Note that $\mathrm{T}^i(M)$ has basis $m_{l_1}\cdots m_{l_i}$, $l_k \in \{1,\ldots,n\}$. Define an A-linear map ϕ_j from $\mathrm{T}^i(M)$ to A by:

$$\phi_j(m_{l_1}\cdots m_{l_i}) = 0 \quad \text{if } \{l_1,\ldots,l_i\} \neq \{j_1,\ldots,j_i\},$$
$$\phi_j(m_{l_1}\cdots m_{l_i}) = \varepsilon(\sigma) \quad \text{if } \{l_1,\ldots,l_i\} = \{j_1,\ldots,j_i\},$$

where σ is the permutation sending j_k to l_k (ε is the sign of a permutation). Then ϕ_j induces the desired alternating i-linear form on M.

3.1.5 Exercise. Let k be a field, V a k-vector space and $d \geq 0$ an integer. Recall that we have defined the Grassmannian $\mathrm{Gr}_d(V)$ of d-dimensional subspaces of V, with the structure of C^∞-manifold if $k = \mathbb{R}$ and V is finite dimensional.

Let W be a d-dimensional subspace of V, and let $w_1,\ldots,w_d$ be a basis of W. Then we get an element $w_1\cdots w_d$ of $\Lambda^d(V)$. Show that this element is non-zero, and that its image in $\mathbb{P}(\Lambda^d(V))$ only depends on W. Hence we have a map ϕ from $\mathrm{Gr}_d(V)$ to $\mathbb{P}(\Lambda^d(V))$. Show that this map is injective. It is called the Plücker embedding.

3.2 Tensor products of vector bundles and locally free sheaves

Let X be a C^k-manifold ($k \geq 0$), and $p\colon E \to X$ a real (or complex) C^l-vector bundle on it (with $l \leq k$). Then we define the dual $E^\vee$ of E, which is also a real (or complex) C^l-vector bundle, as follows. As a set, $E^\vee$ is the disjoint union of the $E(x)^\vee$, x in X. Let $q\colon E^\vee \to X$ be the map with $q^{-1}x = E(x)^\vee$. To give $E^\vee$ the structure of a C^l-vector bundle, choose a covering of X by open subsets X_i, and trivializations $\phi_i\colon F_i \times X_i \to p^{-1}X_i$ of E. Then, for x in X_i, we have an isomorphism of vector spaces $\phi_i(x)\colon F_i \to E(x_i)$. Consequently, we have an isomorphism $\psi_i(x) := (\phi_i(x)^\vee)^{-1}$ from $F_i^\vee$ to $E^\vee(x_i)$. For all i, we have a bijection $\psi_i\colon F_i^\vee \times X_i \to q^{-1}X_i$. These bijections give $E^\vee$ the structure of a C^l-vector bundle. One checks that this structure does not depend on the choice of the X_i and the ϕ_i. Defined like this, we have, for each x in X, a bilinear map $\langle\cdot,\cdot\rangle_x$ from $E^\vee(x) \times E(x)$ to $\mathbb{R}$ (or $\mathbb{C}$), given by evaluation. For $U \subset X$ open, s in $E(U)$ and t in $E^\vee(U)$, the function $\langle t,s\rangle_U$ on U that sends x to $\langle t(x),s(x)\rangle_x$ is in $C^l_X(U)$. We want to show that these maps define an isomorphism from the C^l_X-module $E^\vee$ to the dual of E, as C^l_X-module. But in order to do this, we first have to define what such a dual is supposed to be.

3.2.1 Definition. *Let $(Y,\mathcal{O}_Y)$ be a ringed space, $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_Y$-modules. Then we define the presheaf $\mathbf{Hom}_{\mathcal{O}_Y}(\mathcal{M},\mathcal{N})$ as follows:*

$$\mathbf{Hom}_{\mathcal{O}_Y}(\mathcal{M},\mathcal{N})(U) = \mathrm{Hom}_{\mathcal{O}_Y|_U}(\mathcal{M}|_U,\mathcal{N}|_U), \quad U \subset Y \text{ open},$$

with the obvious restriction maps. This presheaf is actually a sheaf (the rather long verification is left to the reader), and moreover a $\mathcal{O}_Y$-module. The $\mathcal{O}_Y$-module $\mathbf{Hom}_{\mathcal{O}_Y}(\mathcal{M},\mathcal{O}_Y)$ is called the dual of $\mathcal{M}$ and will be denoted $\mathcal{M}^\vee$.

