To prove 3, we may write $x = fdx_{i_1}\cdots dx_{i_r}$ and $y = gdx_{j_1}\cdots dx_{j_s}$. One computes:

$$\begin{aligned} d(xy) &= d(fgdx_{i_1}\cdots dx_{i_r}\, dx_{j_1}\cdots dx_{j_s}) = (fdg+gdf)dx_{i_1}\cdots dx_{i_r}\, dx_{j_1}\cdots dx_{j_s} = \\ &= df\, dx_{i_1}\cdots dx_{i_r}\, gdx_{j_1}\cdots dx_{j_s} + (-1)^r fdx_{i_1}\cdots dx_{i_r}\, dg\, dx_{j_1}\cdots dx_{j_s} = \\ &= dx\, y + (-1)^r xdy. \end{aligned}$$

□

3.4.6 Remark. Note that the previous Proposition only talks about the C^∞-case. It is certainly possible to formulate an analogous result for $k \geq 2$, but I find that too much of a hassle.

Let us consider the case where X is open in $\mathbb{R}^3$. For f in $C^k_X(X)$ with $k \geq 1$ one has $df = (D_1f)dx_1 + (D_2f)dx_2 + (D_3f)dx_3$, which is just an expression for the gradient of f. The reader should verify that $d(f_1dx_1 + f_2dx_2 + f_3dx_3)$ gives the curl and $d(f_1dx_2dx_3 + f_2dx_3dx_1 + f_3dx_1dx_2)$ the divergence. Usually, in calculus, the gradient of a function is a vector field, and not a one-form; this comes from the identification between T_X and Ω^1_X given by the standard Riemannian metric. Likewise, in calculus one applies divergence to vector fields, not to two-forms; here one uses that the multiplication $\Omega^2_X \times \Omega^1_X \to \Omega^3_X$ is a perfect pairing (i.e., it identifies both sides with the dual of the other). A similar remark holds for the curl.

3.4.7 Exercise. Let X be a C^k-manifold with $k \geq 0$. Show that X is the disjoint union (as a topological space) of its arcwise connected components; let $\pi_0(X)$ denote the set of connected components of X. Assume now that $k \geq 1$. Show that the kernel of $d\colon C^k_X \to \Omega^1_X$ is the constant subsheaf $\mathbb{R}_X$ of C^k_X (i.e., the subsheaf of locally constant functions). Conclude that:

$$\ker(d\colon C_X(X) \to \Omega^1_X(X)) = \mathbb{R}^{\pi_0(X)}.$$

3.4.8 Remark. We will see later that the complex of sheaves (Ω_X, d) is exact in all degrees $i > 0$. This means that (Ω_X, d) is a resolution of the sheaf $\mathbb{R}_X$. We will show that the sheaves Ω^i_X are acyclic for the functor $\Gamma(X, \cdot)$ when X is paracompact. It follows that under that condition, the complex $(\Omega_X(X), d)$ computes the cohomology of $\mathbb{R}_X$. Since the cohomology of this complex is by definition the de Rham cohomology, we see that the de Rham cohomology of X is the cohomology of $\mathbb{R}_X$.

3.5 Volume forms, integration and orientation

Thinking about what kind of objects one can expect to be able to integrate over manifolds, one comes to the following definition.

3.5.1 Definition. *Let V be a finite dimensional $\mathbb{R}$-vector space, say of dimension n. Let W be an $\mathbb{R}$-vector space. A volume form on V with values in W is then a map $v\colon V^n \to W$ such that:*

1. *$v(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = v(v_1, \ldots, v_n)$, for all $v_1, \ldots, v_n$ in V and σ in S_n;*

2. *$v(\lambda v_1, v_2 \ldots, v_n) = |\lambda| v(v_1, \ldots, v_n)$, for all $v_1, \ldots, v_n$ in V and λ in $\mathbb{R}$;*

3. *$v(v_1 + v_2, v_2, \ldots, v_n) = v(v_1, \ldots, v_n)$, for all $v_1, \ldots, v_n$ in V.*

