One way to define $\mathrm{T}^i(M)$ is to say that we have a universal $i$-linear map $M^i \to \mathrm{T}^i(M)$, sending $(m_1, \ldots, m_i)$ to $m_1 \otimes \cdots \otimes m_i$. We define the tensor algebra of $M$ to be the $A$-module $\mathrm{T}(M) := \oplus_{i \geq 0} \mathrm{T}^i(M)$, with the $A$-algebra structure defined as follows. Let $i$ and $j$ be $\geq 0$. Consider the map $M^i \times M^j = M^{i+j} \to \mathrm{T}^{i+j}(M)$. Since this map is $i$-linear in the first variable, it induces a map $\mathrm{T}^i(M) \times M^j \to \mathrm{T}^{i+j}(M)$ that is linear in the first variable. This last map is $j$-linear in the second variable, hence induces a map $\mathrm{T}^i(M) \times \mathrm{T}^j(M) \to \mathrm{T}^{i+j}(M)$ that is bilinear and defines our multiplication map. The reader will check that $\mathrm{T}(M)$ becomes an associative graded $A$-algebra. We have $\mathrm{T}^0(M) = A$ and $\mathrm{T}^1(M) = M$. Let $B$ be an associative $A$-algebra, and $\phi\colon M \to B$ a morphism of $A$-modules. Then there exists a unique morphism of $A$-algebras $\tilde{\phi}\colon \mathrm{T}(M) \to B$ such that the restriction of $\tilde{\phi}$ to $\mathrm{T}^1(M)$ is $\phi$. This situation gives us an example of adjoint functors: we have a functor $F$ from the category of associative $A$-algebras to the category of $A$-modules, that sends $B$ to $B$ viewed as an $A$-module, and the functor T in the other direction, such that

$$\mathrm{Hom}_{A-\mathrm{mod}}(M, F(B)) = \mathrm{Hom}_{\mathrm{ass}-A-\mathrm{alg}}(\mathrm{T}(M), B),$$

functorially in $M$ and $B$. If $M$ is a free $A$-module of rank $n$, then $\mathrm{T}^i(M)$ is free of rank $n^i$ (the reader will provide a basis).

### 3.1.3 The symmetric algebra

Let again $A$ be a commutative ring, and $M$ an $A$-module. We define the symmetric algebra $\mathrm{S}(M)$ to be the quotient of $\mathrm{T}(M)$ by the ideal generated by all $x \otimes y - y \otimes x$ with $x$ and $y$ in $M$. Since $\mathrm{T}(M)$ is generated, as $A$-algebra, by $M$, $\mathrm{S}(M)$ is a commutative $A$-algebra. Since the kernel of $\mathrm{T}(M) \to \mathrm{S}(M)$ is generated by homogeneous elements, the grading on $\mathrm{T}(M)$ induces a grading on $\mathrm{S}(M)$. For $i \geq 0$, $\mathrm{S}^i(M)$ is called the $i$th symmetric product of $M$, and denoted $\mathrm{Sym}^i_A(M)$. As in the case of the tensor algebra, the functor S, from the category of $A$-modules to the category $A$-alg of commutative $A$-algebras, is the left-adjoint of the forget functor in the opposite direction:

$$\mathrm{Hom}_{A-\mathrm{mod}}(M, B) = \mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{S}(M), B).$$

If $M$ is free of rank $n$, say with basis $m_1, \ldots, m_n$, then the morphism of $A$-algebras from the polynomial ring $A[x_1, \ldots, x_n]$ to $\mathrm{S}(M)$, sending $x_i$ to $m_i$, is an isomorphism (use the universal property of $\mathrm{S}(M)$ to define the inverse). In particular, $\mathrm{S}^i(M)$ has basis $m_1^{i_1} \cdots m_n^{i_n}$, $i_1 + \cdots + i_n = i$, hence $\mathrm{S}^i(M)$ is free of rank $\binom{i+n-1}{n-1}$. It is left as an exercise to see that, for any $A$-module $M$, the map $M^i \to \mathrm{S}^i(M)$ that sends $(m_1, \ldots, m_i)$ to $m_1 \cdots m_i$ is a universal symmetric $i$-linear map.