We will now first define a morphism of C^l_X-modules from the sheaf of sections of $E^\vee$ to the sheaf $\mathbf{Hom}_{C^l_X}(E,C^l_X)$, and then show that it is an isomorphism. So let $U \subset X$ be open, and t in $E^\vee(U)$. Let $V \subset U$ be open, and s in $E(V)$. Then we have $\langle t|_V,s\rangle_V$ in $C^l_X(V)$. This defines a morphism of $C^l_X(U)$-modules from $E^\vee(U)$ to $\mathrm{Hom}_{C^l_X|_U}(E^\vee|_U,C^l_X|_U)$. One verifies that we get indeed a morphism from $E^\vee$ to $\mathbf{Hom}_{C^l_X}(E,C^l_X)$ as desired. Let us now show that it is an isomorphism. So let $U \subset X$ be open and let ϕ be in $\mathbf{Hom}_{C^l_X}(E, C^l_X)(U)$. Then, for each x in U, ϕ induces, via the equivalence between vector bundles and sheaves of modules (that is, Theorem 2.3.3), an element $\phi(x)$ of $E^\vee(x)$. It is easy to check that these $\phi(x)$ form an element of $E^\vee(U)$.

So we have now seen that the operation $E \mapsto E^\vee$ on C^l-vector bundles corresponds to $E \mapsto \mathbf{Hom}_{C^l_X}(E, C^l_X)$ on C^l_X-modules. Our next goal is to define tensor products, on both sides. Let us first treat the case of vector bundles. So let E and F be C^l-vector bundles on X. We define a vector bundle $E \otimes F$ as follows. As a set, it is the disjoint union of the vector spaces $E(x) \otimes F(x)$, for x in X. The map from it to X is clear. Then $E \otimes F$ is given the structure of C^l-manifold via local trivializations of E and F. Let $U \subset X$ be open, s in $E(U)$ and t in $F(U)$. For each x in U we get $s(x) \otimes t(x)$ in $(E \otimes F)(x)$. These define an element $s \otimes t$ of $(E \otimes F)(U)$. The map from $E(U) \times F(U)$ to $(E \otimes F)(U)$ sending (s,t) to $s \otimes t$ is $C^l_X(U)$-bilinear. Varying U, we get a C^l_X-bilinear map from $E \times F$ to $E \otimes F$. We will prove a bit further that this map is the universal C^l_X-bilinear map from $E \times F$ to C^l_X-modules. Before that, we define the tensor product on the side of sheaves.

3.2.2 Definition. *Let $(Y, \mathcal{O}_Y)$ be a ringed space, $\mathcal{M}$ and $\mathcal{N}$ locally free $\mathcal{O}_Y$-modules of finite rank. By definition, every y in Y has an open neighborhood U on which $\mathcal{M}$ and $\mathcal{N}$ are free, say of ranks m and n. For such open subsets U we put: $\mathcal{T}(U) := \mathcal{M}(U) \otimes_{\mathcal{O}_Y(U)} \mathcal{N}(U)$. For V an open subset of such a U we have a restriction map $\mathcal{T}(U) \to \mathcal{T}(V)$ (note that indeed $\mathcal{M}$ and $\mathcal{N}$ are free on V). Choosing, for such a U, isomorphisms $(\mathcal{O}_Y|_U)^m \to \mathcal{M}|_U$ and $(\mathcal{O}_Y|_U)^n \to \mathcal{N}|_U$, we see that $V \mapsto \mathcal{T}(V)$ is a $\mathcal{O}_Y|_U$-module. The next lemma implies, among other things, that there is a unique $\mathcal{O}_Y$-module $\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$ on Y such that for all U as above we have $(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(U) = \mathcal{T}(U)$. This $\mathcal{O}_Y$-module is locally free of finite rank.*