All volume forms as in the definition are obtained as follows (proof left to the reader). Let w be in W and l in $\Lambda^n(V)^\vee$. Then the map v defined by $v(v_1, \ldots, v_n) := |l(v_1, \ldots, v_n)|\, w$ is a volume form. It follows that for v a volume form, $v_1, \ldots, v_n$ in V and g in $\mathrm{GL}(V)$, we have $v(g(v_1), \ldots, g(v_n)) = |\det(g)|\, v(v_1, \ldots, v_n)$. The set $\mathrm{Vol}(V, W)$ of W-valued volume forms on V his itself an $\mathbb{R}$-vector space (sum and scalar multiplication are defined as usual). For $f \colon W \to W'$ an $\mathbb{R}$-linear map of $\mathbb{R}$-vector spaces, we get an $\mathbb{R}$-linear map f_* from $\mathrm{Vol}(V, W)$ to $\mathrm{Vol}(V, W')$, that sends v to $f \circ v$. In fact, $\mathrm{Vol}(V, \cdot)$ is a covariant functor.

3.5.2 Lemma. *Let V be a finite dimensional $\mathbb{R}$-vector space, and let n be its dimension. Then the functor $\mathrm{Vol}(V, \cdot)$ is representable, by a one-dimensional $\mathbb{R}$-vector space that we denote $|\Lambda^n(V)|$. Equivalently, we have a universal volume form $V^n \to |\Lambda^n(V)|$.*

Proof. Take l to be a non-zero element of $\Lambda^n(V)$, and consider the $\mathbb{R}$-valued volume form $|l|$. The discussion above shows that this volume form is universal. □

The set of $\mathbb{R}$-valued volume forms on V is $|\Lambda^n(V)|^\vee$. An $\mathbb{R}$-valued volume form is called positive if all its values are ≥ 0. We have a map $\Lambda^n(V)^\vee \to |\Lambda^n(V)|^\vee$ that sends l to $|l| := |\cdot| \circ l$. The image of this map is the set of positive volume forms.

3.5.3 Definition. *An orientation on a one-dimensional $\mathbb{R}$-vector space L is a connected component of $L - \{0\}$. The union of this component with $\{0\}$ will be denoted L^+ and it will be called the positive component.*

It is clear from the definition that $|\Lambda^n(V)|^\vee$ has a given orientation, for which the positive component consists of the positive volume forms. We are now ready to apply the notion of a volume form to manifolds.

3.5.4 Definition. *Let X be a C^k-manifold, for some $k \geq 1$. We define the vector bundle Vol_X of volume forms on X to be the C^{k-1}-vector bundle with $\mathrm{Vol}_X(x) = |\Lambda^{\dim_X(x)}\mathrm{T}_X(x))|^\vee$ for all x in X, with local trivializations induced by those of T_X. For W a finite dimensional $\mathbb{R}$-vector space $W \otimes_\mathbb{R} \mathrm{Vol}_X$ is defined to be the C^{k-1}-vector bundle with fibres $(W \otimes_\mathbb{R} \mathrm{Vol}_X)(x) = W \otimes_\mathbb{R} \mathrm{Vol}_X(x)$ for all x in X, with local trivializations induced by those of T_X. For $U \subset X$ open and $l \leq k-1$, a C^l-section of $W \otimes_\mathbb{R} \mathrm{Vol}_X$ over U is called a W-valued volume form on U.*

3.5.5 Remark. The sheaf of C^{k-1}-sections of $W \otimes_\mathbb{R} \mathrm{Vol}_X$ is $W_X \otimes_{\mathbb{R}_X} \mathrm{Vol}_X$, where W_X and $\mathbb{R}_X$ denote the constant sheaves on X associated to W and $\mathbb{R}$.

It follows immediately from the definitions that, for $U \subset X$ open with local coordinates $x_1, \ldots, x_n$, every element of $(W \otimes_\mathbb{R} \mathrm{Vol}_X)(U)$ can be uniquely written in the form $w \cdot |dx_1 \cdots dx_n|$, with $w \colon U \to W$ a C^{k-1}-function.

The finite dimensionality of W is there just because we have decided that vector bundles should have finite rank (they have to be manifolds themselves). Working with sheaves, there is no problem whatsoever to allow W to have infinite dimension.

3.5.6 Definition. *Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups on X and f an element of $\mathcal{F}(X)$. The support of f, denoted $\mathrm{Supp}(f)$, is defined to be the set $\{x \in X \mid f_x \neq 0\}$; it is a closed subset of X.*