### 3.1.4 The exterior algebra

This section is somewhat less trivial than the previous two, since as a special case we construct the determinant of a square matrix. Let again $A$ be a commutative ring, and $M$ an $A$-module. The exterior algebra $\Lambda(M)$ of $M$ is then defined to be the quotient of $\mathrm{T}(M)$ by the ideal generated by the $x \otimes x$, with $x$ in $M$. Then $\Lambda(M)$ is a graded associative $A$-algebra. We have $\Lambda^0(M) = A$ and $\Lambda^1(M) = M$. We claim that $\Lambda(M)$ is what is called "graded-commutative":

$$yx = (-1)^{ij} xy, \quad \text{for all } i, j \geq 0,\ x \text{ in } \Lambda^i(M),\ y \text{ in } \Lambda^j(M).$$

To see this, we first note that this is true when $i$ and $j$ are both equal to one, since then $0 = (x+y)(x+y) - xx - yy = xy + yx$. Then it follows for all $i$ and $j$, since $M$ generates $\Lambda(M)$. The product in $\Lambda(M)$ is called the wedge product, and is sometimes denoted $(x, y) \mapsto x \wedge y$. The exterior algebra $\Lambda(M)$ has the following universal property: if $\phi\colon M \to B$ is an $A$-linear map from $M$ to an $A$-algebra $B$, such that $\phi(m)\phi(m) = 0$ for all $m$ in $M$, then there exists a unique morphism of $A$-algebras $\tilde{\phi}$ from $\Lambda(M)$ to $B$ whose restriction to $M$ is $\phi$. We leave it as an exercise to show that the map $M^i \to \Lambda^i(M)$ that sends $(m_1, \ldots, m_i)$ to $m_1 \cdots m_i$ is a universal alternating $i$-linear map. If $M$ is free of rank $n$, say with basis $m_1, \ldots, m_n$, then $\Lambda^i(M)$ is free of rank $\binom{n}{i}$, and $m_{j_1} \cdots m_{j_i}$, $j_1 < \cdots < j_i$, is a basis. Since this is not so obvious, we will give a proof.

So assume that $m_1, \ldots, m_n$ is a basis of $M$. Let $i \geq 0$. It is clear that the $m_{j_1} \cdots m_{j_i}$, $j_1 < \cdots < j_i$, generate $\Lambda^i(M)$, so it remains to show that they are linearly independent. This follows if we can construct, for all $j = (j_1, \ldots, j_i)$, with $j_1 < \cdots < j_i$, an $A$-linear map $\phi_j$ from $\Lambda^i(M)$ to $A$, such that for all $k = (k_1, \ldots, k_i)$ with $k_1 < \cdots < k_i$ we have $\phi_j(m_{k_1}, \ldots, m_{k_i}) = \delta_{j,k}$.

Note that $\mathrm{T}^i(M)$ has basis $m_{l_1}\cdots m_{l_i}$, $l_k\in\{1,\ldots,n\}$. Define an $A$-linear map $\phi_j$ from $\mathrm{T}^i(M)$ to $A$ by:

$$\begin{aligned}\phi_j(m_{l_1}\cdots m_{l_i}) &= 0 \quad \text{if } \{l_1,\ldots,l_i\}\neq\{j_1,\ldots,j_i\},\\ \phi_j(m_{l_1}\cdots m_{l_i}) &= \varepsilon(\sigma) \quad \text{if } \{l_1,\ldots,l_i\}=\{j_1,\ldots,j_i\},\end{aligned}$$

where $\sigma$ is the permutation sending $j_k$ to $l_k$ ($\varepsilon$ is the sign of a permutation). Then $\phi_j$ induces the desired alternating $i$-linear form on $M$.

**3.1.5 Exercise.** Let $k$ be a field, $V$ a $k$-vector space and $d\geq 0$ an integer. Recall that we have defined the Grassmannian $\mathrm{Gr}_d(V)$ of $d$-dimensional subspaces of $V$, with the structure of algebraic variety over $k$ if $k$ is algebraically closed and $V$ finite dimensional.