3.2.3 Lemma. *Let X be a topological space, $\mathcal{U}$ a collection of open subsets of X that covers X and that is a sieve on X (i.e., $U \in \mathcal{U}$ and $V \subset U$ open imply $V \in \mathcal{U}$). A presheaf on $\mathcal{U}$ is defined to be a contravariant functor from $\mathcal{U}$ (morphisms are just the inclusions) to the category of sets. A presheaf F on $\mathcal{U}$ is called a sheaf if, for all U in $\mathcal{U}$, F defines a sheaf on U. Let $\mathrm{Sh}(X)$ and $\mathrm{Sh}(\mathcal{U})$ denote the categories of sheaves on X and $\mathcal{U}$, respectively. Then the functor $F \mapsto F|_{\mathcal{U}}$ from $\mathrm{Sh}(X)$ to $\mathrm{Sh}(\mathcal{U})$ is an equivalence of categories. A quasi-inverse is described in the proof.*

Proof. Let F be a sheaf on $\mathcal{U}$. Let V be an open subset of X. Let $\mathcal{U}|_V$ be the set of U in $\mathcal{U}$ with $U \subset V$. We define $F^+(V)$ to be the projective limit $\lim\limits_{\leftarrow} F(U)$, taken over the U in $\mathcal{U}|_V$. Concretely, this means that an element of $F^+(V)$ is a compatible system of s_U in $F(U)$, indexed by $\mathcal{U}|_V$. Since the restriction of F to any U in $\mathcal{U}$ is a sheaf, we see that $F^+(U) = F(U)$ for such U. It is left to the reader to define the restriction maps for F^+, and to verify that F^+ is a sheaf. The verifications that $F \mapsto F^+$ is a functor, and that it is a quasi-inverse of $G \mapsto G|_{\mathcal{U}}$, are left to the reader. $\square$

3.2.4 Proposition. *Let $(Y, \mathcal{O}_Y)$ be a ringed space and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_Y$-modules. Let $\mathcal{U}$ be the sieve on X consisting of the U on which both $\mathcal{M}$ and $\mathcal{N}$ are trivial. Lemma 3.2.3 implies that the maps:*

$$\mathcal{M}(U) \times \mathcal{N}(U) \to \mathcal{M}(U) \otimes_{\mathcal{O}_Y(U)} \mathcal{N}(U) = (\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(U)$$

define a morphism of sheaves $\mathcal{M} \times \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$ on X. This morphism is a universal $\mathcal{O}_Y$-bilinear map from $\mathcal{M} \times \mathcal{N}$ to $\mathcal{O}_Y$-modules.

Proof. Everything but the universality is clear. So let $\mathcal{P}$ be a $\mathcal{O}_Y$-module and b a bilinear map from $\mathcal{M} \times \mathcal{N}$ to $\mathcal{P}$. Then, for each U in $\mathcal{U}$, we get a unique morphism of $\mathcal{O}_Y(U)$-modules from $(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(U)$ to $\mathcal{P}(U)$ that, composed with the universal bilinear map from $\mathcal{M}(U) \times \mathcal{N}(U)$, is $b(U)$. Lemma 3.2.3 shows that these maps give the desired unique morphism of $\mathcal{O}_Y$-modules from $\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$ to $\mathcal{P}$ that, composed with the bilinear map from $\mathcal{M} \times \mathcal{N}$, is b. $\square$

Let us now go back to our manifolds and tensor products of vector bundles: we had a C^k-manifold X and C^l-vector bundles E and F. We have already defined the C^l-vector bundle $E \otimes F$. It is easy to check that, for an open subset U of X on which E and F are trivial, we have $(E \otimes F)U = E(U) \otimes_{C^l_X(U)} F(U)$ (or, more precisely, the natural map between them is an isomorphism). Lemma 3.2.3 tells us that (the sheaf of sections of) $E \otimes F$ is the same as (more precisely, uniquely isomorphic to) $E \otimes_{C^l_X} F$.

From what we have done up to now, it is clear how to define, for a C^l-vector bundle E on a manifold X, the bundle analogs $\mathrm{T}(E)$, $\mathrm{S}(E)$ and $\Lambda(E)$ of the tensor algebra, the symmetric algebra and the exterior algebra, and that these constructions coincide with their analogs for C^l_X-modules that are locally free of finite rank. (Here we forget for a moment that vector bundles, as we have defined them, have finite dimension.)

In order to consider complex C^l-vector bundles on a C^k-manifold X it suffices to consider the sheaf $C^l_{X,\mathbb{C}}$ of complex valued C^l-functions on X, where, as explained in §1, a $\mathbb{C}$-valued function on X is called C^l if both its real and imaginary part are. Before indulging in differential forms and de Rham cohomology we take a brief look at metrics on vector bundles.

3.3 Metrics on vector bundles

Let X be a C^k-manifold, with $k \geq 0$, and E a C^l-vector bundle on X. There are various ways to describe what a metric on E is. Viewing E as a bundle, a metric on E is a collection of non-degenerate symmetric bilinear forms $\langle \cdot, \cdot \rangle_x$ on the $E(x)$, "varying C^l with x". This last condition means that after local trivialization, the coefficients of the matrix describing the $\langle \cdot, \cdot \rangle_x$ are C^l-functions. Equivalently, for U open in X and s and t in $E(U)$, $\langle s, t \rangle_U \colon x \mapsto \langle s(x), t(x) \rangle_x$ is in $C^l_X(U)$. Viewing E as a locally free C^l_X-module, a metric is a symmetric bilinear map $b \colon E \times E \to C^l_X$ such that, for all x in X, the symmetric bilinear form $b(x)$ on $E(x)$ is non-degenerate. Equivalently, b induces an isomorphism of C^l_X-modules from E to $E^\vee$. Or also: a metric on E is a symmetric isomorphism of C^l_X-modules from E to $E^\vee$. Considering the universal symmetric bilinear form on E, one sees that a metric on E is an element b of $\mathrm{S}^2(E)^\vee(X)$ such that all $b(x)$ are non-degenerate. To conclude: all various equivalent descriptions of symmetric bilinear forms that one sees in a linear algebra course work in the contexts of vector bundles and C^l_X-modules. For example, a metric b on E has a signature, which is a locally constant function $s \colon X \to \mathbb{Z}^2$ such that $s(x)_1$ (resp., $s(x)_2$) is the number of positive (resp., negative) coefficients of $b(x)$ in any diagonal form.

Usually when working with vector bundles with a metric, the metric comes naturally with the vector bundle. But sometimes it is useful to just choose a metric on a given vector bundle, if one exists (for example, if one wants to split short exact sequences of vector bundles). So a natural question to ask is: under what conditions does a vector bundle E admit a metric b (say with a fixed signature)? We will see, when discussing partitions of unity, that, for X paracompact (i.e., X is separated and every open cover has a locally finite refinement), every vector bundle has a positive definite metric. This has to do with the fact that the set of positive definite symmetric bilinear forms on $\mathbb{R}^n$ is convex. On the other hand, there are topological obstructions against the existence of metrics of signature $(1,1)$, because the set of symmetric bilinear forms on $\mathbb{R}^2$ of that signature is homotopically equivalent to the circle. For example, it can be seen that the tangent bundle of the two-sphere S^2 does not admit a metric of signature $(1,1)$ (namely, from such a metric one can construct a nowhere zero vector field on S^2, and everybody knows that the sphere can't be combed).