Let $W$ be a $d$-dimensional subspace of $V$, and let $w_1,\ldots,w_d$ be a basis of $W$. Then we get an element $w_1\cdots w_d$ of $\Lambda^d(V)$. Show that this element is non-zero, and that its image in $\mathbb{P}(\Lambda^d(V))$ only depends on $W$. Hence we have a map $\phi$ from $\mathrm{Gr}_d(V)$ to $\mathbb{P}(\Lambda^d(V))$. Show that this map is injective. It is called the Plücker embedding.

Suppose now that $k$ is algebraically closed and that $V$ is finite dimensional. Show that $f$ is a morphism of algebraic varieties. Try to describe the image of $f$ in the case $d=2$ and $\dim(V)=4$. (It is possible to describe the image by just one homogeneous equation of degree two.)

## 3.2 Tensor products of vector bundles and locally free sheaves

Let $X$ be a $C^k$-manifold ($k\geq 0$), and $p\colon E\to X$ a real (or complex) $C^l$-vector bundle on it (with $l\leq k$). Then we define the dual $E^\vee$ of $E$, which is also a real (or complex) $C^l$-vector bundle, as follows. As a set, $E^\vee$ is the disjoint union of the $E(x)^\vee$, $x$ in $X$. Let $q\colon E^\vee\to X$ be the map with $q^{-1}x=E(x)^\vee$. To give $E^\vee$ the structure of a $C^l$-vector bundle, choose a covering of $X$ by open subsets $X_i$, and trivializations $\phi_i\colon F_i\times X_i\to p^{-1}X_i$ of $E$. Then, for $x$ in $X_i$, we have an isomorphism of vector spaces $\phi_i(x)\colon F_i\to E(x_i)$. Consequently, we have an isomorphism $\psi_i(x):=(\phi_i(x)^\vee)^{-1}$ from $F_i^\vee$ to $E^\vee(x_i)$. For all $i$, we have a bijection $\psi_i\colon F_i^\vee\times X_i\to q^{-1}X_i$. These bijections give $E^\vee$ the structure of a $C^l$-vector bundle. One checks that this structure does not depend on the choice of the $X_i$ and the $\phi_i$. Defined like this, we have, for each $x$ in $X$, a bilinear map $\langle\cdot,\cdot\rangle_x$ from $E^\vee(x)\times E(x)$ to $\mathbb{R}$ (or $\mathbb{C}$), given by evaluation. For $U\subset X$ open, $s$ in $E(U)$ and $t$ in $E^\vee(U)$, the function $\langle t,s\rangle_U$ on $U$ that sends $x$ to $\langle t(x),s(x)\rangle_x$ is in $C^l_X(U)$. We want to show that these maps define an isomorphism from the $C^l_X$-module $E^\vee$ to the dual of $E$, as $C^l_X$-module. But in order to do this, we first have to define what such a dual is supposed to be.

**3.2.1 Definition.** *Let $(Y,\mathcal{O}_Y)$ be a ringed space, $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_Y$-modules. Then we define the presheaf $\mathbf{Hom}_{\mathcal{O}_Y}(\mathcal{M},\mathcal{N})$ as follows:*

$$\mathbf{Hom}_{\mathcal{O}_Y}(\mathcal{M},\mathcal{N})(U)=\mathrm{Hom}_{\mathcal{O}_Y|_U}(\mathcal{M}|_U,\mathcal{N}|_U),\quad U\subset Y \text{ open},$$

*with the obvious restriction maps. This presheaf is actually a sheaf (the rather long verification is left to the reader), and moreover a $\mathcal{O}_Y$-module. The $\mathcal{O}_Y$-module $\mathbf{Hom}_{\mathcal{O}_Y}(\mathcal{M},\mathcal{O}_Y)$ is called the dual of $\mathcal{M}$ and will be denoted $\mathcal{M}^\vee$.*

We will now first define a morphism of $C^l_X$-modules from the sheaf of sections of $E^\vee$ to the sheaf $\mathbf{Hom}_{C^l_X}(E,C^l_X)$, and then show that it is an isomorphism. So let $U\subset X$ be open, and $t$ in $E^\vee(U)$.