Before going on, let us look a bit at what is happening here. So let E be a vector bundle of constant rank r on a manifold X, and let s be a fixed signature. For x in X, the set of metrics of signature s on $E(x)$ is an open subset $Y(x)$ of $\mathrm{S}^2(E(x))^\vee$. The $Y(x)$ are all (non-canonically) isomorphic, as manifolds, to the open subset F of $\mathrm{S}^2(\mathbb{R}^r)^\vee$ consisting of metrics of signature s. Using that E is locally trivial, it is easy to equip the disjoint union Y of the $Y(x)$ with the structure of a fibration over X with fibre F. The question of whether or not E has a metric of signature s is then the same as the question of whether or not this fibration has a section. Let us now give two fibrations, with non-empty fibre, that have no section. The first example is the complement of the zero section of the Möbius strip, viewed as a fibration over S^1 with fibre $[-1,1]-\{0\}$. The other example is the Hopf fibration of S^3 over S^2; it is obtained as follows. View $\mathbb{P}^1(\mathbb{C})$ as the two-sphere S^2. Then S^2 is the quotient of $\mathbb{C}^2-\{0\}$ by the action of $\mathbb{C}^*$. View S^3 as the quotient of $\mathbb{C}^2-\{0\}$ by the subgroup $\mathbb{R}^*_{>0}$ of $\mathbb{C}^*$. Then we see that S^2 is the quotient of S^3 by the group $\mathbb{C}^*/\mathbb{R}^*_{>0}$, which is isomorphic to the subgroup S^1 of $\mathbb{C}^*$. Since the action of $\mathbb{C}^*$ on $\mathbb{C}^2-\{0\}$ is free, the action of S^1 on S^3 is so too. Hence we have our fibration. To see that there is no section, note that if there is a section, we get an isomorphism from $S^1\times S^2$ to S^3, but the first of these two is not simply connected whereas the second is.

3.3.1 Definition. *Let X be a C^k-manifold, with $k\geq 1$. A Riemannian metric on X is then a positive definite metric on T_X.*

As we have said, we will show later that every paracompact manifold has a Riemannian metric. Suppose now that X is a C^k-manifold and that $\langle\cdot,\cdot\rangle$ is a Riemannian metric on it. Suppose that $c\colon I\to X$ is C^1, with $I=[a,b]$ some non-empty closed interval in $\mathbb{R}$. Then we can define the length of c as follows:

$$\text{length}(c) := \int_a^b \|c'(t)\|_{c(t)}\,dt, \tag{3.3.2}$$

with $\|\cdot\|$ the norm associated to $\langle\cdot,\cdot\rangle$. The reader should note that when X is $\mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ is the standard Riemannian metric, this definition of length coincides with the standard one. An important fact is that if $c_1 := c\circ\phi$ with $\phi\colon J\to I$ a diffeomorphism of closed intervals, say with $J=[a_1,b_1]$, then the length of c_1 equals that of c:

$$\int_{a_1}^{b_1}\|c_1'(s)\|_{c_1(s)}\,ds = \int_{a_1}^{b_1}|\phi'(s)|\,\|c'(\phi(s))\|_{c(\phi(s))}\,ds = \int_a^b\|c'(t)\|_{c(t)}\,dt. \tag{3.3.3}$$

This implies that the length of a curve c is independent of the choice of the parametrization. Suppose now moreover that X is connected. Then X is arcwise connected, hence we can define a real valued function d on $X\times X$ by:

$$d(x,y) := \inf\{\text{length}(c)\mid c \text{ a smooth curve from } x \text{ to } y\}. \tag{3.3.4}$$

It is quite clear that d is symmetric and that it satisfies the triangle inequality. One can show without too much pain that if X is separated, then one has $d(x,y)=0$ if and only if $x=y$. See for example Chapter 9 of Spivak, Volume 1. The problem of finding the shortest path between two given points leads to variational calculus and to the definition of a geodesic.

Not all metrics that arise naturally are positive definite. For example, in the theory of general relativity one studies four-dimensional manifolds with a metric of signature $(1,3)$; so-called Lorentzian manifolds. The path from x to y that corresponds to a free fall is then a path of maximal length from x to y (of course one only considers paths that do respect the speed limit imposed by the speed of light, because otherwise the square root in the definition of $\|\cdot\|$ becomes imaginary). There is an excellent book on this matter, by Sachs and Wu, with the title "General relativity for mathematicians".

To finish this section: not even all bilinear forms that occur naturally on vector bundles are symmetric. For example, anti-symmetric bilinear forms, also called symplectic forms, play an important role in classical mechanics (Hamilton systems). Here the reader should think of formulas such as $\sum_i dp_i \wedge dq_i$.

3.4 Differential forms

Let X be a C^k-manifold, with $k \geq 1$. Then we have the C^{k-1}-vector bundle T_X on X. The dual $\mathrm{T}_X^\vee$ of T_X is called the bundle of 1-forms on X, and is denoted Ω_X^1. Note that with our conventions, Ω_X^1 also denotes the sheaf of sections of Ω_X^1. We define Ω_X to be $\Lambda(\Omega_X^1)$. This Ω_X) is a graded-commutative C_X^{k-1}-algebra, and its degree i component Ω_X^i is called the sheaf or bundle of i-forms. We will do two important things in the next two sections: we define the usual morphisms of sheaves $d\colon \Omega_X^i \to \Omega_X^{i+1}$ that give us the de Rham complex of X, and we define what integration of forms of top degree is.

Our first step is to define a map $d\colon C_X^k \to \Omega_X^1$. But even to do this, we have to go back to T_X itself: namely, we have to give an interpretation of $\mathrm{T}_X(U)$ for $U \subset X$ open. So let $U \subset X$ open, and let ∂ in $\mathrm{T}_X(U)$. We will show that ∂ defines a derivation from $C_X^k|_U$ to $C_X^{k-1}|_U$. So let $V \subset U$ be open, and let f be in $C_X^k(V)$. Let x be in V. Then we have $\partial(x)$ a tangent vector at x, and f_x in the stalk $C_{X,x}^k$. We define: $(\partial f)x := \partial(x) f_x$, which is in $\mathbb{R}$. No matter how we view tangent vectors, this number is simply the derivative at x of f in the direction $\partial(x)$. Looking in a chart, it is clear that the function ∂f from V to $\mathbb{R}$ is in $C_X^{k-1}(V)$, that ∂ is indeed a morphism of sheaves from $C_X^k|_U$ to $C_X^{k-1}|_U$ and that it is a derivation: $\partial(fg) = f\partial(g) + \partial(f)g$. Moreover, for every x in U, the map $\partial(x)\colon C_{X,x}^k \to \mathbb{R}$ that sends f to $(\partial f)x$ is in $\mathrm{Der}'_{\mathbb{R}}(C_{X,x}^k, \mathbb{R})$ (notation as in Section 2). We define $\mathbf{Der}'_{\mathbb{R}}(C_X^k, C_X^{k-1})$ to be the sheaf on X whose sections over U are the morphisms of sheaves $\partial\colon C_X^k|_U \to C_X^{k-1}|_U$ that are $\mathbb{R}$-derivations such that for every x in X the map $C_{X,x}^k \to \mathbb{R}$, $f \mapsto (\partial f)(x)$, is in $\mathrm{Der}'_{\mathbb{R}}(C_{X,x}^k, \mathbb{R})$. Then the construction that we have just given is an isomorphism of C_X^{k-1}-modules:

$$\mathrm{T}_X \longrightarrow \mathbf{Der}'_{\mathbb{R}}(C_X^k, C_X^{k-1}).$$

We can now define our map $d\colon C_X^k \to \Omega_X^1$. Let $U \subset X$ be open and let f be in $C_X^k(U)$. By construction, $\Omega_X^1(U)$ is equal to $\mathbf{Hom}_{C_X^{k-1}}(\mathrm{T}_X, C_X^{k-1})(U)$. We define df to be the element in $\Omega_X^1(U)$ that sends ∂ in $\mathrm{T}_X(V)$, with $V \subset U$ open, to ∂f. The reader will verify that d is a morphism of sheaves, that it is $\mathbb{R}$-linear and that it satisfies:

$$d(fg) = f\,dg + g\,df, \tag{3.4.1}$$

for U open in X and f and g in $C_X^k(U)$. Intuitively, the expression $(df)x$ can be thought of as a measure for the infinitesimal rate of change of f at x in an unspecified direction, and that, when evaluated on a tangent vector at x, it gives the derivative at x of f in that direction. The map d itself can be thought of as a universal derivation from C_X^k to C_X^{k-1}-modules satisfying the third property in the definition of $\mathrm{Der}'_{\mathbb{R}}(C_{X,x}^k, \mathbb{R})$ at all x. One should note that the morphism of sheaves $d\colon C_X^k \to \Omega_X^1$ is not a morphism of vector bundles (except in the case where Ω_X^1 is zero, of course), because it is not C_X^k-linear (it is a derivation, after all).

3.4.2 Proposition. *Let X be a C^k-manifold with $k \geq 1$. Let $U \subset X$ be an open set and $x_1, \ldots, x_n$ local coordinates on U, i.e., the x_i are in $C_X^k(U)$ and the map $x\colon U \to \mathbb{R}^n$ sending u to $(x_1(u), \ldots, x_n(u))$ is an isomorphism of C^k-manifolds from U to an open subset V of $\mathbb{R}^n$. Then $\Omega_X^1|_U$ is a free C_U^{k-1}-module and $(dx_1, \ldots, dx_n)$ is a basis. This basis is the dual basis of the basis $D_1, \ldots, D_n$ of $\mathrm{T}_X|_U$ given by the partial derivatives. For f in $C_X^k(U)$ we have the formula:*

$$df = \sum_{i=1}^{n} (D_i f) dx_i.$$

Proof. The fact that the D_i form a basis of $\mathrm{T}_X|_U$ was proved, point-wise, in §2.1. By construction, the dx_i form the dual basis. The formula above follows from evaluating both sides on the D_j. □

3.4.3 Corollary. *Let X be a C^k-manifold with $k \geq 1$. Let $U \subset X$ be open, $x_1, \ldots, x_n$ local coordinates on U, and $r \geq 0$ an integer. Then $\Omega_X^r|_U$ is a free C_U^{k-1}-module and the $dx_{i_1} \cdots dx_{i_r}$ with $i_1 < \cdots < i_r$ form a basis.*

3.4.4 Proposition. *Let X be a C^∞-manifold. There exists a unique morphism of sheaves $d\colon \Omega_X \to \Omega_X$ such that:*

1. *d is $\mathbb{R}$-linear and maps Ω_X^i to Ω_X^{i+1};*
2. *the restriction of d to Ω_X^0 is $d\colon C_X^\infty \to \Omega_X^1$;*
3. *for $U \subset X$ open, x in $\Omega_X^r(U)$ and y in $\Omega_X^s(U)$, we have $d(xy) = (dx)y + (-1)^r x dy$;*
4. *$d^2 = 0$.*

Proof. Since Ω_X^1 generates Ω_X as a C_X^∞-algebra, there exists at most one such morphism. Because of this uniqueness, it suffices to prove the existence locally. So we may and do assume that X is an open subset of $\mathbb{R}^n$. Let $r \geq 0$ be an integer. The $dx_{i_1} \cdots dx_{i_r}$ with $i_1 < \cdots < i_r$ form a C_X^∞-basis of Ω_X^r. Conditions 2 and 4 of the proposition we are proving imply that we must define

$$d(f dx_{i_1} \cdots dx_{i_r}) = df\ dx_{i_1} \cdots dx_{i_r} = \sum_{i=1}^{n} (D_i f) dx_i dx_{i_1} \cdots dx_{i_r}. \tag{3.4.5}$$

Let us now show that the morphism d defined by this formula satisfies all the conditions of the proposition. Conditions 1 and 2 are clearly satisfied. Let us now do 4. One computes:

$$\begin{aligned} d(d(f dx_{i_1} \cdots dx_{i_r})) &= d(\sum_i (D_i f) dx_i dx_{i_1} \cdots dx_{i_r}) = \sum_i d(D_i f) dx_i\ dx_{i_1} \cdots dx_{i_r} = \\ &= \sum_i \sum_j D_j(D_i f) dx_j\ dx_i\ dx_{i_1} \cdots dx_{i_r} = \\ &= \left(\sum_{i,j} D_j(D_i f) dx_i\ dx_j \right) dx_{i_1} \cdots dx_{i_r} = 0. \end{aligned}$$